An investigation of instruction in two-digit addition and subtraction using a classroom teaching experiment methodology, design research, and multilevel modeling

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An Investigation of Instruction in Two-Digit Addition and Subtraction Using a Classroom Teaching Experiment Methodology, Design Research, and Multilevel Modeling

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Thesis Declaration

I certify that the work presented in this thesis is, to the best of my knowledge and belief, original, except as acknowledged in the text, and that the material has not been submitted, either in whole or in part, for a degree at this or any other university.

I acknowledge that I have read and understood the University’s rules, requirements, procedures, and policy relating to my higher degree research award and to my thesis. I certify that I have complied with the rules, requirements, procedures, and policy of the University (as they may be from time to time).

Pamela Dalton Tabor
March 6, 2008
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\(^1\) American conventions of spelling and punctuation have been used in this thesis since I am an American.
Thank you to the administration, staff, and students of my home elementary school without whom the teaching experiment could not have occurred. Yes, Trish, I’m done now.
Dedication

I dedicate this dissertation to Ron, Jeremy, and Zack, the three most patient, supportive men I know, to God who gave me the strength to complete this endeavor, and in memory of Marissa Faith Handy who loved to learn and brought joy to all.
Abstract

In his keynote address to the National Council of Teachers of Mathematics research presession, Sloane (2006b) challenged mathematics education researchers to “quantify qualitative insights.” This quasi-experimental study used blended methods to investigate the development of two-digit addition and subtraction strategies. Concurrent classroom teaching experiments were conducted in two intact first grade classrooms (n = 41) in a mid-Atlantic American public school. From a pragmatic emergent perspective, design research (Gravemeijer & Cobb, 2006) was used to develop local instructional theory. An amplified theoretical framework for early base-ten strategies is explicated. Multilevel modeling for repeated measures was used to evaluate the differences in strategy usage between classes across occasions and the association of particular pedagogical practices with the emergence of incrementing and decrementing by ten (N10) or decomposition (1010) strategies (Beishuizen, Felix, & Beishuizen, 1990).

The two matched classes were not different in terms of gender, poverty, race, preassessment performance, and special education services. After the first unit of instruction with differentiated pedagogical tools, the collection class was significantly ($p = .001$) more likely to use 1010 than the linear class. No difference was demonstrated during the postassessment. Students in both classes were more likely to use N10 during the last structured interview than in the first ($p < .0001$). Furthermore, there was no difference between the two classes in using any advanced strategy; however, students in both classes were more likely to use an advanced strategy at the conclusion of the study than they were initially ($p = .033$). The order of emergence of 1010 and N10 was not associated with the ability to develop both strategies, but there was an association ($p < .001$) between use of an advanced strategy and success on a district-mandated written assessment of two-digit addition and subtraction.

Two original instructional sequences of contextually-based investigations are presented. Protocols transcribed from videotaped lessons and dynamic assessment interviews are presented to illuminate specific constructs detected and to illustrate the pedagogical techniques. An amplified framework for early place value constructs is proposed. Recommendations for future studies, curricular changes, and the need of early intervention are discussed.
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Chapter 1

Development of the Problem

The recent *No Child Left Behind Act of 2001* (PL 107-110) legislation in the United States mandated that schools utilize teaching methods that are substantiated by a strong body of research or rigorous evidence. High-stakes accountability testing must now be administered before a state can receive federal *Title I* funding which is used for compensatory education (Borman, Springfield, & Slavin, 2001; Gordon, 2003; Lewis, 2003) in impoverished communities. While this legislation has been the source of a great deal of debate, it has highlighted the need to have a strong research foundation to support educational decisions in pedagogy and content.


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1 American conventions of spelling and punctuation have been used in this thesis since I am an American.
thought this was such a critical issue that they developed a computer program called “Buggy” designed for use in training pre-service teachers to conduct error analysis in multi-digit addition and subtraction. It was based on the premise that children make very predictable mistakes in addition and subtraction because they develop “buggy” algorithms. Researchers have conducted studies in error patterns (Reed & Lave, 1981) and “buggy algorithms” (Ashlock, 1972, 1994, 1998). One particular “bug” was the impetus for this study: the smaller-from-larger subtraction bug (Baroody, 1987; Fuson, 1990a; Fuson & Kwon, 1992; Resnick, 1982; Sandrini, Miozzo, Cortelli, & Cappa, 2003) or what Rousham (2003) humorously called “pathological” splitting. This is the error in which a child will always find the difference between the values of the digits, say in the ones place, regardless of whether the greater value is in the minuend or subtrahend. For example, with 42-14, a child employing pathological splitting might find a difference of 32. Rather than regrouping a ten, the child would seem to reason, “I cannot take 4 from 2, so I’ll just take 2 from 4.” This error is observed in both mental calculations and in written algorithms. Why do so many children develop this particular “buggy” procedure? It is obviously divorced from any conceptual meaning. Based on my experiences as a school-based practitioner, this common error illuminated the need for greater understanding of how children learn to solve two-digit addition and subtraction. Particularly, what can be done to help children make sense of two-digit addition and subtraction? Is there one particular instructional sequence that will reduce the occurrence of pathological splitting? Are there instructional materials that should be avoided?

Over the years, many researchers in mathematics education have argued convincingly for the importance of children developing profound understanding (Hiebert & Carpenter, 1992; McClain, Cobb, & Bowers, 1998) as opposed to surface (Davis & McKnight, 1976), instrumental (Skemp, 1976), or procedural knowledge (Baroody, 2003;
Development of the Problem

Carpenter, 1986; Hiebert, 1986) which many have come to refer to as ‘school mathematics’ (Plunkett, 1979). ‘Conceptual learning’ (Simon, Tzur, Heinz, & Kinzel, 2004; Thompson, Philipp, Thompson, & Boyd, 1994) and ‘cognitive perspective’ (Carpenter, Fennema, & Franke, 1996) are terms that have become somewhat synonymous with this notion of teaching for profound understanding. The general consensus is that children make fewer errors and retain knowledge longer when mathematics is learned conceptually as opposed to when they learn ‘procedural knowledge’ apart from any sense-making (Brownell, 1947; Hiebert et al., 1997; Schoenfeld, 1988).

Before children can learn conceptually, a conducive classroom culture (Hiebert et al., 1997) must be developed. Classroom management issues, such as developing certain social norms or classroom routines and sociomathematical norms, must be given proper attention (Cobb, Stephan, McClain, & Gravemeijer, 2001; Stephan & Cobb, 2003). Attention to establishing desirable social norms is critical to establishing an environment in which learning is likely to occur. Social norms guide the social interactions that occur with regard to “…classroom participation structure… Examples of social norms include explaining and justifying solutions, attempting to make sense of explanations given by others, indicating agreement or disagreement, and questioning alternatives when a conflict in interpretations has become apparent” (Cobb, Stephan, McClain, & Gravemeijer, 2001, pp. 122-123). All classes in all subject areas function within a set of social norms. Lambert (1990) suggested that the establishment of these social norms should be an explicit topic of conversation. In other classes, however, social norms are implicitly established over time through a series of interactions within the classroom. In some classes social norms are dictated by the teacher with minimal negotiation. In other classes social norms emerge as teachers and students negotiate a series of conditions.
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within which they can work. Sociomathematical norms are norms that govern actions specific to mathematics such as “…what counts as a different mathematical solution, a sophisticated mathematical solution, an efficient mathematical solution, and an acceptable mathematical explanation” (Cobb, Stephan, McClain, & Gravemeijer, 2001, p. 124).

Pedagogy must shift from traditional, whole-group direct instruction to problem-based inquiry or investigation (Cobb, 1991; Confrey, 1990; Reys, Reys, Nohda, & Emori, 1995). This may take the form of what Fosnot and Dolk (2001) call the ‘math workshop.’ This format might entail opening instruction with a whole-group mini-lesson, followed by a workshop involving inquiry or investigation, and ending with a period of mathematical discourse (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; Hiebert et al., 1997) or ‘math congress’ (Fosnot & Dolk, 2001). The investigation would focus on one or a few engaging, rich mathematical problems at the cutting edge of the children’s mathematical understanding (Wright, Martland, Stafford, & Stanger, 2006). This is contrasted with the traditional pedagogy of giving children a worksheet of multiple problems of “naked computation” or several predictable word problems in which the children apply by rote whatever operation is the current topic of study. When repeated practice is desirable, games can provide the context for meaningful practice (van den Heuvel-Panhuizen, 2001b).

The Problem Statement

Based on my experience with many children, I surmised that working with collections in a base-ten format (i.e. grouped into tens and ones ostensibly to develop an understanding of the place value) contaminates the thinking of children and predisposes them to 1010-type partitioning strategies for mentally solving two-digit addition and subtraction. Children have even been noted to shift from incrementing by ten off the
decuple number to add and subtract (N10) to partitioning both addends into tens and ones, adding each place, and then recomposing the sum (1010) after N10 (see Chapter 3 for a fuller definition of key terms and discussion of the domain of learning) has already been established.

I have observed this shift in strategy type during instruction that was designed to promote N10 but utilized collections in the process. The task involved having an initial screened collection of 34 craft sticks, arranged into three bundles of ten and 4 loose sticks. Additional bundles were placed one at a time in an attempt to build conceptual understanding to the forward number word sequence of incrementing by ten off the decuple (thirty-four, forty-four, etc.). The following excerpt illustrates this activity. The task was a partially screened additive task. The first addend was placed under a cover. The second addend was placed on top of the cover.

**26 November**

T: If I put forty right there [places four bundles on table and covers with the lid] and put one on top [places one bundle on the lid], how much do I have?
S: Fifty. [Snickers with a big grin.]
T: Alright, if I put forty-two right there [adds two single sticks under the lid] and put ten on top, now how much do I have?
S: Fifty-two. [Laughs again].
T: If I have forty-two right there [gestures under the lid] and I put eleven on here [places one bundle and one single on top of the lid], what do I have now?
S: Oh break. … We were… Fifty-three. [Teacher grins] Yeah [while student pumps fist].
T: How did you figure that out? [Moves the lid off the covered collection]
Development of the Problem

S: Because forty, [gestures to four bundles] forty-one, forty-two [points to the two singles], fifty-two, [takes bundle from the eleven and adds it to the forty-two], thirty…no, it’s fifty-three [moves the remaining stick to the pile.]

T: Let’s do another one.

… Non-mathematical dialog…

T: Look, look here. Twenty-four and I’m covering those up. [places two bundles and four singles under the lid]

S: Twenty-four.

T: And I’m putting… [places two bundles on the lid].

S: [Subvocally] Twenty-four, Twenty-four, thirty-four [pause] forty-four [Spoken audibly] It’s forty-four. [with a big smile].

10 December

After working for a couple weeks with the setting of bundles of sticks and covers, the following episode took place.

There were four bundles and one single stick on the table.

T: How much is that?

S: That’s easy. Cover it up. [Teacher places lid on the collection.] Forty-one.

T: Forty-one. Alright, and I want to add to that that much. [Places two bundles and one single stick on top of the lid.] How much is that?

S: [Subvocally] forty, fifty, sixty-one… [audibly] sixty-one.
Development of the Problem

T: Okay, how much did I have under there? [places white board on table beside the lid and taps the lid with one hand while holding the marker in readiness to record his response].

S: Sixty-two.

T: Why is it sixty-two?

S: Because there’s two of these kinds [gestures to the one visible single.]

T: Oh, so how….

S: [Interrupts with] I was counting from the tens to the ones.

Collections Conjecture

I had theorized that the use of bundles would support the conceptual understanding of the N10 strategy. Based on this excerpt and the nature and location of the pauses, I conjecture, however, that the student went from an initial counting-based N10 strategy to a collections-based 1010 strategy. What was it about the design of the task that engendered this metamorphosis in strategy? Since the episode was in a one-on-one instructional setting rather than a classroom setting, it is doubtful that social interaction played a part in this particular instance. The use of 1010 to increment and decrement by ten has also been documented by Cobb and his colleagues (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997). This strategy shift from N10 to 1010 within the course of the same task is a common phenomenon when students are presented with collections. Beishuizen (1993) hypothesized about this phenomenon when he reported about students who used base-ten blocks in conjunction with 33 + 25: “And although the N10 gestalt of the first number is immediately reconstructed (stage 2: 33 + 20 = 53), the suggestion of the alternative 1010 strategy already has been visible (30 + 20 = 50) and seems hardly avoidable” (p. 300). Cobb and Wheatley have argued that 1010
strategies “seem to reflect a prior instructional emphasis on the value of each digit of a two-digit number” (1988, p. 18). They further noted that more than one child “constructed ten as an iterable unit in the [horizontal number] sentence setting and ten as an abstract collectable unit on the tens tasks [that utilized collections of ten]” (p18). Hiebert et al. suggested that tools influence the conception of two-digit numbers which in turn predisposes children for a particular strategy. Once that strategy is established, it may be difficult for children to develop others. Cobb and his colleagues postulated that taken-as-shared mathematical practices and the “collective emergence of mathematical meaning in the classroom” (1997, p. 221) both constrain and facilitate the emergence of future mathematical meanings and practices. As the individual learns within the environment of the classroom microculture, individual constructions are also influenced.

**Focus of the Study**

Testing this contamination conjecture formed the basis of the research design. One classroom was exposed to grouping materials (e.g., collections of tens such as Unifix trains, and bundles of sticks) and contextual investigations while the other utilized what the Dutch refer to as linear models (van den Heuvel-Panhuizen, 2001b) such as the empty number line, bead strings arranged in alternating colors of lots of ten, and measurement contextual investigations. Both classrooms had already used materials such as arithmetic racks that contain both aspects prior to exploring two-digit numbers and operations. Instruction consisted of both context-based problems as well as naked computation or what the Dutch *RME* refers to as ‘bare number problems’ (van den Heuvel-Panhuizen, 1996, p. 61).

The intention of the study was to design instructional sequences within the constraints of using the particular models of number (i.e., collection and linear models)
Development of the Problem

that would lead to optimum conditions for growth in student understanding. Multilevel analysis was used to model the manner in which the base-ten additive strategies emerged in each class. These models demonstrate an association between class membership and the evolution of strategy usage. The models are presented in chapter 5. The instructional sequences that were designed with each class are presented in chapter 6 in an attempt to elaborate the educational experiences that were associated with particular strategies. Following the analysis of the distinctions between each class, chapter 7 presents similarities found in both classrooms. Based on these similarities that appear to transcend the particular instructional context, conjectures are posited about the underlying psychological constructs in the form of an amplified framework for early base-ten strategies. This theory is drawn from careful analysis of the classroom dialog and assessment data. Conclusions, limitations, and recommendations are presented in chapter 8. Chapters 2 and 3 form the literature review. Each chapter presents a different aspect of the relevant literature. Chapter 2 presents the interpretative framework. Chapter 3 presents the relevant literature regarding two-digit addition and subtraction. Chapter 4 details the methodology of the study.
Chapter 2

Theoretical Framework: Research Traditions

Within the general tradition of the assumption of teaching for profound understanding, there is a large body of research coming from two different theoretical perspectives: 1) the psychological perspective which focuses on the development of mathematical concepts within the individual (Davis, Maher, & Noddings, 1990b; Ernest, 1996; Fosnot, 1996; Kamii & DeClark, 1985; Steffe & Kieren, 1994; Steffe & Thompson, 2000a; von Glasersfeld, 1991), and 2) the sociocultural, anthropological perspective (Bishop, 2002; Bliss, Askew, & Macrae, 1996; Crawford, 1996; Forman, 1996; Lave & Wenger, 1991; Lerman, 1996, 2001; Mercer, Wegerif, & Dawes, 1999; Nunes, 1992; Perret-Clermont, Pontecorvo, Resnick, Zittoun, & Burge, 2003; Rogoff, 1995; Van der Heijden, 1994; van Oers, 1996; Wertsch, del Río, & Alvarez, 1995; Zack & Graves, 2001) in which learning is analyzed from the perspective of the collective classroom community. Over the last few years, differences of perspective have somewhat polarized researchers (Packer & Goicoechea, 2000; Vosniadou, 2007). At the risk of trivializing the two, a brief overview will now be attempted.

Psychological Perspective

Researchers from the psychological perspective are concerned with the conceptual, developmental changes that happen within the mind of the individual. They focus on the constructive nature of learning and knowing. Although there is not complete consensus (Davis, Maher, & Noddings, 1990a), they would tend to see interaction with others as a part of the environment that might act as a catalyst for reflective abstraction and reorganization (Driver, Asoko, Leach, Mortimer, & Scott, 1994). “The role of the
community—other learners and teacher—is to provide the setting, pose the challenges, and offer the support that will encourage mathematical construction” (Davis, Maher, & Noddings, 1990a, p. 3). As a part of the constructivist tradition, this perspective would limit ‘knowledge’ to the experiences and interpretations of the individual. According to Confrey, “Put into simple terms, constructivism can be described as essentially a theory about the limits of human knowledge, a belief that all knowledge is necessarily a product of our own cognitive acts” (1990, p. 108). Many constructivists would reject the notion that an individual can know an objective reality (Cobb, 2007); that is, they reject the notion that an individual’s knowledge is a perfect match of an objective reality. In fact, some would go so far as to state: “No one true reality exists, only interpretations of the world” (Clements & Battista, 1990, p. 34). Other constructivists posit the existence of an objective reality. Harlow, Cummings, and Aberasturi exemplified this stance when they wrote:

The authors’ view of constructivism holds that an independent reality does exist outside the mental world of the individual and that mental concepts and schemes are developed through the interplay of the constructive powers of the mind and the independence of the external world. This view is based on the idea that external reality can be observed and, thus, critically evaluated. Without such evaluation, concepts cannot be accepted, rejected, integrated, or refined. (2006, p. 42)

Von Glasersfeld asserted that radical constructivism “… neither engages in nor denies metaphysical considerations” (1995, p. 51).

Regardless of their view on an objective reality, for all constructivists, knowledge is a dynamic, ever-evolving construction (Confrey, 1990; Simon, 1995; Smith, 1995; von Glasersfeld, 1990, 1995). The constructivist perspective generally claims the work of Jean Piaget as its intellectual roots (Clements & Battista, 1990; Cobb, 1995b; Steffe & Kieren, 1994); although, von Glasersfeld traced its origins to the sixth century B.C. philosopher Xenophanes who argued, “even if someone succeeded in describing exactly how the
Theoretical Framework

world really is, he or she would have no way of knowing that it was the ‘true’
description” (von Glasersfeld, 1990, p. 20).

Piaget was originally trained as a biologist. Initially he was looking for biological
evidence of the development of mathematical and logical thought. Since it was
impossible to study the thought processes of pre-historic man, he turned to ontogenesis
and what he felt was the biological record captured in children. “There are children all
around us. It is with children that we have the best chance of studying the development of
logical knowledge, mathematical knowledge, physical knowledge, and so forth” (Piaget,
1971, pp. 13-14). Piaget’s early childhood experiments grew out of this intent.

Piaget developed some theories of knowledge that have greatly influenced later
researchers:

… [T]o my way of thinking the essential aspect of thought is its operative
and not its figurative aspect … I think that human knowledge is essentially
active. To know is to assimilate reality into systems of transformations. To
know is to transform reality in order to understand how a certain state is
brought about. By virtue of this point of view, I find myself opposed to the
view of knowledge as a copy, a passive copy, of reality … To my way of
thinking, knowing an object does not mean copying it—it means acting
upon it. It means constructing systems of transformations that can be
 carried out on or with this object. Knowing reality means constructing
systems of transformations that correspond, more or less adequately, to
reality. (Piaget, 1971, p. 15)

The whole notion of active construction of knowledge within the individual became a
foundational belief of the constructivist philosophy (Confrey, 1990; von Glasersfeld,

The Role of Language

For constructivists, the meaning of words is dependent on each individual’s
construction of meaning. Kamii put it this way:

Words do not represent numbers of objects in external reality. It is the
child, and not the words, who does the representing. People give meanings
to words, and the meaning of eight has to be constructed by each person
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through reflective abstraction ... Although spoken number words and written numerals belong to social knowledge, their meanings belong to logico-mathematical knowledge, which cannot be taught by direct transmission from the environment. (1986, p. 78)

From the constructivist perspective, since one can never truly know the understanding of another, knowledge cannot be captured in words. Indeed, words are interpreted by each individual and thus cannot communicate the exact understanding of the individual who wrote them. Each reader will interpret the words based on his or her individual constructions. However, for constructivists, the role of language is a vital one.

From the constructivist point of view, there can be no doubt that reflective ability is a major source of knowledge on all levels of mathematics. That is the reason why nearly all the contributors to this volume [Radical Constructivism in Mathematics Education] consider it important that students be led to talk about their thoughts, to each other, to the teacher, or to both. To verbalize what one is doing ensures that one is examining it. And it is precisely during such examination of mental operating that insufficiencies, contradictions, or irrelevancies are likely to be spotted … (von Glasersfeld, 1991, p. xviii)

Sierpinska (1998) noted that while language is important to constructivists, it is usually in the background of the research. Constructivists use language as a vehicle for understanding children’s thinking. They build models of children’s thinking based on their communications. A child’s language can provide hints about the child’s mathematical constructions.

The Role of the Teacher

Constructivism is a theory and philosophy about knowing not instruction (von Glasersfeld, 1995); therefore, generalizations about constructivist teaching are difficult to make. There are shades of interpretations that manifest themselves in a variety of pedagogical practices. There are, consequently, exceptions to the following characterization of a “constructivist teacher.”
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Because constructivists reject the notion of an accessible objective reality, the role of the teacher from the constructivist perspective is quite different from the traditional role of the teacher. Rather than disseminate knowledge to the ignorant, the teacher engineers experiences that provide the student with the opportunity to reorganize his or her thinking (Steffe, 2004; von Glasersfeld, 1987). Instruction which is approached from a constructivist philosophy would minimize the use of lecture and direct instruction (Peterson, Swing, Stark, & Waas, 1984). Rather than telling the student what to learn, the teacher would attempt to arrange educational experiences which allow the individual to make sense for him or herself (Clements & Battista, 1990). This issue has been addressed to such an extent in professional literature that a common misinterpretation is that a teacher should never tell anyone anything (Clements, 1997). Because the development of knowledge is an individual process, the development of student autonomy and reflective processes are primary functions of the teacher (Confrey, 1990).

Constructivism does not enjoy universal acceptance (Anderson, Reder, & Simon, 2000; Kirschner, Sweller, & Clark, 2006; Klahr & Nigam, 2004; Sweller, Kirschner, & Clark, 2007). It has been strongly criticized by some conservatives. William J. Bennett, the former Secretary of Education of the United States, warned parents:

If, from visiting the class and talking with the teacher, you find that the philosophy is that students must normally generate their own mathematical understandings, and that an educator should never be the “sage on the stage” who teachers directly, be suspicious. If math class is mostly a series of games, time-consuming projects, and group activities in which children are meant to discover their own way, you probably won’t be satisfied with the results. (Bennett, Finn, & Cribb, 1999, p. 329)

Sociocultural Perspective

In contrast to the psychological camp, the sociocultural perspective (Lave & Wenger, 1991; Rogoff & Lave, 1984; Wenger, 1998; Wertsch, del Rio, & Alvarez, 1995) sees learning as the acculturation (Bishop, 2002) of the individual into a socially
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prescribed body of knowledge (Driver, Asoko, Leach, Mortimer, & Scott, 1994), in this case, mathematics. Mathematics is seen as a cultural practice. Cultural, discursive psychology (Edwards, 1993; Edwards & Potter, 1992; Kieran, 2001; Kieran, Forman, & Sfard, 2001; Lerman, 1996) and activity theory (Engeström, 1987; Toulmin, 1999; Yamagata-Lynch, 2007) are related concepts. From the sociocultural perspective, the contributions of the individual are analyzed in light of how they affect the evolution of the communal thinking or how they are influenced by the culture. Public discourse takes priority over private (Kieran, 2001). The individual cannot learn apart from social interaction either with other individuals or cultural artifacts. Vygotsky wrote:

Every function in the child’s cultural development appears twice: first, on the social level, and later, on the individual level; first, between people (interpsychological), and then inside the child (intrapsychological). This applies equally to voluntary attention, to logical memory, and to the formation of concepts. All the higher functions originate as actual relations between human individuals [italics in original]. (1978, p. 57)

From the sociocultural perspective, learning happens through negotiation with a more learned other, originates in society, and is internalized by the individual. Knowledge can be defined, exists outside the individual, and can be transferred from one person to another (Atweh, Bleicher, & Cooper, 1998); although some would object to the characterization of transference. Van Oers asserted, “…the transmission models of teaching mathematics turned out to be disappointing” (2001, p. 65). Lerman summarized it this way:

The idea that we receive [emphasis in original] knowledge of the world is intended to be a shorthand to emphasize that without the input of other humans and without the potential of an individual to benefit from that input, an individual would not develop as a human in the full sense of the word. (2001, p. 92)

Lave and Wenger posited the notion of apprenticeship as “legitimate peripheral participation in communities of practice” (p. 31) in this manner:
Learning viewed as situated activity has as its central defining characteristic a process that we call *legitimate peripheral participation*. By this we mean to draw attention to the point that learners inevitably participate in communities of practitioners and that the mastery of knowledge and skill requires newcomers to move toward full participation in the sociocultural practices of a community. “Legitimate peripheral participation” provides a way to speak about the relations between newcomers and old-times, and about activities, identities, artifacts, and communities of knowledge and practice. It concerns the process by which newcomers become part of a community of practice. A person’s intentions to learn are engaged and the meaning of learning is configured through the process of becoming a full participant in a sociocultural practice. This social process includes, indeed it subsumes, the learning of knowledgeable skills. (1991, p. 29)

Rogoff also argued for “participatory appropriation” rather than acquisition or transmission (1995, p. 153). In doing so, she contended that learning is an active, dynamic process in which the learner must be engaged. That which is learned is evolving as it is appropriated through social interaction. Rogoff contrasted this with the notion that learning involves acquiring or adoption of a pre-existent, external, static body of knowledge. This appears to be consistent with the work of Vygotsky. Rosa and Montero asserted:

The Vygotskian approach, on the contrary, shatters this dualism and emphasizes the development of the individual in social interaction; specifically, the individual is formed through the internalization of activities carried out in the bosom of society and through the interaction that occurs within the zone of proximal development. Cognition is a social product that is achieved through interaction. (1990, p. 83)

Vygotsky is seen as the intellectual forerunner of the sociocultural perspective (Bliss, Askew, & Macrae, 1996; Cobb & Bauersfeld, 1995a; Leont'ev, 1997; Lerman, 2000; Wertsch, 1985a, 1985b; Wertsch, del Rio, & Alvarez, 1995) although he would have been more likely to use the term “cultural historical” approach (Bruner, 1996; Rosa & Montero, 1990). Wertsch characterized Vygotsky’s theoretical approach as having three major themes.

The three themes that form the core of Vygotsky’s theoretical framework are (1) a reliance on a genetic or developmental method; (2)
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claim that higher mental processes in the individual have their origin in social processes; and (3) the claim that mental processes can be understood only if we understand the tools and signs that mediate them (1985a, pp. 14-15).

These themes are vital threads woven into the fabric of the sociocultural perspective.

The Role of Language

Language is vital from the social perspective (Adler, 1999; Mercer, Wegerif, & Dawes, 1999; Mercer, Wegerif, Dawes, Sams, & Higgins, 2002; Sfard, 2001). Lerman asserted, “Language is not seen as giving structure to the already conscious cognizing mind; rather, the mind is constituted in discursive practices” (1996, p. 137). For the adherents of this perspective, consciousness and language cannot be separated. Vygotsky stated it this way:

We found the unit that reflects the unity of thinking and speech in the meaning [emphasis in original] of the word. As we have tried to show, word meaning is a unity of both processes that cannot be further decomposed. That is, we cannot say that word meaning is a phenomenon of either speech or thinking. The word without meaning is not a word but an empty sound. (1987b, p. 244)

Lerman also stated that language “…provides the tools of thought, and carries the cultural inheritance of the communities (ethnic, gender, class, etc.) in which the individual grows up” (1996, p. 137). Bruner contended:

Although meanings are “in the mind,” they have their origins and their significance in the culture in which they are created. It is this cultural situatedness of meanings that assures their negotiability and, ultimately, their communicability. (1996, p. 3)

Kieran argued:

“…a discursive perspective makes explicit the integration of the [social and the individual] in that both talking and thinking are considered examples of communication – communication with others and communication with self. In other words, the mediation that occurs on the social and individual planes is reconceptualized as two instances of communication. But the view of communication that is espoused is not one of ‘exchange of meanings’ where entities can be transmitted or exchanged
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without losing their identity. Rather, the notion of communication is one that is occasioned and shaped by the situation. (2001, p. 190)

Vygotsky said, “The meaningful word is a microcosm of human consciousness” (1987b, p. 285). For Vygotsky and his followers, language is an integral component of cognition.

The Role of the Teacher

The role of the teacher, from the sociocultural perspective, is to guide the classroom dialogue in such a way that the collective classroom sociomathematical norms and mathematical practices approach those of the society at large or perhaps to improve upon accepted societal norms, if one takes seriously the writings of Vygotsky (Rosa & Montero, 1990). In order to maximize learning, instruction must be within the child’s zone of proximal development (Bliss, Askew, & Macrae, 1996; Brown & Ferrara, 1985; Vygotsky, 1987a). Leont’ev summarized Vygotsky’s idea of zone of proximal development as “…characteriz[ing] the difference between what the child is capable of himself and what he can become capable of with the help of a teacher” (1997, p. 29). Scaffolding (Puntambekar & Hübscher, 2005; Sherin, Reiser, & Edelson, 2004; Stone, 1998; Wood, Bruner, & Ross, 1976) is a critical component in teaching from the sociocultural perspective (Bliss, Askew, & Macrae, 1996; Collins, Brown, & Newman, 1989). According to the sociocultural perspective, students are more likely to make sense of learning experiences that are closely aligned with their own cultural position and are more likely to be successful solving problems that are presented in formats that reflect similarities to the students’ sociocultural background (Carraher, Carraher, & Schliemann, 1987). Therefore a teacher who takes account of a sociocultural perspective would be likely to be mindful of the context of mathematics instruction and its instructional implications for the students.
Mathematics education research from the sociocultural perspective is often conducted in the setting of actual classrooms. Research from this general perspective would also include the anthropological, ethnomathematical (Nunes, 1992) research that has been conducted among street vendors in Brazil (Carraher, 1988; Carraher, Carraher, & Schliemann, 1985) and tailor’s apprentices in Liberia (Reed & Lave, 1981). Although Vygotsky’s work is generally given credit for providing the philosophical basis for the sociocultural perspective, Lerman (2001) actually went so far as to contend that Vygotskian tenets are the educational applications of Karl Marx’s theories. Leont’ev made the connection thusly:

Vygotsky’s goal was to build the foundations of a Marxist psychology, more concretely – a psychology of consciousness. He managed to see that for Marxist psychology human objective activity must become the central category … The first manifestation of this category in psychology was Vygotsky’s cultural-historical theory with the idea of the mediation of mental processes by psychological tools – by analogy with the way the material tools of labor mediates human physical activity. Via this idea Vygotsky introduced the dialectical method into psychology…. (Leont’ev, 1997, p. 32)

The sociocultural perspective has also had its critics. Anderson, Reder, and Simon (1996, 2000) have strongly questioned the empirical basis of the sociocultural perspective. Anderson et al. (1996) identified four claims they assert those from the situated learning perspective have overstated or exaggerated: “… [1] action is situationally grounded… [2] failure of knowledge to transfer… [3] training by abstraction is of little use … [and, 4] instruction is best in a highly social environment…” (p. 6-9). Anderson et al. cited empirical evidence refuting each of these purported claims. Furthermore, they questioned the instructional implications of situated learning research as “misguided” (p. 10). It should be noted, however, that Greeno (1997) rejected Anderson et al. characterization of the claims of situated learning.
Emergent Perspective

A growing number of researchers have begun to admit influence from both the constructivist and sociocultural approaches (Becker & Varelas, 1993; Billett, 1996; Cobb, 1995a, 1995b, 2000b; Cobb, Jaworski, & Presmeg, 1996; Cobb, Wood, & Yackel, 1990a, 1991; Driver, Asoko, Leach, Mortimer, & Scott, 1994; Halldén, Haglund, & Strömdahl, 2007; Hershkowitz, Hadas, Dreyfus, & Schwarz, 2007; Hiebert et al., 1997; Jones, Thornton, & Putt, 1994; Jones et al., 1996; Meira, 1995; Simon, 1995; Vosniadou, 1996, 2007). In fact, the term ‘emergent mathematics’ (Whitebread, 1995) or ‘emergent perspective’ has been coined for the approach that attempts to coordinate aspects of these two perspectives (Bowers & Nickerson, 2001; Cobb, 1995a, 2007; Cobb & Bauersfeld, 1995b; Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; Cobb, Wood, & Yackel, 1993; Cobb & Yackel, 1996; Gravemeijer & Cobb, 2006; Hershkowitz & Schwarz, 1999; Stephan & Cobb, 2003; Stephan, Cobb, & Gravemeijer, 2003; Whitenack, Knipping, & Novinger, 2001). Cobb, Wood and Yackel (1990b) traced the duality of their approach to Vygotsky. Central to this approach is the assumption that each perspective has benefits given the nature of the analysis at hand. Kieren (2000) employed the analogy of using different lenses to view the same object. Indeed, each lens colors the analysis and adds a richness that might otherwise be overlooked. While they would make distinctions between their view and the socioculturalists’ view of acculturation (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997), Cobb and others have been able to utilize this approach to coordinate research on both individual and communal aspects of learning within the same classroom episodes (Cobb, 2000a, 2001; Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; Cobb, Stephan, McClain, & Gravemeijer, 2001; Gravemeijer, Cobb, Bowers, & Whitenack, 2000;
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Stephan & Cobb, 2003; Stephan, Cobb, & Gravemeijer, 2003; Whitenack, 1995; Yackel & Cobb, 1996). The following quote speaks to this coordinated approach:

What is seen from one perspective as an act of individual learning in which a student reorganizes his or her mathematical reasoning is seen from the other perspective as an act of participation in the evolution of communal mathematical practices. In coordinating social and psychological perspectives, the approach we propose therefore seeks to analyze the development of students’ mathematical reasoning in relation to the local social situations in which they participate and to whose emergence they contribute. (Cobb, Stephan, McClain, & Gravemeijer, 2001, p. 125)

Cobb and his colleagues have developed an interpretive framework that highlights the coordination between the individual and social aspects of the emergent perspective (Gravemeijer & Cobb, 2006). Within each aspect of the framework, there are three levels for analysis. Within the social perspective, there are classroom social norms, sociomathematical norms, and classroom mathematical practices. Each of these corresponds to a conjectured individual, psychological level. The social norms correspond to one’s “[b]eliefs about [one’s] own role, others’ roles, and the general nature of mathematical activity in school” (Cobb, Stephan, McClain, & Gravemeijer, 2001, p. 119). Social norms are operational in all subject areas. An example of a social norm is the notion that one is obligated to attend to, participate in, and attempt to make sense out of the dialog that occurs in the class discussions. Sociomathematical norms (Hershkowitz & Schwarz, 1999; McClain & Cobb, 2001; Yackel & Cobb, 1996) correspond to beliefs and values the individual holds specific to mathematics. For example, McClain and Cobb (2001) identified the norm of what qualifies as a different mathematical solution as one of the sociomathematical norms that enabled the children to focus on the numerical quality of the solutions and be much more productive in their mathematical discourse and thinking. Classroom mathematical practices (Stephan & Rasmussen, 2002; Whitenack, Knipping, & Novinger, 2001) correspond to the individual’s interpretations and reasoning.
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about mathematics. The research team uses this framework to guide the analysis of classroom activity. Attention is given to tools (Sfard & McClain, 2002) that are used and taken-as-shared practices (Bowers & Nickerson, 2001; Stephan, Cobb, & Gravemeijer, 2003) that emerge as a part of that classroom activity. Conjectures are made and refuted about an individual’s mathematical learning based on the communal activity.

Distinctions between Sociocultural and Emergent Perspectives

There are at least three significant differences between the sociocultural and emergent perspectives. The first of these differences is in the relative priority of the social interactions. While discursive psychology attempts to attend to the individual and the communal emerging mathematical practices, the social level is always given priority (Kieran, 2001). From the emergent perspective, the social and individual levels are seen as reflexive, each informing and influencing the other. A second difference is the interpretation of enculturation. Although there are many exceptions, (e.g., Rogoff, Sfard, van Oers, and possibly Lerman, Lave, Wenger, and Kieran), in general socioculturalists would adhere to what constructivists would label a ‘transmission’ of meaning from one generation to another. In this respect, each succeeding generation inherits mathematical understandings from the previous generation as they are acculturated into the broader mathematical community. In other words, one is initiated into the discipline or discourse of mathematics. The constructions of the individual are minimized, perhaps even to the point of merely accepting the standards of the culture at large. Students are seen as apprentices learning from more knowledgeable experts (Rogoff, 1990). Lerman went so far as to assert that an individual’s “unique collection of subjectivities and positionings … are not ultimately private … but are each shared with others of common culture; they are communicable even though they may not be communicated” (1996, p. 141). Sfard proposed “…viewing learning mathematics as an initiation to a certain well defined
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The emergent view of enculturation, as opposed to acculturation (Bishop, 2002), is that mathematical understandings emerge and become ‘taken-as-shared’ (Cobb, 1995a, 2000a; Cobb & Yackel, 1996; Stephan & Cobb, 2003; Whitenack, 1995) from the mathematical practice in the microculture of the classroom. In this regard, the contributions of the individuals that constitute the classroom are vital to the evolution of mathematical understandings. Taken-as-shared refers to the constructivist philosophy that one can never be completely certain of what another knows. To accommodate this tenet, certain practices and operational understandings function in the social setting of the classroom. Even though each individual may have slightly different interpretations of concepts and tools, they are considered taken-as-shared when they no longer form a basis for argumentation or discussion and no longer require justification (Stephan, Cobb, & Gravemeijer, 2003) or warrants and backing as outlined in the Toulmin model of argumentation (Toulmin, 2003).

The view of tools or models is another distinction between the two perspectives. From the socioculturalist perspective, tools are embodied with meaning that is culturally dictated. Students must come to understand the proper usage of the tool. From the emergent perspective, tools are not just the objects, but also the manner in which they are utilized. Thus, tools morph as mathematical conceptions and practices change (Cobb, Boufi, McClain, & Whitenack, 1997; Cobb & Gravemeijer, in press). Those from the emergent perspective are not likely to use the term “embodiment” to refer to mathematical tools. Furthermore, the term ‘tools’ is used here, not just to refer to manipulative materials such as bead strings or rulers, but also encompasses models such as the empty number line and even the mathematical ideas themselves. Gravemeijer
(Gravemeijer, 1999, 2007; Gravemeijer & Doorman, 1999) characterized this gradual change in tool usage as a shift from informal to formal mathematics in that the model is originally a model of their thinking and becomes a model for thinking. A model of their thinking primarily communicates the reasoning that has already been employed and has emerged from the reasoning about the context. On the other hand, a student can reason with a model for thinking.

While Rogoff (1995) never employed the term “emergent,” her arguments resonate with the integrated nature of the emergent analytical approach. Rogoff proposed that research from the sociocultural tradition involves analyzing development from three different perspectives: individual, relational, and institutional.

[The] sociocultural approach … involves observation of development in three planes of analysis corresponding to personal, interpersonal, and community processes. I refer to developmental processes corresponding with these three planes of analysis as apprenticeship, guided participation, and participatory appropriation, in turn. These are inseparable, mutually constituting planes comprising activities that can become the focus of analysis at different times, but with the others necessarily remaining in the background of the analysis. I argue that children take part in the activities of their community, engaging with other children and with adults in routine and tacit as well as explicit collaboration (both in each others’ presence and in otherwise socially structured activities) and in the process of participation become prepared for later participation in related events. (p. 139)

From this quote, parallels between Rogoff’s views and the emergent perspective are quite apparent. She suggested that analysis must necessarily include attention to both the development of the individual as well as the cultural aspect of living and acting within a community.

The emergent perspective has not been without detractors. Lerman (1996, 2001) dismissed the emergent perspective because of what he contended to be its inability to account for more than shallow social interactions such as sociomathematical norms. Smith (1995), on the other hand, illuminated the fine balancing act that is necessary to
engage in research that utilizes the emergent approach or even to participate in a productive dialogue between researchers from the two perspectives.

**Summary**

The descriptions of philosophical perspectives contained in this chapter should be seen as broad characterizations of these perspectives. In fact, as Greeno (1997) noted, there is a great deal of variation in interpretation and a number of methodological specializations within each perspective. As one reads the literature, a rich, complex picture emerges with respect to each approach. Some researchers exhibit characteristics of more than one perspective. Furthermore, the adherents evolve in their thinking as time progresses (Ernest, 1996). Von Glasersfeld, who is credited with positing radical constructivism, recently proposed that constructivists should “achieve a far more detailed analysis of the complex area covered by the generic term ‘social interaction’” (2000, p. 4). This seems like a rather startling metamorphosis for the man who said:

…from the pragmatic point of view, we consider ideas, theories, and “laws of nature” as structures that are constantly exposed to our experiential world (from which we derived them), and either they hold up or they do not. Any cognitive structure that serves its purpose in our time, therefore, proves no more and no less than just that – namely, given the circumstances we have experienced (and determined by experiencing them), it has done what was expected of it. Logically, this gives us no clue as to how the “objective” world might be; it merely means that we know one viable way to a goal that we have chosen under specific circumstances in our experiential world. It tells us nothing – and cannot tell us anything – about how many other ways there might be or how that experience that we consider the goal might be connected to a world beyond our experience. The only aspect of that “real” world that actually enters into the realm of experience is its constraints….(1984, p. 24)

The statement von Glasersfeld made in 2000 could be seen as suggesting that constructivists should examine culture, which, by definition, would be greater than the experiences of any single individual. Whether or not that is what he intended to suggest, those from a sociocultural perspective would likely view the statement in that manner.
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This illustrates one of the complexities when one reads the various researchers and philosophers. The language of the discipline takes on certain connotations, depending on the culture of the perspective. This often leads to misinterpretation of statements. As Smith (1995) has argued, in order for researchers to engage in a productive dialog, a taken-as-shared vocabulary must be negotiated. Dialog and collaboration can only strengthen the research endeavors of all involved (Greeno, 1997; Kieren, 2000).

This study was conducted from a pragmatic emergent perspective (see chapter 4 for discussion).
Chapter 3

Domain of Learning

The domain of learning relevant to this study is the development of two-digit addition and subtraction. An overview of the research in this domain of learning will now be presented. Key terms with respect to the domain of learning will be defined as they have been operationalized in this study. Details of characteristics common to the pedagogy and instructional experiences of both classes will be explicated. The relevant portions of the local curriculum that constrained the pacing and scope of the study will be presented. Distinctive features of each instructional sequence will be tied to the relevant literature.

Strategies for Solving Two-Digit Addition and Subtraction

Since the late 1980s, many research studies have investigated the student-devised strategies used to solve two-digit addition and subtraction. Although researchers have chosen to use many different labels for the strategies, they roughly divide into two different categories of strategies (Cobb, 1995b). The first strategy involves keeping the first addend or minuend intact, decomposing the second operand, and adding or subtracting multiples of ten and then the remaining ones. The second strategy decomposes both addends or minuend and subtrahend into tens and units, performs the operations separately among the tens and units, and then recomposes the results. Beishuizen (Beishuizen, 1985, 1993, 1999; Beishuizen & Van Mulken, 1988; Beishuizen, Van Putten, & Van Mulken, 1997; Klein, Beishuizen, & Treffers, 1998; Wolters, Beishuizen, Broers, & Knoppert, 1990) used the abbreviation \( N10 \) to designate the first strategy. ‘\( N \)’ represents the initial number that is kept whole while ‘10’ represents the series of tens that are then added or subtracted. For example, \( 38 + 24 \) might be solved: 38
+ 20 = 58, 58 + 4 = 62. A less sophisticated example of this would add each ten from the twenty individually. Beishuizen uses \(1010\) to refer to both operands having been decomposed into tens and units. For example, with 38 + 24 the student might decompose 38 into 30 and 8 and 24 into 20 and 4. The student would then add 30 plus 20 to get 50 and 8 plus 4 to get 12. Finally the student would add 50 plus 12 to get 62. In Dutch literature written for teaching professionals, the terms *jumping* and *splitting* (Beishuizen, 1993; Menne, 2001) are used. For the purposes of this study I referred to *jump* or *split* in instructional settings and *N10* or *1010* in writing. These strategies have also been referred to as *counting-based* and *collections-based* strategies (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; Cobb & Wheatley, 1988), *aggregation* and *separated* (Heirdsfield, 2001), *collected multiunits* and *sequence multiunits* (Fuson, 1990b, 1992), *sequencing* and *partitioning* (Thompson & Smith, 1999), *base-ten* and *sequential relations procedures* (Blöte, Klein, & Beishuizen, 2000), *hold one addend constant* and *group by tens and ones* (Reys, Reys, Nohda, & Emori, 1995), *begin-with-one-number* and *decompose-tens-and-ones* methods (Fuson & Smith, 1997), *stepwise* and *hundreds, tens, units* (Selter, 2001) and *incremental* and *combining tens and ones* strategies (Carpenter, Fennema, & Franke, 1996). The emergence of these two different types of strategies appears to be based on the conception of two-digit numbers that the child holds (Cobb & Wheatley, 1988; Fuson, 1990b, 1992). Cobb (1995b) specifically argued that the emergence of N10 is dependent on the development of the conception of ten as an abstract composite unit. Kamii (1986) suggested that this is dependent on the ability to conserve the whole.

There are also some variations of these strategies. \(A10\) or *add through ten* is a variation of the N10 strategy that utilizes ‘friendly’ numbers. The problem 58 + 35 might be solved: 58 + 2 = 60, 60 + 30 = 90, 90 + 3 = 93 (Beishuizen, Van Putten, & Van
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Mulken, 1997; Blöte, Klein, & Beishuizen, 2000). N10c or jump too far is N10 with compensation (Beishuizen & Anghileri, 1998; Blöte, Klein, & Beishuizen, 2000; Klein, Beishuizen, & Treffers, 1998), round one or both addends to multiple of ten, then adjust (Reys, Reys, Nohda, & Emori, 1995), wholistic (Heirdsfield, 2001), or sequencing with compensation (I. Thompson & Smith, 1999). In this case, 96 – 29 might be solved: 96 – 30 = 66, 66 + 1 = 67. 1010s splits into tens and then sequences the remaining unit groups. 57 + 32 would be solved 50 + 30 = 80, 80 + 7 = 87, 87 + 2 = 89. Likewise, 86 – 32 would be solved 80 – 30 = 50, 50 + 6 = 56, 56 – 2 = 54. The ‘s’ stands for ‘stepwise’ (Beishuizen, Felix, & Beishuizen, 1990). Fuson and Smith (1997) and Thompson and Smith (1999) would refer to 1010s as mixed methods. Reys, Reys, Nohda, and Emori also described “round both addends to multiples of five, then adjust” (1995, p. 310).

In Beishuizen et al. (1990), 1010R stands for Reduced 1010, that is what Cobb and Wheatley (1988) called an abstract singleton strategy and Fuson labeled a strategy based on a concatenated single digit conceptual structure (Fuson, 1990a, 1992; Fuson et al., 1997). Using this method 24 + 12 would be solved as follows: 2 + 1 = 3, 4 + 2 = 6, so its 36. Beishuizen also uses u-1010 and u-N10 to designate beginning the strategy with the units rather than the tens (Beishuizen, 1993). Fuson noted that this strategy is likely to occur when addition and subtraction problems are presented in a vertical format. She argued, “…solving vertical multidigit addition or subtraction marks [i.e., bare number problems] seems to occur within a separate written marks world” that is devoid of number sense (1992, p. 262). In fact, Resnick argued, “… an algorithm need not include any explicit reference to the semantics in order to be successfully performed” (1982, p. 137).

While Hiebert and colleagues (1997) asserted that neither N10 nor 1010 is better than the other, not all would agree. Heirdsfield (2001, p. 135) classified N10 as an efficient, high level mental strategy and 1010 as a “[l]ower level alternative mental
Beishuizen and his colleagues found that students who use N10 are less likely to make errors and demonstrate greater flexibility in future strategy adaptations (Beishuizen, Van Putten, & Van Mulken, 1997; Wolters, Beishuizen, Broers, & Knoppert, 1990). Furthermore, Beishuizen found that children who demonstrated a greater strength in mathematics were more likely to use N10 than 1010 (Beishuizen, 1993, 1999; Beishuizen & Anghileri, 1998; Wolters, Beishuizen, Broers, & Knoppert, 1990). Once a strategy is established, children often have difficulty acquiring another strategy (Hiebert et al., 1997). Children who begin with 1010 frequently become entrenched in the use of that strategy to the exclusion of others (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997). This appears to be especially true for those weaker students who show preference for 1010 strategies (Beishuizen, Van Putten, & Van Mulken, 1997). Consequently, the Dutch wait a considerable amount of time before introducing instructional sequences that engender 1010 strategies or the standard algorithms (Beishuizen, Van Putten, & Van Mulken, 1997). Rather they begin with instructional sequences designed to facilitate the development of N10. This approach is taken because if the traditional algorithms are introduced too quickly, children stop making sense of the problems and revert to procedural 1010R algorithms (Beishuizen, 1999). Treffers asserted that in “…grades 1, 2 and 3 there is no room for the standard algorithm. Mental arithmetic must be developed first, according to the realistic idea” (1991, p. 48). Kamii also argued that traditional algorithms are detrimental to number sense (Kamii, 1986, 1989; Kamii & DeClark, 1985; Kamii & Dominick, 1998; Schmidt, 1995). She cited the differences between children’s invented algorithms and the traditional algorithms as evidence that the traditional algorithms are counterintuitive. Most notably, she found that children who had not been taught the traditional algorithm invariably begin with the largest place or on the left and work their way right to the
smaller places. This has been substantiated by Mandell (1985), Ian Thompson (1994) and Beishuizen, Van Putten, and Van Mulken (1997). The traditional algorithms for multi-digit addition and subtraction both begin with the smallest portion, the units in the case of whole numbers, and work their way left. Drawing on Piagetian theory, Kamii and DeClark (1985) argued that the traditional algorithms are social or conventional knowledge whereas the invented algorithms arise from logico-mathematical knowledge. Therefore the two types of knowledge form a disconnect. Kamii and Dominick cited “two reasons for saying that algorithms are harmful: 1) They encourage children to give up their own thinking, and 2) they ‘unteach’ place value, thereby preventing children from developing number sense” (1998, p. 135).

Although N10 strategies are frequently used by Dutch children (Beishuizen, Van Putten, & Van Mulken, 1997), it has been noted that American children rarely use N10 strategies to solve two-digit addition and subtraction (Fuson et al., 1997). Wright suggested that lack of ability to produce the number word sequence needed to increment and decrement by ten may be a contributing factor (Wright, Martland, Stafford, & Stanger, 2006). Beishuizen (2001) argued that the rarity of N10 is probably due to the pervasive use of base-ten blocks and other collections materials. At first glance, one might be led to believe that he is referring to all collections materials. However, he may be specifically referring to the pregrouped (Van de Walle, 2004) nature of base-ten blocks. It is worth noting here that pregrouped materials do not allow for the decomposition of one place into ten of the next smaller place. Instead, the “rod” or “long”, for example, must be exchanged for ten unit cubes. This trading process may impede the construction of conceptual understandings. It is reasonable to argue that the young children will see an exchange as very different from a breaking apart. For this
reason, the collections used in this study were groupable base-ten materials such as bundles of sticks with rubber bands and Unifix trains.

Beishuizen further cited the relatively early introduction of the standard written algorithms (Blöte, Klein, & Beishuizen, 2000). Fuson (1992), citing the rule-based approach to problems found in American textbooks, the traditional scope and sequence of instruction, and the artificial way in which regrouping is withheld from initial instruction in addition and subtraction, argued that the pervasive occurrence of the concatenated single-digit concepts is probably due to the instruction children receive.

The hypothetical learning trajectory (Simon, 1995) of two-digit addition and subtraction might be likened to a river. This river branches around an island. One branch is fraught with whitewater and the occasional eddy that entraps unwary travelers in pathological splitting (Rousham, 2003). The other branch gradually meanders along the powerful ‘jump’ current. This route also contains the possible stagnation of low level thinking (Beishuizen, 2001), but a wise guide can likely avoid these difficulties with appropriate instruction. Generally speaking, those who travel this branch are less likely to commit nonsensical errors or bugs such as the smaller-from-larger-bug (Beishuizen, Van Putten, & Van Mulken, 1997). Both routes have the potential to reach the main channel on the far side of the island. Americans have traditionally followed the splitting branch. Beishuizen (1999; 2001) has suggested, on more than one occasion, that Americans and others, like children from the United Kingdom with a similar history in use of base-ten materials, might be better off continuing to follow this route, even though it is so potentially dangerous. He appears to think that the oars Americans have traditionally chosen to use, such as base-ten blocks, have become so iconoclastic that change is nearly impossible. Instead, he suggested that Americans should better learn to tackle the splitting
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whitewater. It seems to me that we might be better served by discarding inadequate oars in favor of more sophisticated methods and tools and, therefore, a safer journey.

**Definitions of Key Domain of Learning Terms**

There are a number of key terms that pertain specifically to this domain of learning. Common definitions for these terms between the author and readers are critical to efficient communication. Therefore, these key terms are defined here. (See Appendix D for other terms and acronyms used in this thesis not specific to this domain of learning.)

**1010** – the partitioning strategy used to solve multi-digit addition and subtraction in which operands are split into quantities based on the place values of the numbers involved (e.g., tens and units). The operations are performed separately on the parts from each place. The results are then recomposed to form the sum or difference. For example, in solving $32 + 24$ the strategy might progress as shown in Figure 3.1:

![Figure 3.1. 1010 Strategy](image-url)
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**A10** – stands for adding through 10 and is a variation of N10 in which the second operand is decomposed in such a way as to enable the use of the decuple numbers. For example, in solving $47 + 25$, one might add $47 + 3 + 20 + 2$. In this manner, the “friendly” decuple numbers are used.

**Abstract Composite Unit of Ten** – the construct of ten that allows one to “... take numerical composites as single entities while simultaneously maintaining their tenness” (Cobb & Wheatley, 1988, p. 5). See also Steffe, Cobb, and von Glasersfeld (1988).

**Abstract Singleton** – when a multi-digit number is seen as individual digits (Cobb & Wheatley, 1988); roughly synonymous with Fuson et al.’s “concatenated single digit” (Fuson, 1990a; Fuson & Smith, 1997; Fuson et al., 1997).

**Algorithm** – a procedure used to solve systematically a particular type of calculation.

**Base-ten materials** – collections of materials that are grouped by place value. Base-ten blocks, Unifix rods of 10, beans in cups, and bundles of 10 straws are some common examples. These can be groupable or pregrouped. Groupable materials are those that can be ungrouped and regrouped at will. Pregrouped materials, referred to by Fuson as “size embodiments… in which the larger multiunit already exists” (1992, p. 264), are proportional materials such as base-ten blocks that cannot be decomposed into smaller units. Instead, they must be traded for equivalent values of the smaller place. For the collections instructional portion of this study, only groupable materials were used.

**Collections-based strategies** – 1010 strategies that are likely to arise from use with base-ten materials (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; Cobb & Wheatley, 1988).
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**Combinations of ten** – sets of addends that have a sum of ten (e.g., 7+3 and 1+9). These are also called number bonds, decompositions of ten, sums of ten, complements of ten, partitions of ten, and hearts of ten (Menne, 2001).

**Composite unit of ten** – the conception of ten as a unit that can be manipulated as a whole. The ten is both ten individual units and one ten simultaneously (Steffe, 2004; Steffe, Cobb, & von Glasersfeld, 1988).

**Counting-based strategies** – N10 and A10-like strategies that utilize number word sequences to solve addition and subtraction (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; Cobb & Wheatley, 1988).

**Iterable unit** – the construct of ten that allows one to use ten as a quantity that can be iterated, incrementing and decrementing. Children at this construct can add and subtract two-digit numbers without the need of manipulative materials. See Steffe (2004) and Steffe, Cobb, and von Glasersfeld (1988) for more detail.

**Jump/jumping** – the instructional name for the N10 strategy.

**Mental calculation** – calculation that is performed mentally without the aid of paper and pencil or any materials.

**N10** – strategy in which ‘N’ represents the initial number that is kept whole while ‘10’ represents the series of tens that are then added or subtracted. For example, 38 + 24 might be solved: 38 + 20 = 58, 58 + 4 = 62.

**Numerical composite of ten** – ten is seen as a collection of ten individual objects of no more significance than a group of seven objects (Steffe, Cobb, & von Glasersfeld, 1988).

**Pathological splitters** – Rousham’s term for students who use splitting all the time (2003). In the context of this study, it generally refers to students who employ the smaller-from-larger “buggy” algorithm (Baroody, 1987).
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**Split/Splitting** – the instructional name for the 1010 strategy

**Strategy** – an approach for solving a problem or calculation

**Elaboration of Two Instructional Sequences or Trajectories**

Whether one uses the term hypothetical learning trajectory (Simon, 1995), learning-teaching trajectory (van den Heuvel-Panhuizen, 2001b), landscape (Fosnot & Dolk, 2001), instructional sequence (Gravemeijer, 1999), teaching and learning cycle (Wright, Martland, Stafford, & Stanger, 2006), or river (Tabor, see above and chapter 8), it is critical to carefully develop a series of learning experiences based on a theory of how children are likely to learn and what activities are likely to invoke particular conceptions. Throughout the course of this dissertation, the terms are used somewhat interchangeably.

**Similarities in Instruction Between Classes**

In this study, many aspects of the instruction in the two classes were very similar. For example, the use of context was common practice in both classes (see chapter 4 for more detailed explanation). The RME reality principle, the notion that instructional investigation contexts should be readily imaginable by children, (van den Heuvel-Panhuizen, 2001b) was often in evidence. Both classes also began each class session with a **Mental Calculation Activity** that functioned as a brief warm-up. These activities were designed to build capacity in the prerequisites that have been identified as contributing to success in two-digit addition and subtraction. Wright, Martland, Stafford and Stanger (2006) have identified two main aspects of early number that are critical to developing facility with two-digit addition and subtraction.

**Structuring Numbers.**

The first is what might be referred to as structuring numbers (Gravemeijer, Cobb, Bowers, & Whitenack, 2000; Wright, Stanger, Stafford, & Martland, 2006) to twenty or
what Treffers called calculating up to twenty (Treffers, 2001). This includes the combinations of ten. Number bonds, decompositions, partitioning ten, or sums of ten are other names for the following combinations: 0+10, 1+9, 2+8, 3+7, 4+6, 5+5, 6+4, 7+3, 8+2, 9+1, and 10+0. Most educators believe children should be able to produce these combinations with automaticity. They should also be facile with counting-on and counting down from by ones as well as non-count-by-one (Wright, Martland, Stafford, & Stanger, 2006) or heuristic (Carpenter, 1980) strategies such as doubles, near doubles, and adding through ten up to twenty.

The second aspect of early number that is critical to success with two-digit addition and subtraction involves facility with number word sequences, both forward and backward, in the range of 1 to 100. This includes not only the ability to produce the number words by ones and identify numbers that come immediately before and after a given number but also the number words associated with the ability to increment and decrement by ten both on and off the decuple.

Mental Calculation Activities.

To address these prerequisites, a set of 180 Mental Calculation Activities (MCAs) for first grade was developed for the local school district. These MCAs were intended to be used as a brief warm-up, one on each day of the school year. The MCAs were developed with the intention of building capacity toward the ultimate goal of using N10 to add and subtract two-digit numbers. They were designed for practitioners who have had little professional development in mathematics and a very limited exposure to research in mathematics education. They were trialed during the 2003-2004 school year in 33 elementary schools in the district for which they were written. From this trial, several modifications were incorporated in a revised version. This new version was used in the current study. The original MCAs used bundles of sticks with covered tasks in an
attempt to connect meaning to incrementing and decrementing by ten. The general pattern followed this progression: materials presented alone in a partially screened setting as in Day 144 (see Figure 3.2), materials connected to the empty number line (Klein, Beishuizen, & Treffers, 1998) in a partially screened setting the next day (see Figure 3.3), materials connected to the empty number line in a totally screened setting the next day (see Figure 3.4), materials connected to the empty number line and a number sentence the next day (see Figure 3.5). A similar progression was followed with removal of tens from decuples (i.e., 40 - 10, 40 - 20, 40 - 30); decuples plus a collection of units (i.e., 50 + 1, 50 + 4, 50 + 2); off decuple plus even tens (i.e., 42 + 10, 42 + 20, 42 + 30); Subtraction of tens from off-decuple numbers (i.e., 62 – 10, 62 – 20, 62 – 30); and adding off-decuple numbers to decuple numbers (i.e., 40 + 21, 40 + 22, 40 + 23); The original MCAs contained no regrouping because it was not a part of the district or state curriculum for first grade.

In addition to these MCAs intended to promote N10, there were also strategies intended to combine 1010 with the doubles strategy (i.e. “Big Doubles” on Day 150; see Figure 3.6) and the sums of ten strategy (i.e. “Sums of 100” on Day 161; see Figure 3.7). Beishuizen et al. (1997) noted that facility with the sums to 100 as related to the sums to ten (e.g., 70 + 30) is associated with facilitating 1010 strategies.

Any MCA that was written for bundles was modified in the linear classroom. Rather than use bundles, a bead string that had ten alternating lots of two different colors of ten beads was substituted (e.g., Revised Days 144 and 167; Figures 3.8 & 3.9). Any MCA intended to promote split strategies such as Days 150 and 161 was eliminated. To complete the 180 days of MCAs, days that included crossing the decuple in both addition and subtraction replaced the eliminated days. This was to avoid a situation in which buggy procedures can become entrenched because of extended exposure to addition and
subtraction situations in which the child never encounters more than nine units in a sum or fewer units in the minuend than the subtrahend (Beishuizen, Van Putten, & Van Mulken, 1997; Carpenter, Fennema, Franke, Levi, & Empson, 1999; Fuson & Smith, 1997). The order of presentation also changed. In the original MCAs, children first worked exclusively with decuples. In the revised version, children were presented with situations that involve crossing the decuple such as \( 67 + 5 \) much earlier. It should be noted here that it was intentionally chosen not to use the term ‘regrouping’ in this instance. The very term ‘regrouping’ brings to mind images of grouping items into tens and remainders. Instead, the term ‘crossing the decuple’ was chosen to indicate that students were likely to use a counting-based strategy (for example, \( 67 + 5 \) as \( 67 + 3 + 2 \)).

In both versions of the MCAs attention was given to the sums of ten, structuring numbers to 20, and various number word sequences such as 7, 17, 27…. This was based on the assumption that if children cannot easily produce the number word sequences and sums of ten, incrementing and decrementing by ten or adding and subtracting through ten would be virtually impossible (Beishuizen, Van Putten, & Van Mulken, 1997).
“Bundles and Sticks” - (Partially Screened Task with Multiples of 10) Use a groupable base-ten model such as bundles and sticks. Briefly show the first collection and then cover. Place the second collection beside the first as illustrated. Leave the second uncovered. Ask, “How many in all? How did you know?”

First Collection

Second Collection

Figure 3.2. Day 144: Partially Screened Bundles
Figure 3.3. Day 145: Partially Screened Bundles with ENL

Day 145:

“Bundles and Sticks” (Partially Screened Task with Multiples of 10) [See Day 144.] Ask, “How many in all? How did you know?” Model students’ thinking on the ENL.

Figure 3.4. Day 146: Totally Screened Bundles with ENL

Day 146: Counting Strategies

“Bundles and Sticks” (Totally Screened Task with Multiples of 10) Use a groupable base-ten model such as bundles and sticks. Briefly show the first collection and then cover. Place the second collection beside the first as illustrated. After a brief time, cover the second collection also. Ask, “How many in all? How did you know?” Model students’ thinking on the ENL.
Day 147: Counting Strategies

“Bundles and Sticks” (Totally Screened Task with Multiples of 10) [See Day 146.] Ask, “How many in all? How did you know?” Model students’ thinking on the ENL. Have students write a number sentence to match.

Cover

Cover

Cover

Cover

+10

30 + 10 = 40

30 + 20 = 50

or

30 + 10 + 10 = 50

30 + 30 = 60

or

30 + 10 + 10 + 10 = 60

Figure 3.5. Day 147: Totally Screened Bundles with ENL and Number Sentence
Day 150: **“Finger Pattern Doubles”** 4+4 to 9+9 [See Day 105.]

**“Doubles and Halves Twins” - Big Doubles** (See Day 120.) Place an equal number of bundles inside the torso of each twin. Students should write the sum of the “Doubles Twins” that would be placed in the oval between them. Use white boards or paper for every pupil response. Repeat with other numbers.

**Think:**

- [2 bundles + 2 bundles]
  - 2 tens + 2 tens = 4 tens
  - 20 + 20 = 40

- [3 bundles + 3 bundles]
  - 3 tens + 3 tens = 6 tens
  - 30 + 30 = 60

- [4 bundles + 4 bundles]
  - 4 tens + 4 tens = 8 tens
  - 40 + 40 = 80

Figure 3.6. Day 150: Doubles

Day 161: **“Numbers that Love to Go Together to Make 10”** (See Day 113.) Have children write an expression for the two numbers that go together to make 10 (i.e. 6 + 4 or 4 + 6). (Extension: Children could use more than 2 terms such as 6 + 3 + 1.)

**“Numbers that Love to Go Together To Make 100”** (See Day 113 only have the two hearts together equal 100.) Place 4 bundles in the first heart. Have students determine the number of bundles that “loves to go with 4 bundles to make 10 bundles.” Write the related expressions.

- 4 bundles + 6 bundles = 10 bundles  ➞  40 + 60 = 100
- 5 bundles + 5 bundles = 10 bundles  ➞  50 + 50 = 100
- 2 bundles + 8 bundles = 10 bundles  ➞  20 + 80 = 100
- 7 bundles + 3 bundles = 10 bundles  ➞  70 + 30 = 100

Figure 3.7. Day 161: Numbers that Love to Go Together
**Revised Day 144: “Bead String”** (Partially Screened Task with Multiples of 10) Briefly show the first collection and then cover. Place the second collection beside the first as illustrated. Leave the second uncovered. Ask, “How many in all? How did you know?”

![Figure 3.8. Revised Day 144 with Bead String](image)

**Revised Day 167: “Bead String”** (Totally Screened Task with Multiples of 10) [See Day 144.] Ask, “How many in all? How did you know?” Model students’ thinking on the ENL and 100 Chart.

![Figure 3.9. Revised Day 167 with Bead String](image)
Instructional Sequences for Two-Digit Number Concepts

Relevant Research

It has long been considered necessary to teach place value concepts prior to teaching two-digit addition and subtraction (Wright, Stanger, Stafford, & Martland, 2006). If not explicitly stated, this belief can be inferred because place value is taught in isolation prior to instruction in multi-digit addition and subtraction (Cotter, 2000; Fuson, Smith, & Lo Cicero, 1997). Most American textbooks follow this approach (Fuson, 1990a, 1990b, 1992). Even the Dutch influenced Young Mathematicians At Work series follows this format; the series has a whole chapter on place value (Fosnot & Dolk, 2001) which precedes the additive contexts. This was also the current practice in every grade level in the district in which the study was conducted.

Numerous researchers have studied place value over the last couple decades (Baroody, 1990; Becker & Varelas, 1993; Fuson, 1990a, 1990b; Fuson & Briars, 1990; Kamii, 1986; Miura & Okamoto, 1989; Varelas & Becker, 1997). Sharon Ross presented two studies in which she conducted structured interviews with children from second through fifth grade (Ross, 1989). These interviews included a number of tasks designed to evaluate the child’s understanding of the base-ten numeral system. In one task the student was given a bag of 25 sticks that were not grouped. The student was asked to determine the number of sticks. Once the quantity was established in writing, the digits 2 and 5 were circled individually and the child was asked, “‘Does this part of your twenty-five have anything to do with how many sticks you have?’” (p.48). Less than half of the participants were successful. One-fifth of the participants thought there was no connection at all. Just over another fifth of the students “invented numerical meanings, such as that the 5 meant ‘half of ten’ . . . or that the 2 meant ‘count by twos.’” (p. 48).
Ross then repeated the procedure using base-ten blocks using 5 rods and 2 units. In this case, over two-thirds of the children were successful. Also, participants were given tasks in which they were asked to identify which digit was in a given place in a two-digit numeral. In a follow-up study, students were given 26 counters and asked contextual problems, such as a problem about wheels on a car that involved arranging the counters in sets of four. When the problem was solved, there were six sets with a remainder of two. After the child had solved the problem, the same questions as above were asked about the digits 2 and 6 from the numeral 26. Almost half of the participants responded that the 2 indicated the remainder or that the 6 indicated the number of sets.

Based on a synthesis of research studies, Ross proposed the following stages:

1. **Whole Numeral** in which 52 is seen as the whole amount. This is what Cobb and Wheatley (1988) referred to as a “numerical composite.”

2. **Positional Property** in which students are also able to identify that the tens place is on the left and the ones place is on the right, but the understanding “…does not encompass the quantities to which each corresponds” (p. 49).

3. **Face Value** in which students make a connection between models and convenient groups (e.g., cars) and may even refer to the tens place but do not “…recognize that the number represented by the tens digit is a multiple of ten” (p. 49).

4. **Construction Zone** in which students understand the concepts of tens and ones but the knowledge is fragile and performance is unreliable.

5. **Understanding** in which children understand tens and ones even in nonstandard decomposition of the whole (i.e., 1 ten and 15 ones is 25).

Ross concluded:

...even extensive experience with embodiments like base-ten blocks and other place-value manipulatives does not appear to facilitate an
understanding of place value as measured by the digit-correspondence... If we introduce materials that have been designed to embody base-ten groupings before students have constructed appropriate quantitative meanings for the individual digits, we may unwittingly provoke or reinforce a stage-3 (face value) interpretation of digits. With these materials the teacher and manufacturer may have “embodied the ten,” but the student need not. (p50)

Kamii (Kamii & Joseph, 1988) and Ian Thompson (Thompson & Bramald, 2002) found similar results in the face value tasks. They (Kamii, 1986; Thompson & Bramald, 2002) also pointed to the difficulty children have in developing place value concepts. Varelas and Becker (1997) and Wright (Aug. 8, 2004, personal e-mail correspondence) have suggested that the reason place value is so difficult for children is that there is both an additive and a multiplicative aspect to place value. Expanded notation has been used instructionally in an effort to address the additive nature. While Bell and Bell (1988 as cited in Fuson, 1992) and Stigler, Lee and Stevenson (1990) did not find evidence that expanded form supports conceptual understanding, Cotter (1996, 2000) found the use of expanded form as experienced with place value cards to be very effective in supporting conceptual understanding. This study (see chapter 6) found children often easily comprehend the additive aspect through the use of arrow cards (see Appendix B) and expanded notation. The multiplicative aspect appears to be much more complex. Wright (Wright, Stanger, Stafford, & Martland, 2006) posited that the additive aspect needs to be in place before children can contemplate the multiplicative aspect. Kamii (1986); Jones, Thornton, and Putt (1994); Thompson and Bramald (2002); and Wright (Aug. 8, 2004, personal e-mail correspondence) each called for a rethinking of the way in which place value has been historically taught. Rather than teach place value in isolation prior to multi-digit addition and subtraction, these researchers proposed that place value instruction be embedded in instruction about multi-digit addition and subtraction.
In spite of these findings, researchers have continued to teach place value concepts with base-ten blocks and Unifix trains. Fuson and Smith claimed that lack of experience using “externally countable tens” (1997, pp. 172-173) leads to the concatenated single digit concept of numbers mentioned earlier. They would include the bead string as one possible form of externally countable tens. The use of grouping materials persists even though Beishuizen (1993) found that the use of base-ten grouping materials greatly increased the likelihood of 1010 strategies. Hiebert et al. (1997) also noted that tools influence the conception of two-digit numbers which in turn predisposed children for a particular strategy. Once that strategy is established, it may be difficult for children to develop others.

To complicate the issue further, the definition of place value varies. Some immediately think of a place value mat and building numbers with collections grouped into sets of tens and singles or face value type tasks. For others, place value involves being able to manipulate numbers in such a way as to take advantage of the base-ten number system (Becker & Varelas, 1993). For these individuals, notions such as ten more, ten less, and benchmarks of decuple numbers broaden the scope of what is meant by place value. For the purposes of this study, the broader perspective was taken.

The notion that place value concepts must be taught in advance of two-digit addition and subtraction has been challenged over the last three decades (Carpenter, Fennema, Franke, Levi, & Empson, 1999; Cobb & Wheatley, 1988). Fuson mentioned this possibility even though she has not always heeded her own advice (Fuson, 1990b). The Freudenthal Institute through RME is forging a new perspective on learning place value. Rather than teaching separate instructional units on place value, notions of place value are embedded within instruction about addition and subtraction in RME. The mathematics is the context. While in RME, “group model[s]” are acknowledged in
addition to the “line” and “combined model[s]” (van den Heuvel-Panhuizen, 2001b, p. 63), collections-based materials occur relatively infrequently. For example, *RME* makes extensive use of the empty number line to model addition and subtraction strategies, but base-ten blocks would seldom if ever be used.

**Local Curriculum**

One of the intents of this study was to function within the local curricular constraints. It is therefore important to document the local curriculum with respect to this study. The district in which the study was conducted had a unit in its curriculum entitled “Grade 1 – Unit 7: Place Value and Patterns” that included the following district indicators:

- **N.1.1** Rote count to 100 forward and backward, and count on and count back from a given number to 100
- **A.1.16** Identify and represent whole numbers to 50 on a number line using manipulatives and symbols
- **N.1.2** Count up to 100 objects, and read, write, and represent whole numbers to 100 using symbols, words, and models
- **A.1.1** Represent and analyze patterns using skip counting by 2 and 10 starting with any whole number using whole numbers to 100 with and without using the 100 chart
- **PR.1.13** Express solutions using pictorial or algebraic methods
- **PR.1.21** Identify the relationship between numerical and physical models
- **N.1.3** Express whole numbers to 99 in expanded form
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- **A.1.1** Represent and analyze patterns using skip counting by 2 and 10 starting with any whole number using whole numbers to 100 with and without using the 100 chart

- **A.1.3** Represent and analyze patterns using skip counting backward by 10s starting with a multiple of 10 with and without using the 100 chart

- **A.1.12** Represent relationships using the terms greater than, less than, and equal to for quantities up to 100

- **N.1.26** Use strategies (including counting on, counting up, counting back, doubles, and making a ten) to develop fluency with basic addition and subtraction facts

Appendix C contains the district-mandated end of unit assessment and the modifications that were employed with the linear classroom.

**Inquiry-Based Approach to Place Value**

Collections Class.

Historically this unit was taught using base-ten collections. It has not been taught in a contextual setting such as inventorying materials by making groups of ten to explore two-digit numbers. Therefore, both instructional sequences outlined earlier in this chapter contained novel approaches for the district. In the collections class, instructional sequences similar to the ‘inventory’ (Cameron, Hersch, & Fosnot, 2004a; Fosnot & Dolk, 2001) and the ‘candy factory’ (Cobb, Boufi, McClain, & Whitenack, 1997; Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; Cobb, Perlwitz, & Underwood, 1996; McClain, 2003; McClain, Cobb, & Bowers, 1998; Whitenack, Knipping, Novinger, & Underwood, 2001; Yackel, 2001) were used to explore two-digit number concepts. In the inventory context, the children in the classroom inventoried various math materials in
preparation for a grant application to purchase additional similar materials. The premise was that we needed to know how many of each item we already had before we would know how many of each item still needed to be purchased. In the candy factory instructional sequence, Cobb et al. developed the scenario in which a proprietor of a candy factory stored different kinds of candies in large jars on the counter. At times the shop became busy and a long line developed. Because candy was sold by the piece, large orders often caused a backup in the shop. The class then brainstormed methods of dealing with this problem. After several possible solutions are proposed, the students were told that the proprietor determined to package the individual candies into rolls of ten candies. The class was then presented with various jars of candy. Each jar (i.e., a bag) had different amounts of candy (i.e., Unifix cubes). The children were asked to estimate how many candies they thought were in the jar. They then “packaged” the candies into rolls by creating Unifix trains of ten blocks each. After exploring this context, the rolls and candy were represented by drawing rectangles for rolls and dots for individual candies. Finally, notation was introduced in which \( r \) represented the roll and \( p \) represented individual pieces of candy. The sequence then proceeded to combining two collections of rolls and pieces as additional candies were baked and removing collections as purchases were made.

One innovation made to this instructional sequence that was unique to this study was the introduction of arrow cards and blank hundred charts that the shopkeeper used as signs to keep track of the amount of candy available between sales. Particular attention was given to presenting situations in which the child must increment and decrement by ten presenting sales in which whole rolls were made and sold (i.e., \( 52 + 20, 31 – 10 \)). Eventually the empty number line was used to record the strategies that were presented during classroom discourse if \( N10 \) or \( A10 \) were used. The empty number line was not
‘taught’ procedurally. It emerged from the discourse as a means of communicating the thinking that had been shared. As the child explained his or her thinking, the teacher modeled the thinking using the empty number line. Eventually, the empty number line became a tool for reasoning about problems. When a counting-based strategy was not presented, the empty number line was not used. In the case of 1010 strategies, multiple number sentences or arrow diagrams were used to represent the thinking. If R1010 was communicated, children were prompted with, “Is that ‘5’ in 57 really 5 or is it something else? What is it?” Children were also challenged to use the arrow cards to prove their thinking.

Linear Class.

In the linear class, an instructional sequence similar to the empty number line measurement sequence used by Fosnot and Dolk in the Mathematics in the City Professional Development Project (Cameron, Hersch, & Fosnot, 2004b; Fosnot & Dolk, 2001) was used. This sequence was developed based on the linear measurement research of Cobb and his colleagues (Stephan, 1998; Stephan, Bowers, Cobb, & Gravemeijer, 2003; Stephan, Cobb, & Gravemeijer, 2003) at Vanderbilt University. This study was presented in detail in the *Journal for Research in Mathematics Education: Monograph number 12: Supporting students’ development of measuring conceptions: Analyzing students’ learning in social context* (Stephan, Bowers, Cobb, & Gravemeijer, 2003). These instructional sequences used nonstandard linear measurement to develop flexible two-digit addition and subtraction. Initially the study began with the context of the king using his foot to measure everything. This became extremely troublesome to the king because he personally had to do all the measuring. Children then began doing the measurement with their own foot. They measured various features around the school. The mathematical concept that emerged from this experience was the notion of measuring as
covering distance. This was negotiated during the discourse that evolved from a question the teacher posed that explicitly drew attention to two different methods of measuring. One method involved counting the shoe prints while the other involved counting the steps taken. The different procedures then became a topic of discussion. Stephan et al. (2003) illustrated how the concept of measuring as covering distance became taken-as-shared. Next, acting on a suggestion from the king’s advisors, the children created a strip of five footprints for convenience sake. Partitioning distance using a collection of units was the mathematical practice that emerged from this portion of the sequence. Several aspects of determining measurement using the foot-strip were negotiated between the students and teacher during classroom discourse.

The researchers then switched to the context of the Smurf village. In this context Unifix cubes became cans of food that the Smurfs used to measure various items around the room. Each pair of children was given a bag of blocks. After some experience with this context, the tool of the Smurf bar (ten Unifix cubes linked together) was introduced. The need for this tool arose from a desire for curtailment. Rather than tote around a bag of fifty cans, the Smurfs could easily carry one Smurf bar. The children grappled with various procedures for accurately measuring the different items. After several days’ experience with the Smurf bar, the measurement strip was introduced. In this case, the needed tool emerged from the mathematical practice of one of the pairs of children. The task was to cut pieces of adding machine tape that represented wood of various lengths for a Smurf raft. During this task, one pair of children actually began to record each iteration of ten on the tape. The teacher then introduced a measurement strip that was 100 cans long, with iterations of each ten recorded on the tape. Thus, the tape was divided into ten sections and labeled with 10, 20, 30, … , 100. Children subsequently used these hundred-strips to measure. The teacher would ask questions such as “Where is twenty-
five?’ and ‘How long would something twenty-seven be?’” (Stephan, Cobb, & Gravemeijer, 2003, p. 91). Eventually hash marks were added to the hundred-strip and children used it as a tool to reason mathematically about the height of classmates. Over time, the importance of the measurement stick decreased and the empty number line served as a scaffold for their mathematical reasoning (Gravemeijer, Bowers, & Stephan, 2003).

Fosnot and Dolk (2001) developed an instructional sequence that modified and truncated those developed by McClain, Cobb, Gravemeijer, and Estes (1999) (see also Stephan, Bowers, Cobb, and Gravemeijer 2003 for a thorough discussion). The context of this instructional setting was the need to produce a pattern or plan for an individual who had agreed to cut paper strips that would be used to label the artwork created by the class. The artwork had the potential to be created on a number of different colors of paper. Each different color of paper was precut to unique dimensions. The teacher posed the task and then suggested that the students might measure each different kind of paper using Unifix cubes, since so many cubes were available in the classroom. Each station had a bin with two colors of Unifix cubes. Children worked in pairs to measure and record their findings on a sheet prepared by the teacher. The sheet listed all of the different kinds of paper available. The teacher suggested using the notation of L to represent the longer side of the paper and S to represent the shorter side of the paper. While children were measuring, several pairs devised methods for demarcating units of ten with the blocks. These different methods became explicit topics of discussion during the whole-group time.

After children were given sufficient time to measure several pieces of paper, the class held a congress to design the plan. The teacher began by asking for the measurement of the shortest side of the paper with the smallest dimensions. The teacher had strung one hundred Unifix cubes, in alternating colors of five cubes, across the blackboard. About
eight inches beneath the cubes, she had taped a long strip of adding machine tape. One measurement at a time was added to the tape. Each measurement was labeled with the color of the paper and an L or S for long or short respectively. Beneath the tape the teacher labeled the numerical value of each measurement. Children used the cubes to justify the placement of the various dimensions. During the course of the discussion, reference points began to emerge. After all the dimensions were plotted, the teacher posed additional measurements for the student to plot. Special attention was given to decuple numbers and numbers that involved incrementing and decrementing by ten the values that were already on the tape.

In subsequent lessons, the teacher posed the situation that she had additional paper at home that was in different dimensions and she was wondering how long the labels would need to be if the pieces were taped together. The teacher presented a carefully chosen sequence of problems that were likely to engender N10 and A10 strategies. As the children shared their solutions, they explained their strategies for determining the combined measurements. As explanations were given, the teacher drew a series of arcs and labels above the adding machine tape to record their thinking. In this manner, the empty number line emerged naturally from their mathematical practice. Eventually, the tape plan was no longer attached to the board. Instead, the teacher verbally referred to the tape, drew an empty line, and proceeded to record the arcs and labels to model the children’s thinking. Eventually, the empty number line became a model for their thinking (Cameron, Hersch, & Fosnot, 2004b).

Since the length of the modified measurement instructional sequence used by Fosnot and Dolk was much shorter in duration and would last approximately the same length of time as the inventory and initial candy factory context, a variation of it was used in the linear class to explore two-digit number concepts in place of the existing place
value unit. While the collections classroom was exploring the candy factory scenario, the linear classroom experienced what became known as the ‘video tapes storage’ context. The scenario was that the researcher had collected a large number of video tapes during the course of the study that needed to be organized. The researcher needed their help to determine what size containers to purchase and how many would need to be bought. They were asked to gather some data so that some decisions could be made. It just so happened that one video tape was exactly 10 Unifix cubes long. The children explored the number of tapes that would fit end-to-end in various-sized containers. This also entailed accounting for left over space in which other various artifacts from the study might be stored. They also determined how long a container would need to be if a certain number of tapes needed to be stored in it. (For this context, the children were exploring the length of the container only. The concepts of volume or area were not addressed in the study and did not arise in classroom dialog.) This scenario eventually became an additive task. The following is an example of the type of questions that were posed to the children: ‘If I wanted to store a book that was 14 cubes long and two video tapes length-wise, how long would the container need to be?’

Measurement Unit.
After these sequences were realized, the study suspended while each class completed a ten-day unit on measurement from the regular curriculum. This unit included more than just linear measurement. The study then resumed with an instructional sequence designed to explore two-digit addition and subtraction. Both classes explored the same context-based problems. They both used the empty number line. During this phase of the study, neither class emphasized the collections. Although the collection classroom made reference to the candy scenario and were free to use the collections as tools, they were not featured in the lessons and were not prominently displayed in the
classroom. They were not available in the N10 classroom at all; however, the bead string and Unifix string continued to be available in the N10 classroom, though not prominently displayed.

**Interstate Driving Scenario.**
Both classrooms explored what has been entitled the ‘interstate driving scenario’. Because the school was located within two miles of a major US interstate, the children were very familiar with interstate travel. The interstate was a limited-access, high-speed freeway which had exits that were named by community and numbered according to the distance the exit was from the state line. The scenario involved such tasks as determining the exit that would be taken if one got on the interstate at Exit 67 and drove 22 miles. The children also explored questions such as ‘If we got on at Exit 77 and got off at Exit 85, how many miles did we drive on the interstate?’ The actual exits that surrounded the school were used. The children were excited about this scenario and reported having driven passed some of the exits that had been discussed. The first grade classes took their annual field trip during this instructional sequence. The buses drove on the interstate and the children were able to see all but three of the exits that had been discussed. Interestingly enough, many of the adults that were exposed to the scenario reported that they had not realized the significance of the exit numbers.

**Homework.**
The school district had a policy that children in first grade should receive approximately ten minutes of mathematics homework four nights a week, Monday through Thursday. Because of the desire to control the strategies that were introduced to children during the study, the decision was made to use games and other activities that allowed for meaningful practice of structuring numbers to 20 and developing facility with number word sequences for the duration of the study. These included games from a
variety of sources including Wright, Stanger, Stafford, and Martland (2006), the Van de Walle (2004) dot cards, and games from the *Nimble with Numbers* series (Childs, Choate, & Jenkins, 1999). See chapter 6 and Appendix B for more information on the games and activities used in lieu of homework.

**Summary**

This chapter on the domain of learning has reviewed the literature with respect to the strategies used for two-digit addition and subtraction, place value, and the prerequisite skills that enable one to employ those strategies with facility. Relevant indicators from the local curriculum were detailed. Overviews of the Mental Calculation Activities and the inquiry-based approach used in the instructional sequences in the classroom teaching experiments were presented. Chapter 6 gives a detailed account of the instructional sequences as they were realized with each class.
Chapter 4

Research Design

This chapter details the research design employed in this study. The research questions developed as a result of the interrogation of the literature with respect to the interpretive framework (see chapter 2) and the domain of learning (see chapter 3). This study was conducted from a pragmatic emergent perspective within the domain of learning of two-digit addition and subtraction.

Research questions

This study addressed eleven research questions.

1. Are children who have not used base-ten grouping materials (i.e., bundles and sticks, Unifix trains of 10, base-ten blocks, etc.) more likely to develop N10 than those who have used the base-ten grouping materials?
2. Are children who have used base-ten grouping materials more likely to develop 1010 than those who have never used the materials?
3. If so, can the predisposition toward 1010 in those children who have experience using collections be overcome through subsequent instruction?
4. Will the children with collections exposure develop N10, given instructional experiences designed to engender N10?
5. Will children who experience collections-based instruction still exhibit the smaller-from-larger bug if adequate attention is given to developing conceptual understanding and then connecting those experiences to symbolic representations?
6. Can ten as an iterable unit (Cobb & Wheatley, 1988) be established without the use of base-ten collections?
7. Will the addition of mental calculation warm-up activities designed to build facility with number word sequences and partitions of ten eliminate some of the difficulties noted in previous studies such as difficulty developing ten as an iterable unit?

8. Will children who have demonstrated N10 in the context of materials or a contextual problem demonstrate 1010 when presented with bare two-digit addition and subtraction sentences?

9. Are there gender differences in the construction of specific strategies?

10. Is it possible to use inquiry-based investigations while operating within the constraints of local curricular and pacing guidelines?

11. Can theory be developed to describe more accurately the range of strategies that emerge for solving two-digit addition and subtraction?

**Plan of the Study**

**Blended Methods**

This study was conducted from a pragmatic emergent perspective (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; Cobb, Jaworski, & Presmeg, 1996; Cobb, Wood, & Yackel, 1993; Cobb & Yackel, 1996; Gravemeijer & Cobb, 2006; Whitebread, 1995) via mixed methods (Johnson, Onwuegbuzie, & Turner, 2007). The study attempted to analyze the contributions of individuals and the taken-as-shared understandings (Cobb, Stephan, McClain, & Gravemeijer, 2001) of the classroom community that emerged with respect to the development of two-digit addition and subtraction as evidenced by the classroom discourse, individual behaviors, and human artifacts such as written representations of strategies. A quantitative analysis was conducted as a exploratory data analysis (Tukey, 1980; Velleman, 1998) to investigate the
association between exposure to collections and development of specific strategies. The data were also analyzed with respect to demographic factors.

**Concurrent Teaching Experiments**

The study took the form of a modified classroom teaching experiment (Cobb, 2000a; Cobb, Stephan, McClain, & Gravemeijer, 2001; Stephan & Cobb, 2003). The primary innovation of this methodology was that two teaching experiments ran concurrently. The purpose of this was not to attempt to demonstrate one instructional method more effective than another but to attempt to control for a Hawthorne effect (Adair, 1984; Bramel & Friend, 1981; Brown, 1992; Sommer, 1968) and to shed additional light on particular mathematical models (Gravemeijer, 1999) or student strategies that emerged in light of certain instructional situations. Both classroom teaching experiments had the goals of maximizing the development of mathematical meaning and of designing new instructional sequences. Data analysis was a qualitative and quantitative blend (Thomas, 2003). The constraints of district-specified order of instructional units, pacing requirements, and the state curriculum content standards governed all instructional sequences realized in the study.

An additional component of analysis was included in the design of this study. While design research does not usually include quantitative analysis between groups, multilevel modeling was used to model the nature of the differences in the strategies that emerged between the classes and to evaluate the association between specific pedagogical decisions and the strategies that emerged in each classroom. This innovation is in line with the argument of Collins, Joseph and Bielaczyc that “any assessment of educational innovations must carry out both quantitative and qualitative assessments, using comparative analysis, as does Consumer Reports” (2004, p. 39). In this regard, this study was quasi-experimental (National Research Council, 2001) in that it used convenience
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samples of two intact classrooms from the same elementary school. This decision was both pragmatic and intentional. Pragmatically, it is extremely difficult to assign randomly children to classrooms even at the beginning of the school year. Several issues influence classroom membership. Some children have individualized educational programs (IEPs) which are legal documents under the auspices of the U.S. federal government for the provision of special education services. These IEPs mandate the subjects and number of minutes per week that a child receives the services of the special educator. Since the special educator cannot be in more than one place at a time, all children with an IEP in mathematics are placed in the same inclusion classroom for math instruction. Furthermore, parents often request a particular teacher for their child. Occasionally, twins are either intentionally separated or kept in the same classroom. Children with difficult behaviors are often intentionally spread across all the classrooms for that grade level to minimize behavioral problems in any one classroom. These factors make randomization of classroom assignment nearly impossible in a public school setting if children receiving special education services are a part of the study. Since it was important that the study be of practical use to public schools, it was vital that the experimental classrooms be typical of those found in American public schools. Therefore it was important for the study to have children receiving special education, children living in poverty, and children from a variety of ethnic backgrounds. This would address a common criticism from practitioners that educational research is often conducted in ideal situations with children who are not representative of the population with which they work.

Design Research

Foundational to the qualitative aspect of this blended study was the design experiment (Brown, 1992; Collins, 1992; Gorard, Roberts, & Taylor, 2004; Gravemeijer & Cobb, 2006; Verschaffel et al., 1999) or design research (Cobb, 2000b, 2003; Cobb &
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Gravemeijer, in press; Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; Collins, Joseph, & Bielaczyc, 2004; Edelson, 2002; Gravemeijer, 2004; Gravemeijer & Cobb, 2006; Gravemeijer & Doorman, 1999; Hoadley, 2002; Sloane, 2006a; van den Akker, Gravemeijer, McKenney, & Nieveen, 2006) model. Collectively, the term design studies has been used to encompass this body of research (Bannan-Ritland, 2003; Brown, 1992; Shavelson, Phillips, Towne, & Feuer, 2003). Barab and Squire broadly defined design research as, “a series of approaches, with the intent of producing new theories, artifacts, and practices that account for and potentially impact learning and teaching in naturalistic settings” (2004, p. 2). Cobb and Gravemeijer (in press) defined design research in the following manner:

…we define design research as a family of methodological approaches in which instructional design and research are interdependent. On the one hand, the design of learning environments serves as the context for research and, on the other hand, ongoing and retrospective analyses are conducted in order to inform the improvement of the design. This type of research involves attempting to support the development of particular forms of learning, and studying the learning that actually occurs in these designed settings.

Design research is an emerging form of research (Kelly & Lesh, 2002; Rittle-Johnson & Koedinger, 2005). Researchers within this field are still grappling with terminology and methodological issues as the January/February 2003 (Vol. 32(1)) special issue of Educational Researcher attests.

Originally Collins proposed eight methodological ideas that he envisioned guiding the development of design research in the area of technological innovations in education. These ideas were:

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With the exception of his fourth idea which dealt specifically with innovations in technology, this study addressed all of Collin’s ideas. Within his idea of teachers as co-investigators is the idea that “experiments must work within the constraints that the teachers involved think are necessary to be successful” (p. 17). This study explicitly set out to operate within all of the normal operational constraints under which typical teachers must work. Furthermore, the cooperating teachers were members of the research team. The design of the study tested two instructional sequences with systematic variation within one school. The study used multiple evaluations including the quantitative analysis of the association between strategy usage and instructional sequence.

The goal of the design research aspect of this study was to produce local instructional theory (Cobb, 2000a; Gravemeijer, 1998, 1999, 2004) with respect to an instructional sequence for two-digit addition and subtraction. It is hoped that this study might contribute to the on-going dialog from a collection of research studies that comprise the developmental research (Gravemeijer, 1998; van den Akker, 1999) shaping early childhood mathematics education in the area of numeracy. It is further hoped that this study might provide a practical example of how design research might be coupled with more traditional research methods as suggested by Shavelson, Phillips, Towne and Feuer (2003).

Prior to the commencement of the instructional sequences in the classrooms, a thought experiment (Dolk, den Hertog, & Gravemeijer, 2002; Freudenthal, 1988 as cited by Gravemeijer, 1998; Gravemeijer, 1998, 2004; Gravemeijer & Cobb, 2006) was conducted for each classroom. This thought experiment attempted to conjecture a local instructional theory (Gravemeijer & Cobb, 2006) that would anticipate the development of taken-as-shared classroom practices and mathematical concepts and the tasks that would engender these practices (see chapter 2 for elaboration of the emergent
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Furthermore, the thought experiment sought to anticipate individual reorganizations that would be constructed as the individuals participated in the classroom practices. The study used a cyclic process of reflection, experimentation, analysis, and modification. This process was conducted on a daily basis as the instructional sequences were realized in the classrooms (Gravemeijer, 1999). Modifications were made in the planned instructional sequences based on the mathematical practices that emerged in each classroom (Cobb, 2000b). This occurred within the course of the lesson in the form of micro-adjustments (R. J. Wright, Martland, Stafford, & Stanger, 2006) that were made in response to the behaviors observed by the teacher. Changes were also made in the instructional plan from one lesson to the next based on the mathematical practices that emerged in the previous lesson. These changes were made as a result of intentional reflection on the field notes taken during the lesson. The lessons were taught by the researcher with the participation of the classroom teachers and special educator. The researcher facilitated any whole group dialog, and the classroom teacher made occasional contributions, generally in the form of a question, when he or she perceived that it would be beneficial to the evolution of the dialog. During the closure of the lessons, the cooperating teachers frequently made comments endorsing particular behaviors that reinforced the social or sociomathematical norms (Cobb, Wood, & Yackel, 1993; McClain & Cobb, 2001) (see chapter 2 for detailed discussion) that had been established in the classroom prior to the beginning of the study or were of particular value to the cooperating teacher. The cooperating teachers and the researcher met on a regular basis. The classroom teachers were fully aware of the aims of the research and the philosophy behind the study.

This study was based on the conjecture that tools and experiences to which children are exposed affect their development of and preference for particular strategies.
Any differences between the two classes during the final instructional sequence were analyzed in light of the prior instructional experiences of each classroom.

**Realistic Mathematics Education Influence**

The design research methods and pedagogy of this study were greatly influenced by the principles and philosophy of the *Realistic Mathematics Education* (RME) developed at the Freudenthal Institute in The Netherlands (Gravemeijer, 1994a, 1999; Treffers, 1993; Treffers & Beishuizen, 1999; van den Heuvel-Panhuizen, 2001a, 2001b). RME has a set of guiding principles that govern the didactic practices associated with their research. Van den Heuvel-Panhuizen lists these principles as: the “activity principle,” the “reality principle,” the “level principle,” the “intertwinement principle,” the “interaction principle,” and the “guidance principle” (2001a, pp. 51-55). The activity principle might also be called the constructivist principle. It is the notion that children should be actively involved during instruction. Mathematical knowledge is constructed through this activity. The reality principle reflects the pedagogical strategy in which children investigate mathematical ideas through the use of contextual problems that can easily be imagined by the children. The term reality here is not meant to convey the notion that only real life scenarios should be used as contexts. A fairy tale might present such a readily imaginable context. The key here is that the context provides a means for children to reason about the mathematical concepts and practices at hand. The level principle highlights the need for contextual problems to be rich problems that allow for a range of solution strategies. Children must be able to develop more sophisticated strategies due to engaging in dialog about the context. The intertwinement principle is the notion that nothing should be taught in isolation. Connections must be made in order to optimize learning. The interaction principle highlights the notion that social interaction is a part of learning. The guidance principle is also known as the reinvention principle.
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Gravemeijer stated that the reinvention principle “…is basically the Socratic method, although the role of the student is more active in realistic mathematics education…entailing a gradual elaboration and sophistication of the subject matter” (Gravemeijer, 1998, pp. 284-285).

A related notion that has emerged from RME research is the concept of progressive mathematization (Buys, 2001; Klein, Beishuizen, & Treffers, 1998; Van Putten, van den Brom-Snijders, & Beishuizen, 2005). This is the idea that models are initially tied very closely to the context and reflect the mathematical reasoning that has occurred. Gradually they are generalized and become models for reasoning about mathematics (Gravemeijer, 1999; Gravemeijer & Doorman, 1999). Vertical mathematization occurs when “…the students spontaneously abbreviate and schematize their way of working… Another indication of shift in levels is when the students themselves bring up the matter of the efficiency of an approach” (Gravemeijer, 1998, p. 290).

Both the RME guiding principles and the notion of progressive mathematization guided the design of the instructional sequences in this study.

Multilevel Analysis

Sloane proposed uniting “mathematics education research by quantifying qualitative” research findings (Sloane, 2006b). Shavelson, Phillips, Towne and Feuer (2003) challenged those utilizing design research to attend to warrants for their research claims. One approach Shavelson et al. proposed for providing warrants was the integration of quasi-experiments within design research. Gorard, Roberts and Taylor (2004) and Yamagata-Lynch (2007) also proposed strengthening design research through the integration of quantitative methods. This study did just that. The concurrent classroom teaching experiment allowed for the development of different instructional sequences in
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each classroom. The data were then analyzed to identify associations between the probability of strategy usage and the instructional sequence experienced.

A repeated measures model of multilevel analysis was used to analyze the behavior that emerged during the course of the concurrent classroom teaching experiments with respect to the instructional sequences that were realized. This model was chosen over that of a growth curve model. A growth curve model assumes a natural maturation curve within each subject over the course of the study. The growth curve model expects deviations from that trajectory as a result of treatment. The repeated measures model assumes a stable level of development in the absence of treatment. Growth curve models often involve measures that are not necessarily assessed at the same point in time for each subject (Laird & Ware, 1982). In this study, each of the three structured interview assessments was administered after a particular phase of instruction. Both classes were undergoing treatment concurrently. The study was design research; it was exploratory in nature. Therefore, no established growth curve trajectories existed. One purpose of the study was to attempt to establish initial descriptions of those theoretical trajectories. It was assumed that any change in strategy usage was the result of the treatment (i.e., the instructional sequences realized in the microculture of the classrooms) rather than due to natural maturation. Therefore, a repeated measures model was a better fit for the data in this study.

The multilevel analysis of change over time deals with two general questions (1) What is the nature of the change over time for each individual? (2) What predictor variables explain the differences in change between individuals? (Bryk & Raudenbush, 1987; Rogosa & Willett, 1985; Singer & Willett, 2003) Each child in the two classrooms was given a preassessment and three measures of a dynamic assessment (Berman, 2001; Lidz, 2003, 1987; Lidz & Gindis, 2003; Tzuriel, 2001) using a structured clinical
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interview (Ginsburg, 1997) (see Appendix A) modified from the Math Recovery Assessment 1.2 (R. J. Wright, Martland, & Stafford, 2006). Each of the three instructional sequences was followed by an assessment. Each individual assessment was videotaped for subsequent analysis.

The variable of interest for this analysis was the strategy or strategies that the individual used to solve two-digit addition and subtraction problems presented with and without materials. Each student was presented with up to 12 tasks. The assessment analysis involved classifying and coding each student’s strategy usage in solving each task. The individual was given a summative code that characterized all strategies demonstrated during the course of the assessment. The codes used were: none or counts by one, 1010 only, N10 only, both 1010 and N10. A series of dummy codes were then developed to analyze the data since the codes were not mutually exclusive. For example, for one model, 1010strategy was developed. The codes were: 1 = any evidence of 1010, 0 = no evidence of 1010. This allowed for the analysis of change in the use of the two major strategies: 1010 and N10. Other dummy variables were developed to allow investigation of other questions. For example, AdvStra was used to investigate patterns of development of any strategy that capitalized on the base-ten structure of the number system contrasted to those who displayed no strategy that used the ten structure of numbers. The codes for this variable were 1) some advanced strategy 0) no evidence of any advanced strategy. The statistical models developed during the analysis did not take account of accuracy of students’ solutions because of the sparsity of data (Cohen, Cohen, West, & Aiken, 2003) this variable would have added to the models. This sparseness would have contributed numerous zero cells that would have caused the models to fail to converge. However, availability of an advanced strategy (1010 or N10) was analyzed as a
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predictor of success (as defined by the district) on a paper-and-pencil, district-based assessment of two-digit addition and subtraction.

Through the use of the dummy variables, several exploratory models were constructed to evaluate the usefulness of explanatory variables such as gender, SES, minority status, special education status, pretest level, and class membership. Only those variables making a significant contribution to the models were included in the final version of the models (see chapter 5 for detailed discussion of the statistical models).

The data were arranged in a “person-period” format (Singer & Willett, 2003). The data from these assessments were analyzed statistically through the use of Fisher’s exact test for dichotomous random variables as performed by SPSS and multilevel generalized linear models using the logit link function in a multilevel analysis of repeated measures of categorical data. The logit is the natural logarithm of odds and was chosen due to the fact that the response variable and all of the predictors were binomial. Logit is frequently used in epidemiology research to establish the odds or an odds ratio comparing the chance of two different groups contracting a particular condition. For example, a logit link function might be use to calculate the odds of smokers compared to odd of non-smokers contracting lung cancer.

In this study, a generalized linear model uses the proportion of individuals with successful strategy use at each time in each class to estimate the logit that an individual within a particular class would use that strategy at a given time. In this study, level 1 (with-in child) represents variation among repeated measures of the variables. (The term “repeated measure” is used in the sense of a measure used more than once with each child). Level 2 represents variations between individual change trajectories in relation to predictors (Rogosa & Willett, 1985; Singer & Willett, 2003). The model is said to be generalized because the data are binomial and not linear in nature. Sciandra, Muggeo, and
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Lovison (2008) reported that generalized linear mixed models are the most common form of model for dichotomous response data. Mixed refers to having both dichotomous and continuous variables. The models presented in this study are generalized linear models rather than generalized linear mixed models because all of the variables are dichotomous.

The logit estimates with confidence intervals (CIs) were the transformed into the odds for each class at each time through the use of exponentiation. The odds were further transformed into predicted probabilities by dividing the odds by 1 + the odds. The range of the probability for each class at each time is between zero and one inclusively.

These quantitative exploratory data analyses (Turkey, 1980; Velleman, 1998) were conducted in an attempt to corroborate apparent associations and trends noted during the course of the teaching experiment and initial qualitative analysis. The multilevel analysis was conducted using MLwiN 2.02 (Rasbash, Steele, Browne, & Prosser, 2005). When the model would converge, the penalized quasi-likelihood (PQL) (Hox, 2002; Ng, Carpenter, Goldstein, & Rasbash, 2006; Rodríguez & Goldman, 2001) Taylor series expansion approach was used due to its better estimations in simulation studies (Goldstein & Rasbash, 1996; Rodríguez & Goldman, 1995). This entails estimating both the fixed effects as well as the residuals (Hox, 2002). However, when the PQL model would not converge, marginal quasi-likelihood (MQL) approaches using only the fixed effects were used (Goldstein, 1995). Similarly, second order Taylor series expansion was used when the model would converge. When this was not possible, the first term was estimated in a first order expansion. Because the measures were repeated twice only, the data were characterized by underdispersion or data that was not as regularly distributed as one would expect if the data were a typical binomial distribution. This is likely due to the fact that the distribution was not due to chance but influenced by the instructional variables experienced by the students. To correct for this underdispersion
in the multilevel models, the extra binomial variation (Goldstein, Healy, & Rasbash, 1994; Rasbash, Steele, Browne, & Prosser, 2005; D. B. Wright, 1997) was used. The output of MLwin is in the form of equations (see Appendix E). The logit estimates and standard errors from each equation were then recorded in a table. The Wald test ($z = \text{estimate/standard error}$) was used to test the parameter estimates for significance (Hox, 2002). Logit estimates were transformed to odds and predicted probabilities of strategy use in each class at each time. CIs were calculated for each model. Graphic representations of the approximated probabilities with CIs were constructed in order to analyze each model graphically as recommended by Tukey (1980).

Sets of pairwise comparisons (Wald tests based on $\chi^2$ with 1 df), between classes within times and between times within classes, were conducted. These tests were not adjusted for their multiplicity. A Bonferroni correction (Hox, 2002) was applied separately to each set. In the 3 comparisons between classes within times, each test was considered significant if the $p$-value < .017 for an overall alpha of .05. In the 6 comparisons between times within classes, each test was considered significant if the $p$-value < .008 for an overall alpha of .05.

It was important to use multilevel modeling for this problem due to the nested nature of the data. Fisher’s exact test was not sufficient to analyze the statistical differences between the two classes on repeated measures. When analyzing the change over time, it became necessary to employ multilevel modeling to account for the nested nature of the hierarchical data (Hox, 2002; Lee, Herzog, Meade, Webb, & Brandon, 2007; Rogosa & Willett, 1985; Singer & Willett, 2003; Snijders & Bosker, 1999). Because there were three repeated measures for each student, each set of three scores was no longer characterized by the same expectation of independence since they represent longitudinal data on the same individual. If the multilevel nature of this data was ignored, it would
potentially lead to an inflation of statistical significance due to an overestimate of standard error (Atkins, 2005; Hox, 2002; Lee, Herzog, Meade, Webb, & Brandon, 2007; Sloane, 2008; Snijders & Bosker, 1999).

In this study, level 1 (within-child) represented the variation among repeated measures of the variables within the same child. Level 2 (between children) represented the variation between children (Rogosa & Willett, 1985). A multilevel generalized linear model for dichotomous response data (Goldstein, 1991, 1995; Wong & Mason, 1985) using a quasi-likelihood estimation method with a logit link function with extra binomial variation with restricted iterative generalized least squares (RIGLS) option (D. B. Wright, 1997) was used to estimate the model. RIGLS was used in each model since it is “generally preferred. . . particularly with small samples” (D. B. Wright, 1997, p. 22).

Because the study used convenience samples rather than random sampling, an initial analysis was conducted in an attempt to evaluate the extent of the similarities between the two classrooms in several identified characteristics. SPSS Fisher’s Exact Test for Dichotomous Random Variables was used to compare the two groups with respect to gender, minority status, SES, special education status, and pretest scores.

**Modified Meta-Analysis**

Each lesson was videotaped to enable careful review, documentation of activity, and transcription in protocol format of selected episodes critical to making the arguments. Analysis was conducted in multiple phases or tiers (Lesh & Kelly, 2000) and involved the use of a modified meta-analysis from a retrospective stance (Cobb & Gravemeijer, in press; Cobb & Whitenack, 1996; Dolk, den Hertog, & Gravemeijer, 2002; Gravemeijer & Cobb, 2006). Initially video of each lesson was viewed. Careful notes were compiled in conjunction with the written lesson plan used to guide instruction that day and artifacts gathered from that day’s activity. Initial conjectures were made about the mathematical
practices that were emerging within the activity and dialog from that day. Also during this phase of analysis, the video of each individual’s three structured interview dynamic assessments was carefully coded on an assessment schedule. Student strategies for solving two-digit addition and subtraction were identified for each task. Conjectures were made about the student’s individual interpretations and reasoning about mathematics.

After all lessons and assessments were initially viewed, a second wave of analysis was conducted. In this phase of analysis, the data compiled from each lesson in the initial phase of analysis were analyzed to form conjectures with respect to the mathematical practices that were emerging in each class over the course of time (multiple teaching episodes). Interesting segments that illustrated these conjectures were transcribed in protocol format. In the third phase of analysis the social trends observed were juxtaposed (Lesh & Kelly, 2000) with the strategies that were identified from the data of each individual’s assessments. Individual assessment performance was used to evaluate conjectures made about individual constructions from the communal taken-as-shared mathematical practices (Cobb, 2000a; Cobb, Stephan, McClain, & Gravemeijer, 2001; Cobb & Yackel, 1996; Gravemeijer, Cobb, Bowers, & Whitenack, 2000; Stephan & Cobb, 2003; Stephan, Cobb, & Gravemeijer, 2003; Whitenack, 1995) identified in the second phase of analysis.

Each classroom constituted a distinct aspect of the study. The instructional outcomes in each classroom were essentially the same in content except for 1) the materials available to the class and 2) the initial instructional sequences used to introduce the concept of numbers from 20 to 100 and initial multi-digit addition. These instructional sequences (see Table 4.1 for a summary) are described in detail in the instructional sequences results (see chapter 6).
In the final phase of the analysis, the analysis from each classroom became the data for comparison between the classes. In this phase of the analysis, the influence of the use of collections-based materials and contexts was analyzed. An idealized instructional sequence and an amplified framework for early base-ten strategies were developed. This phase of the analysis is a substantial modification to the approach developed by Cobb and his colleagues (Cobb & Whitenack, 1996; Gravemeijer & Cobb, 2006) since their research design involved working in one classroom at a time.

The researcher maintained detailed field notes and reflections throughout the course of the experiment. Classroom artifacts such as work samples and pedagogical tools (i.e., games, chart paper documenting notation of strategies discussed during whole group discussions, record sheets used by students) were collected (see Appendix B for samples of pedagogical tools). Because the researcher had multiple roles in the classroom teaching experiments (i.e., teacher and researcher roles) it was important to capture observations from both perspectives. For this reason, after the researcher had debriefed the lesson with the cooperating teacher(s) immediately following the lesson, the researcher spent time immediately documenting initial thoughts that occurred during the whole group discussions. Because the researcher was facilitating these discussions, it was impossible to simultaneously document in field notes preliminary conjectures. Therefore, it was vital to record systematically these reflections to prevent loss of data. These data were later used in planning for the next day’s instruction. Audio tapes and notes documenting the weekly meeting with the cooperating teachers to discuss the emerging mathematical practices, the progress of the children, and the effectiveness of the instructional sequences were maintained.
Participants

Several major studies have explored various aspects of this study, but often they were not conducted within the confines of a ‘typical’ American public school. Because of this, teachers are often tempted to dismiss research results because of the perception that the population is not a representative group, or the conditions were contrived; therefore, the same results would never be achieved with a ‘real’ class. For this reason I felt the need to conduct this study in conditions that were as authentic as possible. The following list constitutes issues that American teachers face on a daily basis in public schools: Title 1 schools with a high rate of children living in poverty, classrooms with 20+ students, highly mobile populations, funding limitations, limited amounts of teacher planning, students with special education IEPs and behavioral issues, cooperating teachers who have had no particular prior experiences above standard inservice professional development offered to all teachers through the school and district, and unrealistic pacing and curricular guidelines established by the district and state.

The last issue is particularly problematic. Teachers indicate that pacing schedules in curriculum guidelines allow insufficient time for adequate instruction focusing on development of students’ concepts. Not withstanding the Third International Mathematics and Science Study (TIMSS) report (Calsyn, Gonzales, & Frase, 1999; Mullis et al., 2000) and the Trends in International Mathematics and Science Study (TIMSS 2003) which found that American mathematics instruction typically attempts to cover far more topics in much less depth than its international counterparts, the curriculum is still ‘an inch deep and a mile wide’ (Schmidt et al., 1999/2002). This contrasts with Stephan and her colleagues who took over six weeks (31 instructional days) to develop the notion of linear measurement in a first grade classroom (Stephan, Cobb, & Gravemeijer, 2003). The district in which this study was conducted, on the other hand, allows for three weeks in
the pacing guide to teach all measurement concepts in first grade. Conducting the study under the conditions described above afforded the opportunity to determine if the instructional sequences were viable in a typical, everyday educational setting.

**Population**

The population of the study consisted of two intact first grade classrooms from an elementary school with over 600 students, grades pre-kindergarten (four-year-olds) through fifth grade (ten-year-olds), located in the U.S. Mid-Atlantic States. The school was located in a suburban community on the fringe of a large urban area. The school served several mobile home parks, two upper middle class neighborhoods, and a major military installation. Approximately 85% of the children had at least one parent on active duty in the military. As a result, there was a high rate of mobility. As of November of the study year, 32.6% of the children in first through fifth grade had enrolled since the end of the last school year in June. Among the new children were those who had transferred from 24 different states, one U.S. territory, and three foreign countries. This mobility presents tremendous academic challenges. Another 10.9% of the school population had a boundary exception for child care provided on the military post because at least one of their parents was a civilian government employee working on the military post. This means that they attended a school outside the area in which they lived and received before and/or after-school care at the military post youth center or child development center. The parents must apply for and be granted a boundary exception in order for this to occur. The exception must be renewed annually.

The population of the school was diverse. At the time of the study, 47.1 % were Caucasian, 38.2 % were African American, 10.4 % were Hispanic, 2.6 % were Asian, and 1.7 % were other including Native American, Hawaiian, and Alaskan Natives. The faculty was also fairly diverse: 42.9 % of the teachers at the school held a master’s
degree, an additional 8.2% had completed a planned program of 30 hours beyond the master’s degree, and one was currently a PhD candidate. Approximately 65.3% of the teachers had six or more years of teaching experience. One hundred percent of the teachers were teaching within their area of certification and taught subjects in which they held either a minor or major.

The Collections Classroom

The cooperating teacher in the collections-based classroom was a female teacher with 18 years of teaching experience. The bulk of this experience was spent in second, third or fourth grade. Only the last two years of experience had been spent in first grade. She held a master’s degree, and was considered a good teacher by the school administration, her peers, and the community. At the beginning of the experiment there were 23 children in the classroom, ten boys and thirteen girls. Eighteen of the children resided on the military post. Eight of the 19 children who completed the study received free or reduced meals and 2 children received special education services which involved being pulled into the inclusion classroom for math and reading instruction. After the completion of the teaching experiment but before the conclusion of the post assessment, an additional student from the classroom had been identified for special education services. Another 26.3% received special education services in speech and language only. These IEPs were managed by the speech pathologist and could include both articulation as well as language communication goals. Since they did not have an IEP in reading, writing, or mathematics, they received their instruction in the collections-based classroom with their regular homeroom peers. This class was designated the collections-based class because they had already had incidental exposure to grouping materials in their daily calendar activities. Before the completion of the experiment, two of the boys
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had moved from the school and the two lowest functioning children had begun the process for referral for special education services.

The Linear Classroom

The cooperating teacher in the other classroom was a male teacher with two years of experience teaching first grade. He held a bachelor’s degree and was considered a good teacher by the school administration, his peers, and the community given his level of experience. He was eager to learn and expressed enthusiasm in this research project from the beginning. There were 24 children in the class, nine male and fifteen female. Sixteen of the children resided on the military post. Ten out of the 22 children in the classroom who completed the study received free or reduced meals. In this classroom there were included three children who received special education services in both reading and mathematics and one who received services only from the speech language pathologist. Another five children with special needs joined the class for reading small group and all mathematics instruction.

There was a first-year special educator in the linear classroom during mathematics instruction. She was charged with providing direct services to the children who received special education. The special educator and the classroom teacher had been team teaching since the beginning of the year. The special educator held a master’s degree in special education. She was considered conscientious and effective by the administration and her peers and had an excellent rapport with the children. Before the completion of the study, two girls, one of whom was a special education student, and one boy had withdrawn from the school and one of the special education male students had been absent from school for two months. By the end of the study, four additional children from the class had been referred to special education testing. The results of that testing was yet to be determined at the conclusion of the experiment.
**Similarity of Classes**

If one includes the children with an academic IEP, a speech-language only IEP, and children who were in the referral process, 36.4% of the linear classroom and 42.1% of the collections classroom were either already receiving or being referred for special education. This seems like an incredibly high percentage. There were several factors involved in this high percentage. The two classrooms that participated in the study were two of five first grade classrooms in the school. All of the children with an academic IEP in mathematics or reading were a part of the linear classroom because it functioned as the inclusion classroom. The other classroom had a particularly high concentration of children who received speech-language services. Another factor was that this particular military post was one of four in that branch of the service that carried the designation of special needs post. That designation indicated that this post with the surrounding medical facilities was one of four equipped to meet the needs of any military dependent with special medical needs. These special medical needs frequently necessitate or coexist with special educational needs.

All three teachers who participated in the study had a previous working relationship with the researcher and expressed no concern about participating in the study before it began. All reported positive feelings about the study at the completion of the teaching experiment and stated that they would participate again if given an opportunity to participate in another study. Neither class had the top performing children in first grade. The top performing children were clustered together in one classroom so that they could benefit from the services of the teacher of gifted and talented. That classroom was not chosen for the study due to the dissimilarity of classroom populations. This study involved approximately two-fifths of the school’s first grade population. The two classroom populations were not statistically different in terms of pretest levels. The pre
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assessment scores and characteristics of the two classrooms were analyzed using Fisher’s exact test for dichotomous random variables using SPSS. The details of this analysis are reported in the quantitative analysis in Chapter 5.

Parent Meeting

The researcher conducted two meetings with parents of the prospective participants to discuss the study prior to the commencement of the teaching experiment. Each parent was sent an invitation with a request for an R.S.V.P. The researcher made phone contact with each family prior to the meetings. One meeting was held on a Friday morning prior to the commencement of the school day. The other was held at 5:30 PM after the end of the business day. It was hoped that this would increase the number of parents able to attend. The parents were given an information sheet detailing the purpose of the study, the procedures, the potential risks, the right of the participants and their parents, and how confidentiality would be protected. Parents and participants were informed that they would maintain the right to discontinue participation in the study at any time. Since the school maintained an “open door” policy, parents were assured that they would be welcome to visit the classroom at any time during the course of the study. Parents were requested to delay teaching their children how to perform the standard algorithms for multi-digit addition and subtraction until after the conclusion of the study since this would potentially influence the results of the study. Parents who were unable to attend the meeting received an additional telephone call to explain the project. If they agreed for their child to participate, they were asked to sign the consent form in the presence of a witness and return the form to school. There was a 100% participation rate in the two classes.
Role of the researcher

The role of the researcher in this study was multi-faceted. The researcher designed or modified instructional sequences in order to facilitate students’ construction of two-digit addition and subtraction concepts. This involved the planning, implementing, and reflecting upon the instruction in both experimental classrooms. Furthermore, it was the researcher’s role to document the instructional sequences as they were realized in each classroom.

In order to control the instructional environment as much as possible, the researcher taught the whole group portions of each lesson in each classroom. Both classrooms functioned under the same basic instructional format. Each class session began with a warm-up designed to improve underlying skills necessary for facile mental computation. These warm-up activities included practicing various forward and backward number word sequences, counting materials in conjunction with recording the counts on a hundred chart or empty number line (Beishuizen, 1999; Gravemeijer, 1999; Klein, Beishuizen, & Treffers, 1998; Menne, 2001; Rousham, 1997; van den Heuvel-Panhuizen, 2001b) during Mental Calculation Activities (see chapter 3 for detailed discussion), and playing games using the Van de Walle dot cards (Van de Walle, 2004). The instructional sequences results chapter (see chapter 6) elaborates on these settings since they were a result of the design experiment. Following the warm-up, the class as a whole group was presented with a contextual problem to explore. These problems were either modified from the work of previous researchers or were designed by the researcher. Gravemeijer called this process of modification of pedagogical tools “theory-guided bricolage” (Gravemeijer, 1998, p. 279; Gravemeijer & Cobb, 2006) or tinkering (Gravemeijer, 1994b). Wright (Jan. 24, 2008, personal e-mail correspondence) used the term “pedagogical engineering” in much the same way.
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In each case, the problems were from contexts (van den Brink, 1984) that were readily imaginable by the children as in the philosophy of the Realistic Mathematics Education (RME) (Gravemeijer, 1998; Streefland, 1991; Treffers & Beishuizen, 1999; van den Heuvel-Panhuizen, 1999, 2001a, 2001b) developed in the Netherlands. Once the problem had been presented, children worked in pairs or small groups to investigate the problem. After a time of investigation in which the researcher and the classroom teacher were observing and documenting strategies, the class would return to a whole group setting for a mathematical dialog about the investigation. During this discussion the researcher asked children in a predetermined order to share their strategies for solving the problem. It was the role of the researcher to choose carefully the order in which the various arguments were presented in order to lead the discussion into increasingly more sophisticated mathematical conceptions (Cobb, Boufi, McClain, & Whitenack, 1997; Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; Gravemeijer & Cobb, 2006; McClain, 1995). In this way, new ideas emerged (Cobb & Bauersfeld, 1995b; Cobb, Jaworski, & Presmeg, 1996) from the mathematical dialog that resulted from children sharing their mathematical thinking. Some days the context was a game designed for meaningful practice (van den Heuvel-Panhuizen, 2001b) that was related to a previously introduced context. The researcher differentiated the problem difficulty by manipulating the range of numbers on the dice or spinners. Special dice and spinners were routinely created to restrict the problem difficulty to the zone of proximal development (Leont'ev, 1997; Vygotsky, 1987a) or zone of potential construction (Steffe, 1992, 2004; Steffe & Thompson, 2000b) of each student.

Throughout the course of the study, the researcher was working under the constraint of following the district’s curricular content topics and pacing (i.e., exploration of two-digit numbers for three weeks, break for three-week district-generated unit on
measurement, followed by adding and subtracting numbers to 20, break for three-week
district-generated unit on money, followed by adding and subtracting two-digit numbers).

An additional unanticipated role of the researcher was that of professional
developer for the cooperating teachers. In addition to debriefing for a few minutes after
each lesson, the researcher met weekly with the cooperating teachers for the duration of
the teaching experiment. During this time, instructional sequences were evaluated,
student progress was discussed, and design rationales were shared. These meetings
quickly took on the nature of informal, on-going professional development or mentoring.
Each teacher observed during the whole group portions of each class. During the
meetings, teachers would inquire about specific instructional practices that were observed
and pedagogical decisions that were made during the course of a class session.

In addition to teaching both classes during the experiment, the researcher
conducted all the structured interview assessments for each child in the experiment. These
assessments were modified from the Math Recovery Assessment 1.2 (R. J. Wright,
Martland, & Stafford, 2006). Each assessment was videotaped for later scoring. These
assessments which were administered periodically throughout the course of the study
informed the researcher’s pedagogical decisions regarding the design of the instructional
sequences. The results of the assessment also formed the basis of the data for the
quantitative analysis.

Throughout the course of the study, the researcher collected artifacts from the
study such as chart papers used in the mathematical dialogue and student work samples.
The researcher also insured that each lesson was videotaped for later analysis. During the
investigation portions of the lessons, the researcher took field notes documenting
observations of student behaviors and strategies. After each lesson, the researcher
reflected on the lesson, recorded working conjectures about the emerging mathematics,
and added further field notes to document observations made during the whole group portions. After both lessons were documented, the researcher used the reflections to evaluate the hypothetical learning trajectory as it related to the proposed instructional sequence and if needed, redesign the instruction for the next day’s lessons. The next day’s lesson plans for each class were then modified as needed and sent to the cooperating teachers. Each cooperating teacher received a copy of all lesson plans and support materials for all instructional sequences which they independently chose to implement the following year.

**Plan for data analysis**

The post experiment data analysis began with a careful viewing of each teaching episode focusing on whole group instruction and mathematical dialog. During this review, detailed notes to augment field notes were made. These were coordinated with the classroom artifacts for clarity. Careful notes were made about the mathematical contributions of each child and how those constructs were indicated by the mathematical behaviors displayed during both whole group and small group interactions. Transcription of particularly interesting episodes were made to capture exact dialog and actions. There were forty-six lessons in each classroom. Table 4.1 summarizes the major instructional sequences as they were realized in the two classrooms and the three waves of assessment data collection within the instructional context. Two of the contextual investigations were original to this study, the video tape storage problem and the interstate driving context (see chapter 6 for detailed descriptions of the instructional sequences).

During the course of the study, the researcher had noted what appeared to be inconsistencies between an individual’s contributions during the classroom instructional
### Table 4.1 Major Instructional Sequences and Assessments

<table>
<thead>
<tr>
<th>Collections Classroom</th>
<th>Date</th>
<th>Linear Classroom</th>
</tr>
</thead>
<tbody>
<tr>
<td>Preassessment administered to all children</td>
<td>Last full week of January</td>
<td>Preassessment administered to all children</td>
</tr>
<tr>
<td><strong>Partitioning Sequence</strong>&lt;br&gt;Contextual Investigation: Orange Crate (Cobb, Gravemeijer, Yackel, McClain, &amp; Whitenack, 1997; Whitenack, 1995)</td>
<td>Day 1 – 5 (Jan. 31 – Feb. 4)</td>
<td><strong>Partitioning Sequence</strong>&lt;br&gt;Contextual Investigation: Orange Crate (Cobb, Gravemeijer, Yackel, McClain, &amp; Whitenack, 1997; Whitenack, 1995)</td>
</tr>
<tr>
<td><strong>Exploring Numbers to 100</strong>&lt;br&gt;Contextual Investigation: Inventory (Fosnot &amp; Dolk, 2001)</td>
<td>Day 6 – 10 (Feb. 7 – 11)</td>
<td><strong>Exploring Numbers to 100</strong>&lt;br&gt;Contextual Investigation: Art Gallery Part I (Fosnot &amp; Dolk, 2001)</td>
</tr>
<tr>
<td>District Generated Assessment: Place Value: Exploring 2-digit Numbers</td>
<td>Day 19 (Mar. 4)</td>
<td>District Generated Assessment: Place Value: Exploring 2-digit Numbers</td>
</tr>
<tr>
<td><strong>Wave 1 Data Collection:</strong>&lt;br&gt;1&lt;sup&gt;st&lt;/sup&gt; Structured Interview Administered Individually to Each Child (Mar 7 – 9)</td>
<td>2-Week Break in Teaching Experiment while classroom teachers taught a 2-week unit on Measurement.</td>
<td><strong>Wave 1 Data Collection:</strong>&lt;br&gt;1&lt;sup&gt;st&lt;/sup&gt; Structured Interview Administered Individually to Each Child (Mar 7 – 9)</td>
</tr>
<tr>
<td><strong>Structuring Numbers to 20 – Part II</strong>&lt;br&gt;Variety of Games to build facility with Structuring Numbers to 20</td>
<td>Day 20 – 29 (Mar. 21 – Apr. 4) [Spring Break: Mar. 24 – 28]</td>
<td><strong>Structuring Numbers to 20 – Part II</strong>&lt;br&gt;Variety of Games to build facility with Structuring Numbers to 20</td>
</tr>
<tr>
<td>District Generated Assessment: Adding and Subtracting to 20</td>
<td>Day 28</td>
<td>District Generated Assessment: Adding and Subtracting to 20</td>
</tr>
<tr>
<td><strong>Wave 2 Data Collection:</strong>&lt;br&gt;2&lt;sup&gt;nd&lt;/sup&gt; Structured Interview Administered Individually to Each Child (Apr. 20 – 26)</td>
<td>3-Week break in teaching experiment while classroom teachers taught a 3-week unit on money.</td>
<td><strong>Wave 2 Data Collection:</strong>&lt;br&gt;2&lt;sup&gt;nd&lt;/sup&gt; Structured Interview Administered Individually to Each Child (Apr. 20 – 26)</td>
</tr>
</tbody>
</table>
setting and that same individual’s performance on the individual structured interviews. Therefore, a comparative analysis (Krummheuer, 2007) between individual assessment performance and that individual’s contributions to the classroom dialogue was conducted in phase two of the meta-analysis described above.

During the analyses of the classroom instruction and the assessment data, certain patterns of behaviors emerged that some might label as misconceptions or errors. These patterns were analyzed to attempt to identify mathematical constructs that might be common to the development of children in both classrooms. Chapter 7 gives detailed descriptions of these constructs.

Based on the qualitative analysis, it appeared that there were associations between classroom membership and particular strategy use. Therefore, the quantitative analysis described above was conducted to corroborate those conjectured associations. The intention was to model the nature of the differences between the two classes. The purpose
Research Design

of this modeling was exploratory data analysis (Tukey, 1980; Velleman, 1998) and not inferential statistics which would attempt to claim generalization to other groups of students. These quantitative findings were subsequently used to guide the further qualitative analysis. Attempts were made to describe qualitatively the apparent quantitative differences and similarities found between the two classes and over time.

Summary

This study constitutes a blended study conducted in a typical American elementary school. Qualitative and quantitative analyses were used reflexively. Design research practices via concurrent classroom teaching experiments were complemented with multilevel analysis in order to gain a more robust picture of the social mathematical practices and individual mathematical constructs that emerged and developed over the course of the teaching experiments. The aim of the study was to contribute theory with respect to the pedagogical practices designed to engender the emergence of two-digit addition and subtraction strategies.
Chapter 5

Results 1: Quantitative Analysis

As one contemplates the notion of attempting to quantify phenomena involving humans, it becomes apparent that these phenomena are not particularly linear. Rather, the phenomena are a bit more like children playing with a parachute. The children stand around the circumference of the chute. Each child holds the edge of the parachute. They raise and lower their hands to change the shape of the chute. Then on cue, they all raise their hands and walk forward. This creates something similar to a dynamic dome. The children function as pillars and anchors simultaneously. Some of the children are taller or stronger than others. Some might even let go of the chute. All of these factors change the nature of this very complex, dynamic structure.

Any one individual participating in the chute activity cannot view the whole. Any observer also cannot view the whole. The observer may be able to view a greater percentage of the whole, but must still choose a perspective: ground level from a distance outside the chute, ground level from within the chute, from a distance above the chute. From any of these perspectives, some of the activity is missed. At best, any model or representation of the activity will approximate a portion of the activity at a moment in time. Even if the models and representations are executed with precision and artistry, they are at best, a limited approximation of the real activity.

Thus too is the case with any statistical model. This data set is a series of snapshots: three pictures of the chute, if you will. In each picture are the group activity that forms the chute structure and the individuals holding the chute who are changing over time in individual ways. This chute structure cannot exist without the children and their efforts. However, the participation in the group activity also changes the individuals. The
extent of their growth depends on variables such as the effort they exert during the activity, their prior fitness level and experience, perhaps genetic predispositions, and their understanding of the purpose and intent of the activity. Trying to find a way to quantify and measure all of those factors would be very difficult if not impossible. How can one measure another’s private thoughts and motivations? At best, one can determine a limited number of characteristics and behaviors which might lend insight into the private world of the individual. Any inferences made about that private world are, by their very nature, suspect.

Much is the same when attempting to understand the complex workings of a classroom. The models that follow are an approximation of an extremely complex situation. Like any model, they capture only a portion of what truly was. The usefulness of the models relates to the extent to which they provide insight into the dynamic nature of mathematics education and direction for future studies.

**Purpose of the Quantitative Analysis**

Four of the research questions of this study were such that they were best addressed through longitudinal quantitative analysis. These questions were: 1) Are children who have not used base-ten grouping materials (i.e., bundles and sticks, Unifix trains of 10, base-ten blocks, etc.) more likely to develop N10 than those who have used the base-ten grouping materials? 2) Are children who have used base-ten grouping materials more likely to develop 1010 than those who have never used the materials? 3) If so, can the predisposition toward 1010 in those children who have experience using collections be overcome through subsequent instruction? 4) Will the children with collections exposure develop N10, given the appropriate supporting experiences? While anecdotal evidence might be cited to answer these questions, a multilevel analysis allows for confidence levels to be established and for predicted probabilities to be estimated. The
Quantitative Analysis

probabilities can then be graphed. These exploratory data analyses enable the qualitative findings to be quantified.

**Classroom Populations Comparison**

The two study classroom populations were not different in terms of pretest levels. The ability to count-on to add in January of first grade was an expectation for all children in the district. Those who had not yet demonstrated the ability would be considered as ‘not yet meeting grade level expectations’. As indicated in Table 5.1 below, 41.5% of the children in the study were unable to count-on to add prior to the beginning of the teaching experiment. The percentages of children who could not count-on in the linear and collections classes were 40.9% and 42.1% respectively. Using Fisher’s Exact Test of two-sided significance, pre-level strategy use was not distributed differently in the two classes ($p = 1.00$).

<table>
<thead>
<tr>
<th>Class</th>
<th>Pre-Level</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear</td>
<td>Not Yet CO</td>
<td>9</td>
</tr>
<tr>
<td>Collections</td>
<td>CO</td>
<td>8</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td></td>
<td>17</td>
</tr>
</tbody>
</table>

Table 5.2 shows the distribution of gender among the linear and collections classes as 36.4% and 36.8% male respectively. Using Fisher’s Exact Test of two-sided significance, there was no difference between these two classes ($p = 1.00$). Table 5.3 shows the distribution of socio economic status (SES) as measured by free and reduced lunches for the linear and collections classes as 45.5% and 42.1% respectively. According to Fisher’s exact test, there was no difference ($p = 1.00$).
Table 5.2. Distribution of Gender by Class

<table>
<thead>
<tr>
<th>Class</th>
<th>Male</th>
<th>Female</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear</td>
<td>8</td>
<td>14</td>
<td>22</td>
</tr>
<tr>
<td>Collections</td>
<td>7</td>
<td>12</td>
<td>19</td>
</tr>
<tr>
<td>Total</td>
<td>15</td>
<td>26</td>
<td>41</td>
</tr>
</tbody>
</table>

Table 5.3. Distribution of Free and Reduced Lunch by Class

<table>
<thead>
<tr>
<th>Class</th>
<th>No</th>
<th>Yes</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear</td>
<td>12</td>
<td>10</td>
<td>22</td>
</tr>
<tr>
<td>Collections</td>
<td>11</td>
<td>8</td>
<td>19</td>
</tr>
<tr>
<td>Total</td>
<td>23</td>
<td>18</td>
<td>41</td>
</tr>
</tbody>
</table>

Children’s race was not distributed as evenly as pretest performance, gender, and SES, but again the difference was not significant according to Fisher’s Exact Test ($p = 0.34$). Table 5.4 indicates how the parents of each child self-reported their racial status. This information was taken from school district documents. At the time of the school registration, there was no way for a parent to indicate that the child was from a mixed racial background. For many of the children the racial group of one of their parents differed from that of their other parent.

Table 5.4. Distribution of Race Code by Class

<table>
<thead>
<tr>
<th>Class</th>
<th>Native Am/ Alaska Nat</th>
<th>Asian/ Pacific Is</th>
<th>African Am</th>
<th>White</th>
<th>Hispanic</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear</td>
<td>2</td>
<td>1</td>
<td>7</td>
<td>9</td>
<td>3</td>
<td>22</td>
</tr>
<tr>
<td>Collections</td>
<td>1</td>
<td>12</td>
<td>4</td>
<td>2</td>
<td>19</td>
<td>19</td>
</tr>
<tr>
<td>Total</td>
<td>3</td>
<td>1</td>
<td>19</td>
<td>13</td>
<td>5</td>
<td>41</td>
</tr>
</tbody>
</table>

The classrooms were not different in regards to number of students receiving some form of special education services (Fisher’s Exact Test of two-tailed probability: $p = .50$). See Table 5.5 for the distribution of services. If one dichotomizes the special education designations to either no special education services or some services (in the referral process, Individual Educational Plan (IEP) for Speech and Language only, or
Quantitative Analysis

Math IEP), then the classes are not different according to Fisher’s exact test ($p = .76$). See Table 5.6 for distribution.

Table 5.5. Distribution of Special Education Designation by Class

<table>
<thead>
<tr>
<th>Class</th>
<th>Special Ed Services</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>None</td>
<td>Referral</td>
</tr>
<tr>
<td>Linear</td>
<td>14</td>
<td>1</td>
</tr>
<tr>
<td>Collections</td>
<td>11</td>
<td>2</td>
</tr>
<tr>
<td>Total</td>
<td>25</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 5.6. Distribution of Collapsed Special Education by Class

<table>
<thead>
<tr>
<th>Class</th>
<th>Special Ed Collapsed</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>No SE</td>
<td>SE</td>
</tr>
<tr>
<td>Linear</td>
<td>14</td>
<td>8</td>
</tr>
<tr>
<td>Collections</td>
<td>11</td>
<td>8</td>
</tr>
<tr>
<td>Total</td>
<td>25</td>
<td>16</td>
</tr>
</tbody>
</table>

Even though the students were not randomly assigned to the classrooms, the two populations of the classes were arguably very similar on five criteria (i.e., pre-level, gender, poverty, race, special education services) before the beginning of the study. It is therefore reasonable to propose that any later perceived differences are an effect of the instruction that occurred during the intervening times.

Data Analysis

Preassessment Strategy Use

The ability to count-on at the outset of the teaching experiment was significantly associated with (Table 5.7, $p < .01$) later developing any advanced ten strategy (1010 or N10) to solve two-digit addition and subtraction problems. Of the 41 children in the study, 17 were unable to count-on to add during the preassessment that was administered prior to the commencement of the teaching experiment. Of those 17, only 5 (29.4%) ever
developed any advanced ten strategy compared to 83.3% of those who could initially count-on.

Table 5.7. Number of Students Ever Advanced by Pre-level Strategy

<table>
<thead>
<tr>
<th>Pre-Level</th>
<th>Ever Advanced</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Not Yet CO</td>
<td>12 No, 5 Yes</td>
<td>17</td>
</tr>
<tr>
<td>CO</td>
<td>4 No, 20 Yes</td>
<td>24</td>
</tr>
<tr>
<td>Total</td>
<td>16 No, 25 Yes</td>
<td>41</td>
</tr>
</tbody>
</table>

Of the five children who were initially unable to count-on yet later developed an advanced strategy for solving two-digit addition and subtraction, two were in the linear classroom and three were in the collections classroom. This was not related to classroom membership ($p = .62$). Of the 24 children who demonstrated the ability to count-on during the preassessment, 20 (83.3%) developed at least one advanced ten strategy. Thus, only 4 of the 24 children who were initially able to count-on failed to use an advanced strategy to solve two-digit addition and subtraction problems. All four of those children were from the linear classroom. Among the children who were initially able to count-on to add but did not develop an advanced strategy, classroom membership was not significant ($p = .10$). The ability to count-on was associated with later developing an advanced strategy regardless of class was significant ($p < .01$). For children in the linear classroom, the ability to count-on at the outset of the study was not a significant association ($p = .08$) developing an advanced strategy by the end of the study. However, within the collections classroom, the ability to count-on was significantly associated with the development of an advanced strategy ($p < .01$). See Table 5.8 for distribution.
Quantitative Analysis

Table 5.8. Number of Students Ever Advanced by Pre-Level Strategy Within Class

<table>
<thead>
<tr>
<th>Class</th>
<th>Pre-Level</th>
<th>Ever Advanced</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Linear</td>
<td>Not Yet Counting On</td>
<td>7</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>Counting On</td>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td>Collections</td>
<td>Not Yet Counting On</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>Counting On</td>
<td>11</td>
<td>11</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>16</td>
<td>25</td>
</tr>
</tbody>
</table>

Change in Strategy Usage

The lack of change in strategy use during all the assessments could be called static strategy usage over time. The number of students with static strategy usage was different (Fisher Exact Test for two-tailed probability; \( p = .01 \)) between the two classes. Table 5.9 indicates the count of children who made a change in strategy usage over the three assessments. Students in the linear classroom were much less likely to change over time in their strategy usage. Of the children who made no change in strategy usage, 75% of the 20 students were from the linear classroom. Three of the children from the linear classroom were static in strategy usage because at the time of the first assessment, they were already demonstrating facility with both the 1010 and N10 strategy. They chose between the two strategies according to the nature of the problem presented to them. When presented with a problem that did not require regrouping, they typically used 1010. When regrouping was required, they used N10.

Table 5.9. Static Strategy Usage Distribution by Class

<table>
<thead>
<tr>
<th>Class</th>
<th>Strategy Usage Change</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Linear</td>
<td>7</td>
<td>15</td>
</tr>
<tr>
<td>Collections</td>
<td>14</td>
<td>5</td>
</tr>
<tr>
<td>Total</td>
<td>21</td>
<td>20</td>
</tr>
</tbody>
</table>
Quantitative Analysis

Exclusion of Participants from Further Analysis

Because the majority of the children who could not count-on to add at the beginning of the teaching experiment did not develop any strategy that utilized ten (1010 or N10) for solving two-digit addition and subtraction, their removal was necessary in order to determine if there were differences between the classrooms of the children who did develop an advanced strategy. Therefore, a new dummy variable called pre0nev10 was created. This variable was used to exclude children who were not counting-on during the preassessment and developed neither N10 nor 1010. The decision was made to exclude children who were pre-level 0 and did not develop any ten strategy from any further analysis. In this manner, the further analysis focused on how the advanced strategies developed in each classroom among the 28 children who developed an advanced strategy at some point during the teaching experiment.

Strategy Coding and Use

During any assessment, children’s strategy usage was coded as “none or counting-on by ones”, “1010 only”, “N10 only”, or “both 1010 and N10”. Due to the small sample size, the multicategorical nature of the data created multiple zero cells that made analysis difficult. Figure 5.1 shows two bar graphs that indicate the frequency of strategy usage during each of the three assessments for both classrooms. Intuitively, there appear to be differences between the strategy usage of the two classes. From the graphs, there appear to be real differences in the trajectories over time of change in strategy usage between the two classrooms. Therefore a more sophisticated analysis that allows for the hierarchical nature of the data is warranted (Cohen, Cohen, West, & Aiken, 2003; Hox, 2002, nd; Lee, Herzog, Meade, Webb, & Brandon, 2007; Rogosa & Willett, 1985; Singer & Willett, 2003; Snijders & Bosker, 1999).
Figure 5.1. Bar Graphs of Observed Strategy Usage over Time
Strategies that Capitalize on Ten as a Unit

Strategy 1010

Because of the relatively small sample size and because the variables were not mutually exclusive, the decision was made to analyze the development of each strategy independently. A series of dummy variables was created that dichotomized each strategy option. This avoided having cells with zeros which would have caused the multilevel analysis to fail to converge. For each dummy variable, a coding of 0 indicated no evidence of the strategy in question on that occasion. A coding of 1 indicated evidence of that strategy usage. Figure 5.2 shows two bar graphs that visually illustrate the observed usage of 1010 in the two classes. The count indicates the number of individuals observed. Intuitively, one can see the associations demonstrated in the following analysis.

In the first phase of the analysis, the probability that a child would use 1010 was analyzed. Time 1, 2, and 3 indicate the repeated measures. Class was coded 0 for the linear classroom and 1 for the collections classroom. A composite multilevel model (Singer & Willett, 2003) was fitted using MLwiN 2.02 (Rasbash, Steele, Browne, & Prosser, 2005). The extra binomial variation was used as a correction for underdispersion (Lindsey, 2004). The parameter estimates, their standard errors and $p$-values from Wald ($z = \text{estimate/standard error}$) are reported in Table 5.10 (see Appendix E for Equation 1 and Table E for equation details).

Likelihood ratio tests could not be applied to assess the significance of effects represented in the model by more than one parameter because log likelihood statistics are not available for this model. However, both interaction parameters were significant by Wald tests and it follows that the class by time interaction effect as a whole was significant. That is the change over time in the probability of using the 1010 strategy was significantly different between groups.
Figure 5.2. Pair of Bar Graphs Indicating Observed 1010 Strategy Usage over Time
Table 5.10. Results from the Multilevel Model for 1010 Strategy by Class and Time

<table>
<thead>
<tr>
<th>n=28</th>
<th>Variable</th>
<th>Estimate</th>
<th>se</th>
<th>z</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Fixed Effects</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Intercept^1</td>
<td>-2.27</td>
<td>1.23</td>
<td>-1.84</td>
<td>.07</td>
</tr>
<tr>
<td></td>
<td>Time_2</td>
<td>1.91</td>
<td>0.85</td>
<td>2.24</td>
<td>.03</td>
</tr>
<tr>
<td></td>
<td>Time_3</td>
<td>0.97</td>
<td>0.81</td>
<td>1.20</td>
<td>.23</td>
</tr>
<tr>
<td></td>
<td>Collections Class</td>
<td>5.90</td>
<td>1.79</td>
<td>3.29</td>
<td>&lt;.01</td>
</tr>
<tr>
<td></td>
<td>Time_2 by Class</td>
<td>-2.88</td>
<td>1.18</td>
<td>-2.45</td>
<td>.01</td>
</tr>
<tr>
<td></td>
<td>Time_3 by Class</td>
<td>-5.21</td>
<td>1.40</td>
<td>-3.72</td>
<td>&lt;.001</td>
</tr>
<tr>
<td></td>
<td>Random Effects</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Level 1^2 Within Individual</td>
<td>0.33</td>
<td>0.06</td>
<td>5.33</td>
<td>&lt;.001</td>
</tr>
<tr>
<td></td>
<td>Level 2 Interindividual</td>
<td>14.22</td>
<td>4.57</td>
<td>3.11</td>
<td>&lt;.001</td>
</tr>
</tbody>
</table>

^1Linear class at Time 1
^2Dispersion parameter

The estimated logits, odds and probabilities and their 95% confidence intervals (CIs) of using the 1010 strategy in each class at each time are reported in Table 5.11. The estimated probabilities with their 95% CIs are plotted in Figure 5.3. The shaded regions represent the 95% CIs for the probability of each class using 1010. The regions represent the range of probability that is encompassed by the CIs. Intuitively, one can see that initially, the probability that the Collections class would use 1010 was much greater than in the linear classroom. Even when one considers the confidence bands, at the time of the first administration, one can see with 95% confidence that the probabilities of 1010 use in the two classrooms are different. However, at the time of the second and third administrations, the confidence bands overlap. Intuitively, one would assume that any difference in probabilities may be due to random chance. Therefore, a statistical analysis to determine the extent of any perceived differences is warranted.
### Quantitative Analysis

**Table 5.11. Logits, Odds, & Probabilities (Prob) with Confidence Intervals (CI) for 1010 Strategy Usage Within Class by Time**

<table>
<thead>
<tr>
<th>Class</th>
<th>Time</th>
<th>Logit</th>
<th>Logit 95 % CI</th>
<th>Odds 95% CI</th>
<th>Prob 95 % CI</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear</td>
<td>1</td>
<td>-2.27</td>
<td>2.42</td>
<td>-4.68 0.15</td>
<td>0.10 1.16</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-0.36</td>
<td>2.33</td>
<td>-2.69 1.97</td>
<td>0.70 7.17</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>-1.30</td>
<td>2.38</td>
<td>-3.68 1.09</td>
<td>0.27 2.96</td>
</tr>
<tr>
<td>Collections</td>
<td>1</td>
<td>3.63</td>
<td>2.55</td>
<td>1.08 6.18</td>
<td>37.68 2.93 484.44</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>2.65</td>
<td>2.50</td>
<td>0.15 5.15</td>
<td>14.15 1.16 172.95</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>-0.61</td>
<td>2.37</td>
<td>-2.98 1.76</td>
<td>0.55 5.84</td>
</tr>
</tbody>
</table>

**Figure 5.3. Graphic Representation of 1010 Probabilities with CI by Class**
Quantitative Analysis

Two sets of pairwise comparisons (Wald tests based on $\chi^2$ with 1 df), between classes within times and between times within classes, are reported in Tables 5.12 and 5.13 respectively. These tests are not adjusted for their multiplicity. A Bonferroni correction (Hox, 2002) was applied separately to each set. In the 3 comparisons between classes within times, each test should be considered significant if the p-value < 0.017 for an overall alpha of .05. In the 6 comparisons between times within classes, each test should be considered significant if the p-value < 0.008 for an overall alpha of .05.

Table 5.12. Between Classes Within Times Comparison for 1010

<table>
<thead>
<tr>
<th>Collections – Linear Class</th>
<th>Time</th>
<th>Estimate</th>
<th>$\chi^2$</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>5.897</td>
<td>10.795</td>
<td>0.001</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>3.013</td>
<td>2.975</td>
<td>0.084</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>0.689</td>
<td>0.161</td>
<td>0.688</td>
</tr>
</tbody>
</table>

The probability that an individual would use 1010 was significantly greater in the collections than the linear class at Time 1 (0.974 to 0.094 respectively). While the estimated probability of use of 1010 was still considerably greater in the collections than the linear class at Time 2 (0.934 to 0.410), it was not significantly so. There was no difference in the estimated probability of 1010 use between the classes at Time 3 (0.353 to 0.215).

Table 5.13. Between Times Within Classes Comparison for 1010

<table>
<thead>
<tr>
<th>Class</th>
<th>Times</th>
<th>Estimate</th>
<th>$\chi^2$</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear</td>
<td>2-1</td>
<td>1.906</td>
<td>5.050</td>
<td>0.025</td>
</tr>
<tr>
<td></td>
<td>3-1</td>
<td>0.972</td>
<td>1.431</td>
<td>0.232</td>
</tr>
<tr>
<td></td>
<td>3-2</td>
<td>-0.933</td>
<td>1.377</td>
<td>0.241</td>
</tr>
<tr>
<td>Collections</td>
<td>2-1</td>
<td>-0.979</td>
<td>1.435</td>
<td>0.231</td>
</tr>
<tr>
<td></td>
<td>3-1</td>
<td>-4.236</td>
<td>13.858</td>
<td>0.0002</td>
</tr>
<tr>
<td></td>
<td>3-2</td>
<td>-3.257</td>
<td>9.217</td>
<td>0.002</td>
</tr>
</tbody>
</table>
Quantitative Analysis

Although the increase in the estimated probability of 1010 use in the linear class between Time 1 and Time 2 was substantial (.094 to .410), it was not significant after adjustment for multiple comparisons. There was a small and non-significant decrease in the probability of 1010 use in the collections class in the same period (.974 to .934). While there was a moderate and non-significant decrease in the estimated probability of 1010 use between Times 2 and 3 in the linear class (.410 to .215), there was a large and significant decrease in the collections class (.934 to .353) in the same period.

The question of whether these changes were different is addressed in the following discussion. When using multiple tests, one must adjust the alpha test level for 3 tests with a Bonferroni correction (Hox, 2002). One would test each against the level of \( p < .017 \). There was a decrease in the probability of 1010 use between Times 1 and 2 (.974 to .934) in the collections class and an increase in the linear class (.094 to .410). The changes were significantly different with a \( \chi^2 = 5.993 \) which has \( p = .014 \). There were decreases in both classes between Times 2 and 3 (Collections .934 to .353; Linear .410 to .215). These changes had a \( \chi^2 = 3.028, p = .082 \). There was an overall decrease in the use of 1010 in the collections classes between Times 1 and 3 (.974 to .353) and an increase in the linear class (.094 to .215). There was a highly significant overall difference between the two groups with a \( \chi^2 = 13.873 \) which has \( p < .001 \).

**Strategy N10**

Figure 5.4 is a pair of graphs that illustrate the number and percentage of students who used N10 in each classroom on the three assessments. Intuitively, the graphs appear to indicate that N10 takes time to develop regardless of classroom membership. As with the 1010 strategy, a multilevel model was fitted to the N10 dichotomized data. The parameter estimates, their standard errors and \( p \)-values from Wald (\( z = \text{estimate/standard error} \)) are reported in Table 5.14 (see Appendix E for Equation 2). For children who were
counting-on to add at the outset of the teaching experiment, there was an effect of time on the development of N10 strategy but no significant effect for class or the interaction between time and class (see Appendix E for Equation 3).

That is, while there was no significant difference between the two classes at any given time of administration or on the changes over time, there were significant increases in the probability of usage of N10 over time in both classes (i.e., the whole sample).

The estimated logits, odds and probabilities and their 95% CIs of using N10 strategy in both classes (all participants excluding Pre0Nev10) at each time are reported in Table 5.15. The estimated probabilities with their 95% CIs are plotted in Figure 5.5. Intuitively, one can see the lack of a significant difference between the two classes in N10 use from the graph of the estimated confidence bands.

Since class designation was not a significant predictor in the model, it was eliminated from the model. A new model was generated that estimated the effect of time for all participants. The parameter estimates, their standard errors and $p$-values from Wald ($z = \text{estimate/standard error}$) are reported in Table 5.16 (see Appendix E for Equation 4).

### Table 5.14. Results from the Multilevel Model for N10 Strategy Usage

<table>
<thead>
<tr>
<th>$n=28$</th>
<th>Variable</th>
<th>Estimate</th>
<th>se</th>
<th>$z$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed Effects</td>
<td>Intercept$^1$</td>
<td>-1.91</td>
<td>0.90</td>
<td>-2.11</td>
<td>.02</td>
</tr>
<tr>
<td></td>
<td>Time_2</td>
<td>2.53</td>
<td>0.63</td>
<td>4.04</td>
<td>&lt;.001</td>
</tr>
<tr>
<td></td>
<td>Time_3</td>
<td>5.36</td>
<td>0.79</td>
<td>6.83</td>
<td>&lt;.001</td>
</tr>
<tr>
<td></td>
<td>Collections Class</td>
<td>-0.87</td>
<td>1.19</td>
<td>-0.73</td>
<td>.23</td>
</tr>
</tbody>
</table>

| Random Effects          | Within Individual    | 0.45     | 0.08 | 5.38 | <.001|
| Level 2                 | Interindividual      | 8.36     | 2.64 | 3.16 | <.001|

$^1$Linear class at Time 1  
$^2$Dispersion parameter
Figure 5.4. Pair of Bar Graphs Indicating Observed N10 Strategy Usage over Time
Quantitative Analysis

Table 5.15. Logits, Odds, & Probabilities (Prob) with Confidence Intervals (CI) for N10 Within Class by Time

<table>
<thead>
<tr>
<th>Class</th>
<th>Time</th>
<th>Logit</th>
<th>Logit 95% CI Lower</th>
<th>Logit 95% CI Upper</th>
<th>Odds 95% CI Lower</th>
<th>Odds 95% CI Upper</th>
<th>Prob 95% CI Lower</th>
<th>Prob 95% CI Upper</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear</td>
<td>1</td>
<td>-1.91</td>
<td>-1.77</td>
<td>-1.68</td>
<td>0.15</td>
<td>0.03</td>
<td>0.87</td>
<td>0.47</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.62</td>
<td>-1.16</td>
<td>2.41</td>
<td>1.87</td>
<td>0.31</td>
<td>11.09</td>
<td>0.24</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>3.46</td>
<td>1.88</td>
<td>5.34</td>
<td>31.66</td>
<td>4.82</td>
<td>207.89</td>
<td>0.83</td>
</tr>
<tr>
<td>Collection</td>
<td>1</td>
<td>-2.77</td>
<td>1.89</td>
<td>-4.66</td>
<td>-0.89</td>
<td>0.06</td>
<td>0.01</td>
<td>0.06</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-0.24</td>
<td>1.71</td>
<td>-1.95</td>
<td>1.47</td>
<td>0.79</td>
<td>0.14</td>
<td>0.44</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>2.59</td>
<td>1.85</td>
<td>0.74</td>
<td>4.44</td>
<td>13.33</td>
<td>2.10</td>
<td>84.61</td>
</tr>
</tbody>
</table>

Figure 5.5. Graphic Representation of N10 Probabilities with CIs by Class
Table 5.16. Results from the Multilevel Model for N10 Strategy for Both Classes

<table>
<thead>
<tr>
<th></th>
<th>Variable</th>
<th>Estimate</th>
<th>se</th>
<th>z</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed</td>
<td>Intercept</td>
<td>-4.30</td>
<td>0.81</td>
<td>-5.29</td>
<td>&lt;.001</td>
</tr>
<tr>
<td></td>
<td>Time_2</td>
<td>2.85</td>
<td>0.58</td>
<td>4.90</td>
<td>&lt;.001</td>
</tr>
<tr>
<td></td>
<td>Time_3</td>
<td>5.67</td>
<td>0.71</td>
<td>7.98</td>
<td>&lt;.001</td>
</tr>
<tr>
<td>Random</td>
<td>Level 1(^1) Within Individual</td>
<td>0.32</td>
<td>0.05</td>
<td>6.40</td>
<td>&lt;.001</td>
</tr>
<tr>
<td></td>
<td>Level 2 Interindividual</td>
<td>15.91</td>
<td>4.03</td>
<td>3.94</td>
<td>&lt;.001</td>
</tr>
</tbody>
</table>

\(^1\)Both classes at Time 1
\(^2\)Dispersion parameter

The estimated logits, odds and probabilities and their 95% CIs of using the N10 strategy in both classes at each time are reported in Table 5.17. The estimated probabilities with their 95% CIs are plotted in Figure 5.6.

Table 5.17. Logits, Odds & Probabilities (Prob) with Confidence Intervals (CI) for N10 Strategy Usage for Both Classes by Time

<table>
<thead>
<tr>
<th>Class</th>
<th>Time</th>
<th>Logit</th>
<th>Logit 95% CI</th>
<th>Odds 95% CI</th>
<th>Prob. 95% CI</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Logit</td>
<td>Lower</td>
<td>Upper</td>
<td>Odds</td>
</tr>
<tr>
<td>Both</td>
<td>1</td>
<td>-4.30</td>
<td>-5.89</td>
<td>-2.71</td>
<td>0.01</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-1.23</td>
<td>-2.90</td>
<td>&lt;0.01</td>
<td>0.24</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>1.37</td>
<td>-0.06</td>
<td>2.81</td>
<td>3.95</td>
</tr>
</tbody>
</table>

A set of pairwise comparisons (Wald tests based on \(\chi^2\) with 1 df) between times for both classes, is reported in Table 5.18. These tests are not adjusted for their multiplicity. A Bonferroni correction (Hox, 2002) was applied separately to the set. In the 3 comparisons between classes within times, each test should be considered significant if the \(p\)-value < 0.017 for an overall alpha of 0.05.

Table 5.18. Significance of Difference Between Times for N10 Usage for Participants in Both Classes

<table>
<thead>
<tr>
<th>Class</th>
<th>Times</th>
<th>Estimate</th>
<th>(\chi^2)</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>Both</td>
<td>2-1</td>
<td>2.852</td>
<td>23.993</td>
<td>&lt; .0001</td>
</tr>
<tr>
<td></td>
<td>3-1</td>
<td>5.671</td>
<td>63.563</td>
<td>&lt; .0001</td>
</tr>
<tr>
<td></td>
<td>3-2</td>
<td>2.819</td>
<td>27.256</td>
<td>&lt; .0001</td>
</tr>
</tbody>
</table>
Quantitative Analysis

The probability of use of strategy N10 was significantly greater at Time 3 than either Time 2 or Time 1 (.798 to .191 and .013 respectively). The probability was also significantly greater at Time 2 than Time 1. The increase from Time 1 to Time 2 was highly significant with $\chi^2 = 23.993, p < .0001$. The increase from Time 1 to Time 3 was highly significant with $\chi^2 = 63.563, p < .0001$. The increase from Time 2 to Time 3 was highly significant with $\chi^2 = 27.256, p < .0001$. Adjusting for 3 tests, one would test each against $p < .017$. Therefore all three were highly significant.
Figure 5.6. Probability of N10 Strategy Usage with CIs over Time
**Advanced Strategies**

At first, the order in which the advanced strategies developed also appeared to have been significantly different between the classes. Among the children who were initially counting-on, \( p = .02 \). It also appeared to be nearing a level of significance \( (p = .06) \) for those who were not yet counting-on at the beginning of the study (see Table 5.19).

<table>
<thead>
<tr>
<th>Advanced Strategy Order</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Not Yet Counting On</td>
<td></td>
</tr>
<tr>
<td>Linear</td>
<td>7 0 2 0 9</td>
</tr>
<tr>
<td>Collections</td>
<td>5 3 0 0 8</td>
</tr>
<tr>
<td>Counting On</td>
<td></td>
</tr>
<tr>
<td>Linear</td>
<td>4 1 4 4 13</td>
</tr>
<tr>
<td>Collections</td>
<td>0 7 2 2 11</td>
</tr>
</tbody>
</table>

When one analyzes the order in which the advanced strategies were acquired (see Table 5.20), at first there appeared to be a highly significant difference \( (p < .001) \) in developing facility with both strategies given initial strategy developed. This apparent significance is because children who had no advanced strategy were 100 % likely never to develop both. Likewise, 100 % of the children with both had both. Therefore, the significance of the difference in advanced strategy order in Table 5.20 is very misleading.

When one analyzes the order in which the advanced strategies were acquired only for N10 and 1010 (see Table 5.21), there was not a significant association \( (p = .17) \) between the ability to become facile with both strategies and the initial strategy acquired.
Table 5.20. Advanced Strategy Order by Ever Both Strategies

<table>
<thead>
<tr>
<th>Advanced Order</th>
<th>Ever Both</th>
<th>No</th>
<th>Yes</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>none</td>
<td>16</td>
<td>0</td>
<td></td>
<td>16</td>
</tr>
<tr>
<td>1010 First</td>
<td>4</td>
<td>7</td>
<td></td>
<td>11</td>
</tr>
<tr>
<td>N10 First</td>
<td>6</td>
<td>2</td>
<td></td>
<td>8</td>
</tr>
<tr>
<td>both</td>
<td>0</td>
<td>6</td>
<td></td>
<td>6</td>
</tr>
<tr>
<td>Total</td>
<td>26</td>
<td>15</td>
<td></td>
<td>41</td>
</tr>
</tbody>
</table>

Table 5.21. Advanced Strategy Order by Ever Both for 1010 and N10 Only

<table>
<thead>
<tr>
<th>Advanced Order</th>
<th>Ever Both</th>
<th>No</th>
<th>Yes</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1010 First</td>
<td>4</td>
<td>7</td>
<td></td>
<td>11</td>
</tr>
<tr>
<td>N10 First</td>
<td>6</td>
<td>2</td>
<td></td>
<td>8</td>
</tr>
<tr>
<td>Total</td>
<td>10</td>
<td>9</td>
<td></td>
<td>19</td>
</tr>
</tbody>
</table>

There also appeared to be a difference between the two classes in the probability that the students would develop an advanced strategy by the end of the teaching experiment. Figure 5.7 is a pair of bar graphs that depict the raw data by class for advanced strategy usage during each assessment occurrence.
Figure 5.7. Pair of Bar Graphs for Observed Data of Advanced Strategies
When one analyzes the development of any advanced ten strategy (N10 or 1010) among children who were either counting-on at the outset of the teaching experiment or developed an advanced strategy ($z = 2.03; p = .04$) (see Appendix E for Equation 5), there was a main effect of class membership on the development of an advanced strategy. However, MLwiN 2.02 would not converge with first order Taylor expansion penalized (or predictive) quasi-likelihood model with the effect for time. This appears to be due to the presence of a zero cell in the collections classroom during the third administration. Therefore, this potential association warrants further investigation. Table 5.22 is the model for advanced strategy estimation by second order marginal quasi-likelihood with extra binomial excluding Pre0Nev10 (see Appendix E for Equation 6). The interaction effect would not converge because all children in the collections (excluding Pre0Nev10) class used an advanced strategy. The computer model cannot deal with a probability of 1.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Estimate</th>
<th>se</th>
<th>z</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept$^1$</td>
<td>-0.75</td>
<td>0.85</td>
<td>-0.89</td>
<td>.38</td>
</tr>
<tr>
<td>Time_2</td>
<td>1.87</td>
<td>1.05</td>
<td>1.78</td>
<td>.08</td>
</tr>
<tr>
<td>Time_3</td>
<td>2.84</td>
<td>1.34</td>
<td>2.12</td>
<td>.03</td>
</tr>
<tr>
<td>Collections Class</td>
<td>2.79</td>
<td>1.43</td>
<td>1.96</td>
<td>.05</td>
</tr>
</tbody>
</table>

Initially the difference in probability that children in the collections classroom would use an advanced strategy over children in the linear classroom was not significant according to Fisher’s Exact Test of 2-sided probability ($p = .06$). As time progressed, the difference in estimated probability between the two classrooms was further reduced with each subsequent assessment. During the final assessment, there was no difference
Quantitative Analysis

between children in the collections classroom and those in the linear classroom with respect to use of an advanced strategy \((p = 0.10)\). The estimated logits, odds and probabilities and their 95% CI of advanced strategy usage at each time by class are reported in Table 5.23. Figure 5.8 graphically approximates these estimates of probability for advanced strategy usage with 95% confidence bands from Table 5.23.

### Table 5.23. Logits, Odds Probability (Prob) with Confidence Intervals for Advanced Strategy Usage

<table>
<thead>
<tr>
<th>Class</th>
<th>Time</th>
<th>Logit</th>
<th>Logit CI</th>
<th>Odds</th>
<th>Odds CI</th>
<th>Prob</th>
<th>Prob CI</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear</td>
<td>1</td>
<td>-0.75</td>
<td>-2.41</td>
<td>0.91</td>
<td>0.09</td>
<td>2.48</td>
<td>.32</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1.11</td>
<td>-0.69</td>
<td>2.92</td>
<td>0.50</td>
<td>18.54</td>
<td>.75</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>2.09</td>
<td>-0.27</td>
<td>4.45</td>
<td>0.76</td>
<td>85.29</td>
<td>.89</td>
</tr>
<tr>
<td>Collections</td>
<td>1</td>
<td>2.04</td>
<td>-0.36</td>
<td>4.45</td>
<td>0.70</td>
<td>85.37</td>
<td>.89</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>3.91</td>
<td>0.99</td>
<td>6.83</td>
<td>2.68</td>
<td>926.12</td>
<td>.98</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>4.88</td>
<td>1.54</td>
<td>8.22</td>
<td>4.68</td>
<td>3721.94</td>
<td>.99</td>
</tr>
</tbody>
</table>

Since the model contained no interaction between time and class, two analyses were conducted, one to test significance of time and the other to test significance of class membership. A set of pairwise comparisons (Wald tests based on \(\chi^2\) with 1 df) between times for both classes, is reported in Table 5.24. These tests are not adjusted for their multiplicity. A Bonferroni correction (Hox, 2002) was applied separately to the set. As before, in the 3 comparisons between classes within times, each test should be considered significant if the \(p\)-value < .017 for an overall alpha of .05.

### Table 5.24. Difference Between Times for Advanced Strategy for Participants in Both Classes

<table>
<thead>
<tr>
<th>Class</th>
<th>Times</th>
<th>Estimate</th>
<th>(\chi^2)</th>
<th>(p)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Both</td>
<td>2-1</td>
<td>1.865</td>
<td>3.152</td>
<td>0.076</td>
</tr>
<tr>
<td></td>
<td>3-1</td>
<td>2.840</td>
<td>4.506</td>
<td>0.033</td>
</tr>
<tr>
<td></td>
<td>3-2</td>
<td>0.975</td>
<td>0.661</td>
<td>0.421</td>
</tr>
</tbody>
</table>

The difference between Time 3 and Time 1 in the probability of use of an advanced strategy in either class was not significant once the correction was applied \((p = .033)\). Test of difference between classes (same at all points in time) from parameter
Quantitative Analysis

estimate/se (i.e., 2.794/1.425) was $z = 1.961$, $p$(two-tailed) = .050. Thus, individuals in the collections class had significantly higher probability of advanced strategy use overall than individuals in the linear classroom.

Figure 5.8. Graphic Representation of Advanced Strategy Probability with CIs
Quantitative Analysis

**Development of Both Strategies**

Another question of interest was whether one instructional sequence was more highly associated with the development of facility with both advanced strategies. This was measured by the child displaying evidence of use of both 1010 and N10 during the same administration. There was no difference ($p = 1.0$) between the two classrooms with respect to children who used both strategies during the same assessment. The graphs in Figure 5.9 illustrate the count and percentages of children who used both strategies in each classroom during each wave of data collection.
Figure 5.9. Pair of Bar Graphs for Observed Data for Use of Both Strategies
Quantitative Analysis

Association Between Advanced Strategy and District Test Performance

A potential criticism of this study might be that just because children can perform on an oral, one-on-one clinical interview does not necessarily mean that they can be successful on more traditional assessments. Therefore, it was important to see if evidence of an advanced ten strategy was a good predictor of performance on a more traditional assessment. An analysis was conducted to determine if the presence of any advanced strategy would be associated with a higher score on the district-mandated paper and pencil assessment of two-digit addition and subtraction (see Appendix C for district mandated assessments). This assessment was administered at the end of the teaching experiment. Percentage scores on the unit assessment for exploring two-digit addition and subtraction were dichotomized somewhat arbitrarily with 0-74% being defined as lack of success and 75-100% being defined as demonstrating success. This was then cross tabulated with the presence of any advanced strategy that exploits the ten structure of the number system on the third structured interview assessment for all participants (see Table 5.25). There was a highly significant difference in performance on the district unit assessment between those who demonstrated an advanced ten strategy as compared to those who did not (p < .001). The one child who demonstrated an advanced strategy during the structured interview but did not meet with success on the district assessment scored a 70% on the assessment.

Table 5.25. Association between Advanced Strategy Use and Success on District Test

<table>
<thead>
<tr>
<th>Advanced Strategies</th>
<th>% Correct</th>
<th>0-74%</th>
<th>75-100%</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>No 10 Strategy</td>
<td>11</td>
<td>5</td>
<td>16</td>
<td></td>
</tr>
<tr>
<td>Some 10 Strategy</td>
<td>1</td>
<td>24</td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>12</td>
<td>29</td>
<td>41</td>
<td></td>
</tr>
</tbody>
</table>
Quantitative Analysis

**Summary of Quantitative Analysis**

While the two classes were not different in terms of five descriptive characteristics (i.e. pre-level, gender, poverty, race, special education services) at the beginning of the teaching experiment, differences in the development of strategies that capitalize on ten as a unit were found between the classes during the course of the experiment. After the first unit of differentiated instruction, the collections class was more likely to use 1010 than the linear class. However, after subsequent instruction, the difference between the two classes was no longer evident. There was no difference between the two classes in the use of N10. However, students in both classes were more likely to use N10 at the end of the study than they were after the first unit of study. Furthermore, there was no difference between the two classes in the likelihood of use of an advanced strategy; however, students in both classes were more likely to use an advanced strategy at the conclusion of the study than they were after the first unit of study. The order of acquisition of 1010 and N10 was not associated with the ability to develop both strategies. There was a strong association between the use of an advanced strategy during the structured interview and achieving a high score on the district-generated, paper and pencil unit assessment on two-digit addition and subtraction which was mandated by the district mathematics curriculum.
Chapter 6

Results 2: Instructional Sequences

This chapter details the instructional sequences as they were realized in each classroom. Table 4.1 summarizes the significant phases of instruction. The two descriptions are presented in tandem so that pedagogical differences could be contrasted easily. Instructional sequences developed and reported by other researchers are briefly overviewed with attention given to adaptations distinctive to this study. Citations are given in order to provide access to additional information about the original instructional sequence. Contextual scenarios developed in this study are presented in greater detail. Pedagogical tools such as recording sheets and games used by the children are referenced in the text and included in Appendix B. The goal in reporting these instructional sequences is to explicate the instruction as it was realized in these two classrooms. It is not expected that instruction would be realized in an identical manner in any other classroom. Rather, these descriptions should serve as examples of the type of instruction that can be achieved in a typical first grade classroom. It would be expected that adaptations and improvements would be made in future implementations. Specific recommendations for such changes are discussed in chapter 8. Chapter 7 details specifics about individual student behaviors observed throughout the course of the teaching experiment and their related conjectured constructs.

Each child in the two classes was assessed before the outset of the teaching experiment. The pre-assessment was an abbreviated structured interview (Wright, Martland, & Stafford, 2006). From this interview each child was classified as either counting-on or not yet counting-on. Each of the two classes was then taught a unit of study by the researcher. The instructional format in both classes was structured in the
same manner. Each class began with a warm-up activity called a Mental Calculation Activity (See chapter 3 for examples). Each activity was intended to build capacity by developing facility with skills necessary to use more sophisticated strategies to solve addition and subtraction problems. This was done as a whole group. The class was then introduced to an investigation or a game. The class worked in pairs to investigate the problem or play the game. Finally, the class met to discuss strategies that were used during the investigation or game. The role of the researcher was to guide the discussion by carefully ordering the contributions of each participant so that increasingly sophisticated strategies emerged. The pedagogical difference between the two classrooms was that only one class was exposed to collections-based models of two-digit numbers. This classroom has been designated the “collections class.” The other class was not exposed to any collections-based models. Instead, that class was exposed to linear models, thus, gained the designation the “linear class.” Some of the investigations and games were common to both classes.

**Partitioning Instructional Sequence**

Initially both classrooms began with investigations, activities, and games intended to encourage partitioning numbers to ten and development of the forward and backward number word sequences (FNWS & BNWS) by tens off the decuples. In the development of number word sequences the collections classroom used bundles of sticks or Unifix trains to model groups of ten. The linear class used a two-color bead string with a ten-catcher (Menne, 2001; van den Heuvel-Panhuizen, 2001b) to represent incrementing and decrementing by ten. The investigation in this section was the orange crate investigation developed by Cobb et al. (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; Whitenack, 1995). The children used ten frames and counters to explore various ways to display ten or fewer oranges in a two by five display in the produce section of the
supermarket. The purpose of this investigation was to partition numbers into two parts. Particular attention was given to notating partitions.

**Inventory and Art Gallery Contexts**

On Day 6 the pedagogy in the two classes substantially differentiated for the first time. In each class, different investigations were used to promote the development of ten as an iterable unit (Steffe, Cobb, & von Glasersfeld, 1988). The collections class began a variation of the *inventory* investigation (Cameron, Hersch, & Fosnot, 2004a; Fosnot & Dolk, 2001) and the linear classroom began the *art gallery* investigation (Cameron, Hersch, & Fosnot, 2004b; Fosnot & Dolk, 2001).

In the inventory investigation the children were asked to assist the school’s math specialist in preparation for a grant application requesting funding for additional materials for use in math instruction. The premise of the investigation was that the specialist needed to have an accurate count of the current available materials so that she would know the quantity of each item that needed to be purchased. The students were presented with a cart with various numbers of different math materials (e.g., 67 rulers, 107 blocks, 32 protractors, 17 plastic bowls). They were also given a recording sheet (see Appendix B) that listed each kind of material with a blank on which to record the number. Within the context, accuracy of the inventory was stressed. The children worked in pairs to count and record the number of the various items. During the class discussion the researcher requested volunteers to tell the inventory of each item. When the inventories of different pairs were not identical, the class discussed methods for making sure that each pair had an accurate count. The researcher had noted during the investigation that one particular pair of students had grouped the rulers in groups of ten during their counting. That pair was asked to explain how they kept track of their count. As a whole group the class then evaluated the likelihood of success if they all used that procedure. It was decided that all
Instructional Sequences

groups would try making groups to ensure accurate counts during the inventory. The next
day pairs recounted the items using the grouping strategy. They were given a modified
recording sheet that had three blanks, one for number of groups of ten, one for “extra
ones,” and one for total (see Appendix B). The investigation continued until the class was
able to agree on the count for each material being inventoried.

Meanwhile, the linear classroom was working with the art gallery investigation.
The premise of this context was that the children were going to use pre-cut paper on
which to create works of art. Each color of paper was cut to different dimensions (i.e.,
dark blue = 5x8, pink = 9x14, gray = 15x23, yellow = 24x96, green = 21x27, white =
18x30, light blue = 33x48, newsprint = 25x32 as measured in Unifix cubes). A group of
parent volunteers had agreed to cut labels from sentence strips that would exactly match
each dimension of the different sized papers. The class’s task was to create a plan on
adding machine tape that the volunteers would follow in order to cut the sentence strips to
the correct length. In order to accomplish this task, students used Unifix blocks as
measuring tools. The blocks were stored in trains of ten blocks of a single color. Two
colors of trains were kept in each container. In this way, the measuring tools were
designed to allow children to capitalize easily on the use of ten when counting the number
of blocks although this was not explicitly stated at the beginning of the investigation.
Students were given a recording sheet (see Appendix B) with the paper colors and a blank
in which to record the paper’s dimensions. The students were instructed to use L and S to
notate “long side” and “short side” of the paper. Beside the adding tape plan was a string
with 100 blocks that were arranged in alternating groups of five green and five white
blocks. Students used this block string to measure the adding machine tape and plot the
various lengths on the plan. As the days passed, all the various dimensions were added to
the strip of adding machine tape and labeled with the appropriate measure.
Instructional Sequences

During the first day of the investigation, some pairs alternated colors of ten cubes and then used that structure to determine easily the length of the paper by counting by tens and then continuing to count by ones for the cubes beyond the last whole decade. Other pairs used only a single color of cubes and then counted by ones to determine the length. Still other pairs alternated colors of ten blocks but then counted single cubes to determine the length. During the whole class discussion at the end of the session, students directed the researcher how to label the plan. The different procedures that students used to measure were discussed. The students evaluated the strategies for efficiency and made recommendations about which strategy they thought would be best to use the next day.

On the 100th day of school members of the collections classroom used two colors of pony beads, pipe cleaners, and foam boards to make a 100-bead abacus whereas members of the linear classroom used two colors of pony beads and yarn to make individual two-color bead strings. Students were allowed to choose to arrange their strings in either alternating groups of five or ten beads. These materials were then used during home activities to create models of various numbers. In one home activity entitled numeral tile draw, children drew a numeral tile (created by cutting apart hundred charts), read the numeral, and then built the number on their abacus or bead string. If children had difficulty reading the numeral, they verified the number by building the number using arrow cards and then expanding the arrows (see Appendix B for direction sheets).

Double-Decker Bus Context

On Day 11 both classrooms were introduced to a modification of the double-decker bus investigation (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997) which used arithmetic racks, an apparatus that has two rows of ten beads, five red and five white, to simulate children sitting on the bus. The children were given problems that explored various ways up to twenty students could be arranged on a bus (see Appendix
B). The purpose of this investigation was to partition numbers to twenty while privileging five, ten, fifteen, and twenty. Problems were posed in the manner of missing sum, missing addends, missing difference, and missing subtrahends. Emphasis was given to problems that were likely to engender doubles and compensation strategies. The bus investigation continued for several days. Each day the warm up continued to address incrementing and decrementing by ten and structuring numbers to twenty. On Day 19 both classes took their first district-generated assessment (see Appendix C for Investigating Two-Digit Numbers to 100).

For the next 2 weeks both classes did a district-generated unit on measurement. This unit was taught by the classroom teacher. During the first week of that time, each individual was given the first of the three structured interviews (See Appendix A for interview assessment schedules for March).

On March 21 the teaching experiment resumed with Day 20. For the next 8 instructional days the children did a collection of games and activities intended to develop facility with structuring numbers to twenty and incrementing and decrementing numbers by ten off the decade (see examples “Jack Rabbit Jump” and “20 Minus” in Appendix B). Following the district pacing guide, on April 5 the children in each classroom took the next district-generated unit assessment (Addition and Subtraction to 20). Each class then took 3 weeks to complete a district-generated unit on money which was taught by the classroom teacher. During this unit the classes did warm-up activities that used dimes (10¢ coins) to increment and decrement thus continuing to develop facility with the number word sequences. During this time each child was administered the second structured interview (See Appendix A for the interview assessment schedules for April).
Instructional Sequences

Candy Factory and Video Tape Storage Contexts

On May 2 both classrooms began Day 30 of the teaching experiment. The collections classroom was introduced to the candy factory scenario (Cobb, Boufi, McClain, & Whitenack, 1997; Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; Yackel, 2001) while the linear classroom began the video tape storage scenario. The candy factory context involved the premise of the class helping the local candy factory devise a better method of storing candy so that orders could be filled and candy inventories could be conducted quickly. Pairs investigated making rolls of ten pieces of candy. They then filled orders using rolls and left over pieces to make various two-digit numbers (see Appendix B for sample Candy Factory record sheets). The investigation was differentiated based on the child’s performance on Interview 2. Most children were asked 1) to determine the number of rolls of candy (tens) that could be formed from various multiples of ten and 2) to determine the number of pieces of candy (ones) in a given number of rolls of candy (tens) (see Candy Inventories 1 and 2 respectively in Appendix B). Children who had already demonstrated this ability on the assessment were given an off-decuple number and asked to identify the number of rolls of candy (tens) and left-over pieces (ones) (see Appendix B for Candy Inventory 3). Other children were adding decuple numbers with single digit numbers to identify a total number of pieces of candy. They were then asked to identify how many more pieces of candy (ones) would be needed to complete another roll of candy (ten) (see Appendix B for Candy Inventory 4). Children who were not yet using ten as a unit were asked to determine the number of pieces of candy needed to make one roll of candy (missing addend to find a sum of ten) (See Appendix B for Candy Inventory 5).

The second phase of the Candy Factory investigation involved additive tasks that were again differentiated based on the child’s performance. All problems were presented
Instructional Sequences

within the context of the candy factory. Most children were adding an off-decuple two-digit number with a multiple of ten. Children who were not facile in counting-on by ones to add were adding fewer than ten to a number of less than or equal to 30. Children who were not yet counting-on by ones to add were adding numbers generated with a 1-20 numeral icosahedron die and a 1-3 dot die (see Appendix B for Candy Order Record Sheets 1, 2, and 3 respectively).

The video tape storage problem was a non-standard measurement problem original to this study and inspired by the art gallery scenario (Cameron, Hersch, & Fosnot, 2004b; Fosnot & Dolk, 2001) and the Smurfs context (Gravemeijer, Bowers, & Stephan, 2003; McClain, Cobb, Gravemeijer, & Estes, 1999; Stephan, Cobb, & Gravemeijer, 2003). The children were shown a VHS video tape. The children measured the tape and discovered that the length of the video tape was exactly ten Unifix cubes. They were then presented with the problem that the researcher had many video tapes that needed to be stored. The researcher wanted to purchase some containers at the local department store for video tape storage. The children were asked to determine how many blocks long a container would need to be in order to store various numbers of tapes. The researcher was then going to take that information to the store to purchase the storage containers. Children used Unifix cubes to measure VHS tapes and simulate container sizes. The cubes were stored in trains of ten. Each train was one color, but two colors of trains were available at each work station. Some children used a different color for each successive train of ten to enable them to keep track easily of the number of tens. Other children were oblivious to the advantage of alternating colors of ten. The children were provided with a record-keeping sheet that listed various numbers of tapes and a blank in which to record the length each container needed to be (see Tape Storage 2 in Appendix
B). At the conclusion of the lesson, the students met to generate a master shopping list for the researcher.

The next day the children were told there was a problem with the measurements. All of the measurements that the children found were multiples of ten, but the containers at the store were not even tens. The researcher gave the children a list of the container measurements that were available for purchase. The question that was posed to the children was, “How many tapes would fit end to end in the container and how much extra space would be left over?” Children were provided with a record-keeping sheet (see Appendix B Tape Storage Problem) that listed the various lengths of containers that were available (i.e., 22, 24, 36, 48, 54 cubes) and had several blanks for children to investigate other measurements of their choosing. Children recorded the number of tapes that would fit, end to end, and the amount of left over space (measured in cubes).

The next day the teacher posed a new task to the children. The teacher presented the children with several options for storage needs and asked how long the container would need to be. For example, if a book was 19 cubes long and one tape was placed beside it, how long would the container need to be? During the discussion, it was established that the tape measurement under investigation was ten cubes in length. The three dimensional aspect of the tapes was not addressed in the context and did not arise in the discussions. The children were presented one of four differentiated record-keeping sheets (see Appendix B). Each sheet used various dice to generate data. The record sheets followed a parallel structure of problems as presented to the collections class in the candy scenario. The children then had to determine the solution to one of four problems involving the number of tapes that would fit in that amount of space or the additional amount of cubes that would be needed to fit one additional tape. On Tape Storage 1 sheet, children used a decahedron with multiples of ten. They then needed to determine the
Instructional Sequences

number of tapes that would fit in that container. Tape Storage 2 sheet involved rolling a 0-9 decahedron to determine the number of tapes that would need to be stored in a container. Their task was to determine the length in cubes of the needed container. Tape Storage 3 sheet involved rolling a 1-30 number generator. The children identified the number of tapes that would fit in the container and the amount of extra space in the container. Tape Storage 4 sheet involved rolling a multiples of ten decahedron die and a 0-9 decahedron die. They recorded the value of each die, added to determine the length of the container. They then had to determine how much longer the container would need to be in order for one additional tape to fit exactly in the container. Tape Storage 5 sheet involved rolling a 0-9 decahedron. The children then had to determine how many cubes longer the container would need to be to hold one tape. Combined Storage 1 rolled a 1-30 and a tens decahedron. The first value determined the length of the book that would be stored. The multiple of ten represented the amount of space need to store the tapes. Combined Storage 2 involved rolling a 1-30 and a 0-9 number generator. This sheet was given to children who needed to practice adding through ten. Combined Storage 3 involved a 1-20 icosahedron and a 1-3 dot cube. This was given to children who were still developing the ability to count-on to add. Students worked in pairs and progressed through the various problems at their own pace.

Interstate Driving Context

On Day 36 (May 10), both classes were introduced to the interstate driving context which was original to this study. In the United States interstate exit numbers correspond to the number of miles away from the state line the exit is. Thus, Exit 89 is 89 miles from the state line. In the state in which the research was conducted, exit numbers become greater as one heads north. On the first day of the context, children were supplied with a list and diagram of the exit numbers and exit names from the local interstate. Many
children reported prior experience with particular exits. The class explored a number of questions about those exits. For example, the researcher posed, “If I got on the interstate at Exit number 80 and drove five miles north before getting off, which exit did I take when I left the interstate?” The day after the context was introduced all first grade children took a field trip to the zoo. During the field trip they rode on a school bus on the interstate. They were very enthusiastic the next day when they were reporting seeing the exits that we had discussed in class the day before. The following ten lessons were spent extending the interstate driving context, connecting it to the empty number line (Gravemeijer, 1999; Treffers & Buys, 2001) sometimes called the open number line (Fosnot & Dolk, 2001), and mathematizing (Cobb, Boufi, McClain, & Whitenack, 1997; Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; Fosnot & Dolk, 2001; Treffers & Beishuizen, 1999; Whitenack, 1995) away from the specific context. The students explored problems that involved addition, subtraction, and missing addends. Instruction was modified for the few children who were not yet counting-on to add. They were given a list and diagram of fabricated exit numbers representing landmarks within the District of Columbia that involved smaller addends within their zone of proximal development. For example, “If I got on the interstate at the Lincoln Memorial which is exit 9 and drove two miles north, where would I be?” Since students had been learning about Washington, D.C. in social studies, this was an imaginable context for them.

**Protocol from Interstate Driving Scenario in the Collections Classroom**

The following protocol is presented to exemplify the typical progression of the whole group, inquiry-based instruction portion of the lesson. It is not intended that the episode presented be seen as ideal instruction but as an example of the type of pedagogy that was typical in the study. This protocol illustrates several aspects of the social and
Instructional Sequences

sociomathematical norms (Cobb, Wood, & Yackel, 1993; McClain & Cobb, 2001) that functioned in the microculture of the classroom. Several times during the course of the dialog the teacher reinforced the expectation that every member of the community had the responsibility to attempt to solve all problems, attempt to communicate their solutions to the group, and attempt to understand another’s solution explanation. Students questioned procedures that led to solutions different from their own and held each other accountable for accurately communicating their thinking. It was also expected that one listened carefully to the solution procedures presented in order to keep from repeating an identical solution strategy. Furthermore, it was normative that explanations would describe experientially real objects. It was not acceptable that children given explanations about merely manipulating digits divorced from meaning in a procedural manner. This excerpt also illustrates how operations were mathematized (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; Dolk, den Hertog, & Gravemeijer, 2002; Gravemeijer, 1997; Klein, Beishuizen, & Treffers, 1998; Treffers & Beishuizen, 1999; Van Putten, van den Brom-Snijders, & Beishuizen, 2005; Whitenack, 1995). Initially the problems were couched in the context of the interstate driving problem. By the end of the discussion, the discussion had decontextualized to dialog about abstract addition. In these instances, the object became the quantity. For example, a child might say, “I added a ten” as if ten were an object. Throughout the course of the dialog, solution strategies were notated with number sentences, empty number line models, and diagrams. The empty number line and other notation systems functioned as tools to further the communication in the classroom. Initially the model was a model of their thinking. Eventually, it became a model for thinking (Gravemeijer, 1999).

This protocol from the interstate driving scenario that took place in the collections classroom on May 17 after approximately 10 minutes of warm up activities. The first
warm up activity involved love hearts for combinations of ten (Menne, 2001) with an every pupil response of bunny ears (Wright, Martland, Stafford, & Stanger, 2006). The teacher displayed a graphic with two hearts that touch. In one heart was displayed a whole number of ten or less such as seven in numeral form. Student were asked to display the number that “loves to go with seven to make ten” using bunny ears (e.g., finger patterns held up at the back of the head like bunny ears). The second warm up activity was hill clapping by tens from 28 to 128 back to 28. In this activity, the children squat with their hands close to the floor. They then chorally count by tens beginning at 28. With each count they raise their hands until they are standing on tip toes for the last count in the sequence (e.g., 128) they continue the activity by counting back by ten to 28. As they count back they lower their hands until they are in the original squatting position.

The teacher was sitting in a chair on the rug with and easel of chart paper beside her. The children were sitting on the rug during this whole group discussion. In the protocol ENL means empty number line, T stands for teacher.

May 17.
T: If we are thinking about our driving, OK, and you know how we talked about the interstate and how the exit numbers tell us how many miles we’ve driven and that sort of thing? And…and if we know the exit numbers we can figure out how many miles between the two exits. We’ve talked about some of that kind of thing. If there was an exit number twenty-eight, OK, and I got on the interstate at Exit number twenty-eight and I drove ten miles [writes 28 + 10 on chart paper as the story is told], and then I got off the interstate at the exit. What would the interstate [exit] number be? Megan?

Megan: Thirty-eight. [Teacher records =38 next to the expression]

T: How did you figure that out so quickly?

Megan: Because there is an eight on twenty-eight and the thirty doesn’t have another three. It’s thirty-eight.

T: OK. How did you know? [Points to Niara]

Niara: Because if you have twenty-eight and you have to add ten more you’re just going down a column.
T: So, if you’re thinking about the hundred chart, you would just be going down.

Jessica interrupting teachers summary: I have a….

Jessica raises hand.

T: Jessica?

Jessica: I was thinking like her, but I pictured it in my head. Twenty-eight is just under it.

T: So, you’re thinking about the hundred chart as well?

Jessica: Nods yes.

T: Tre?

Tre: If you had twenty-eight and you add […] to that eighteen without the ten, you would add eighteen and it would go to thirty-eight … Oh, I don’t know… [Cradles head in hands]

T: I’m not sure I understand what you are telling us. Are you breaking them apart? Is that what you are doing?

Tre: Yeah . . .

T: OK, well let’s think about this problem. What if I got on that same exit twenty-eight, but instead of driving ten miles, I drove twenty miles? [T writes 28+20 as story was told] Where would I be? I want you to think about it. When you have an answer, put your thumb up. [20 second pause] Lindsey, do you have an answer?

Lindsey: No…

T: I want you to move your chair over next to Tynasia where you can see.

[Another 15 second pause] Chase?

Chase: Forty-eight.

T: How did you figure that out, Chase?

Chase: ‘Cause, ‘cause if it’s ten miles it would be thirty-eight and if it is twenty miles then it would be ten more and it would be forty-eight [with emphasis].

T: OK. [T recaps as it is recorded on ENL.]

Niara: Ms. Tabor, if they go ten, twenty, you might put thirty next [pointing to board].

Several students: Yeah!
T: You think? OK.

Bria: Because you’ve had twenty-eight and thirty-eight so the next one, it would be the same eight and it would be forty-eight.

T: Well, if we were predicting patterns, that would be a good one to predict. But you know what? I want to do this problem. [Writes 28+21]

Several students murmur about new problem.

T: Show me when you know. [T holds up thumb to indicate signal.]


Gregory: Forty-nine.

Teacher: How do ya know, Gregory?

Gregory: ‘Cause if twenty-eight and twenty is forty-eight and you add one more, it would be forty-nine.

T: If we hadn’t had this one here [points to previous problem] could we have done the same thinking to figure it out?

Jessica: Oh, I have another way. I just used these [points to the 21] and it would be like adding twenty and one more.

T: So you split them apart?

Jessica: yes.

Aalyah: I, umm, I looked at the ten … uh tens column.

T: Uh-ha.

Aalyah: and then I […] and I got the answer of forty-nine.

T: So, we’re thinking about what we used here [points to the previous problem solution on the ENL]. So I heard a lot of you say, I know what twenty more was so I just added one more. Does that make sense?

T: How ‘bout if I had twenty-eight plus twenty-two [writes 28+22]? What would that be? [Several hand immediately pop up. Teacher pauses for a few more seconds.] Devan?

Devan: Umm, fifty.

T: How do you know, Devan? [while recording 50]

Several students murmur distress about not being asked to respond.
Instructional Sequences

T: [Teacher quietly says] if that was your answer, show me [makes sign language for same here].

Devan: ‘Cause you just added one more.

T: So, I could do it like this [adds one more jump of 1 to the previous ENL]. Is there another way I could have done it?

Aalyah: You start on twenty-eight and add ten and then another ten and then a one and another one.

T: [Teacher draws ENL with two ten hops as the explanation is given. Pauses after that and replies:] Isn’t that exactly what we have here? [points to the ENL above.]

Niara: Mrs. Tabor, I thought you said on an ENL we don’t have to label jumps of one.

T: If it’s one, I usually don’t, but because we were having this discussion, I did. Usually I label them if it is a greater jump than one. Which, I like to do it more than one, because doing it just by one can be a pain sometimes and it’s easy to get off. So, is there another way I can jump besides these two jumps of one.[ENL already shows two jumps of ten]. Tre?

Tre: Add by two.

T: So where would that get me?

Tre: Fifty.

T: So, is there anything that we know for sure that we can use that helps us to know that that’s fifty?

Aalyah: Because, if you add two more it will be fifty because if you add one more, you add one more and it will be forty-nine and then if you add another one would be fifty.

T: OK. So the fact that this is forty-eight [underlines 8] and two more, does that ring any bells?

Several students: Ohhh.

T: Forty-eight and two [with emphasis] more?

Alayah: Cause if you are in the ones column and add two more it would go to fifty because after the forties comes the fifties.

T: [Picks up the Hearts that Love to Go Together card used during warm-up and gestures with them] Anything else we can use?

Jessica: It would be fifty because if you added one jump to forty-eight it would be forty-nine and if you added another one it would be fifty.
T: OK, I was just wondering if there was anything else we could use [waves Love Cards in a very obvious way.]

Several students: Our Love Hearts.

T: OK, how could we use our Love Heart to help us figure out what forty-eight and two more are?

Tynasia: Twenty-eight and two more love to go together to make fifty

T: [Interrupting] Twenty-eight and two more love to go together to make fifty… Oh, I think I know what you are trying to say.

Tynasia: Yeah, twenty-eight and two more go together to make thirty.

T: Ahh. And then what would I have left to add from thirty?

Tynasia: and then forty to fifty…plus ten and plus ten.

T: So there’s more than one way I can solve that, huh?

Bria: You could just add two to forty-eight. If you added one more it would be forty-nine and another one would be fifty.

T: Ok. I think we talked about that up here [indicated previous ENL]. Let me give you another problem. What if we had twenty-eight plus twenty-five? [writes 28+25 on paper]. How might we solve that one? I want you all to think about it and when you have a solution, show me your thumb. [Pause of 35 seconds. Draws line for ENL] This is the last problem we’re going to do today before we talk about the games we are going to do today. I want everybody thinking. [Pause of several seconds more. T reminds several children to sit properly so that everyone can see] Devan, how would you solve it?

Devan: Umm I just add…if those two were twenty and twenty, I knew it was going to be forty and I added the eight and it was forty-eight and I added five more and it was fifty-three.

T: So, you said, twenty and twenty is forty and eight more is forty-eight [T records ENL 20 + 20 +8] and then what did you do?

Devan: I added five more.

T: So how did you add the five? Did you add it all in one chunk or did you break it up? How did you add the five?

Devan: I just counted them.

T: So you just counted on five more? So you made forty-nine, fifty, fifty-one, fifty-two, fifty-three [drew five jumps of one on ENL]
T: Did you have a different way? [Indicating Bria who had raised her hand]

Bria: Because I know that two plus two was four and then I add the eight and five and then if I add the eight and five it would be fifty-three.

T: Is this a two and two [indicates the digit 2 in 28 and the 2 in 25].

Bria: I mean the two and the twp at the beginning of the second number.

T: Right, but is it really a twp? Is it?

Nia: A one.

T: This is a one?

Bria: It was a twenty. ‘cause I knew that twenty plus twenty was forty and if I add the eight it would be forty-eight and then I plus the five it would be fifty, I mean fifty-three.

T: Think you may be explaining the same kind of process that Devan did here [indicating previous ENL]. Tre, how did you do it?

Tre: I had a different answer. Forty-five.

T: OK. Show us how you got your answer.

Tre: I added twenty plus twenty.

T: Alright. [Drew ENL with an arc beginning on 20 and labeled +20] and what did you get?

Tre: Forty. And then added a five. [T labels 40 and adds an arc of +5].

T: What did you get?

Tre: Forty-five. [Teacher labels 45]. And that’s the answer.

T: So you added twenty plus twenty and then you added a five. [T points to the 20, 20 and underlines the 5 in 45.] So did you get it all?

Tre: No

T: So how are we going to get that eight on there? Can we break up eight in any way to make it easier?

Tre: Add a three?

T: So you want to break up eight into five plus three?

Tre: And that would equal forty-eight. ‘Cause five and three is eight.
T: OK, we could do that. [records on the ENL]

Tre: And it would be forty-eight.

Ms. R.: I think he’s thinking that he’s already added the five.

T: Let’s get another piece of paper so we can see what we’re doing here. OK. You are saying that we can add [writes 20 + 8 + 20 + 5 as says] twenty plus eight plus twenty plus five. Right. [Looking at the whole group] Is that what he’s saying?

Several students: Yeah.

T: OK. Does it matter which order we add these together?

Several students: No.

T: That’s the switcheroo, right? We can switcheroo addition. We can do it with more than two addends. So, you said you added these two together [draws lines from the 20s and writes 20+20=40.] So you said twenty plus twenty equals forty, right?

Tre: Then add eight plus five.

T: So I’m going to add these together separately? [Tre nods yes]. So what is eight plus five? [Pause of several seconds]. I wonder, can we break this eight apart and make it friendlier? Nia, what do you think?

Nia: Thirteen?

T: How do you know?

Tre: I can count on by five.

T: OK Is there another way that we can do it that is easier, quicker?

Bria: I know what eight plus five is … fifteen.

Several other students: thirteen.

T: Why do you say fifteen?

Bria: Because, [holding up eight fingers] eight and two [raises two more fingers] makes ten and that leaves three …It is thirteen.

T: Ahhh. She said…

Bria: If you have eight [holds up eight fingers] and two and three is five, if you add the two and three it makes ten ’cause five and five more is ten and three more is thirteen.

Tre: I’m confused.
T: Do you see what she did? She broke five into two and three so that she could use her love hearts. She broke this five here [points to the expression] into two plus three. Can we do that? If we had five cubes, could we break it into two different trains of two and three? [Several children nod.] OK. So she said that she had an eight train over here and a five train, OK [indicates the expression]. Well, she wanted to make this eight train into a ten train because we can work with a ten train a lot easier because we know the ten plus facts, right? Remember the ten plus cards we played and we played the ten plus game and all those? So she broke the five into a two and a three. And then she used her love hearts. eight plus two is ten and three more is thirteen. Now, back to Tre. You’ve got your forty and you have thirteen. Now what?

Tre: Add two jumps of five to forty.

T: Is there an easier way to do it?

Tre: A jump of ten.

T: And then what?

Tre: A jump of two and then a jump of one.

T: Is there another way I could have done it? Nia?

Nia: A jump of three.

T: Yeah, I could have done it in a jump of three, right, if I know that fifty plus three is going to be fifty-three? We can think about the candy shop, right? How would I get together an order of fifty pieces of candy in the candy shop? As quick as can be? How many packages?

Tre: Five packages.

T: Five packages and then if I had three left over pieces, how many would I have?

Several students: Fifty-three.

T: So there are lots of ways I could play with these numbers and break them apart, right?

Discussion of Protocol

The above protocol is typical of the mathematical discussions that occurred among students in both classes. This excerpt highlights the social and sociomathematical norms (McClain & Cobb, 2001; Voigt, 1995; Whitenack, 1995; Yackel & Cobb, 1993, 1996) that were in effect in both classrooms. Student strategies were visually represented
using the empty number line and other systems of notation appropriate to the strategy. Students were often prompted to clarify their statements and defend their solutions. Children frequently challenged another’s solution when the explanation was not clearly understood. It was established that it was the responsibility of each individual to attempt to understand the solution strategies presented. If an individual did not grasp the argument, he or she had the responsibility to seek clarification.

The norms operating in the class were most obvious within an isolated episode when one member of the community violated a norm. For example, at one point during the discussion Bria violated the norm for what constituted an acceptable solution strategy. She referred to the face value of the tens place as a two rather than giving the value of the place as twenty. She was challenged to amend her explanation into a normative response.

Mathematical discussions were initially grounded in a context but would gradually move away from the context through a process of mathematization (Cobb, Boufi, McClain, & Whitenack, 1995; Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; Fosnot & Dolk, 2001; Strom, Kemeny, Lehrer, & Forman, 2001; Treffers & Beishuizen, 1999; Whitenack, 1995).

Occasionally throughout the teaching experiment, games were used to continue to develop facility with underlying skills such as structuring numbers to ten and twenty, and developing number word sequences. These games were sent home in lieu of traditional homework pages. At the end of the interstate driving scenario, children were given the third of the structured interviews (See Appendix A for June interview assessment schedules).

**Summary**

The intention of the presentation of these instructional sequences and the protocol is not to establish a proscribed instructional sequence or pedagogical style. Rather, the
The intention of the protocol is to demonstrate that young children can participate in inquiry-based dialog. Clearly, improvements could be made in the pattern of dialog. The episode presented follows a format of exchange in which the teacher is the central figure. The level of sophistication of the dialog could be improved if the students reacted more to each other’s statements. However, the classroom teacher had developed social norms surrounding the pattern of contributions within the classroom participation. A primary goal of the study was to establish the norms of articulating their strategies and striving to understand the strategies of others. Therefore, changing the pattern of interaction was subjugated to these goals.

The intention of the elaboration of the instructional sequences is to set forth a series of instructional sequences that can be modified by teachers to fit the needs of their students. The instructional sequences presented here were “real” and meaningful to these students. It is conceivable that the same contexts may not be equally appropriate to children in other locales. The interstate driving context worked well because of the close proximity to the interstate. The children had personal knowledge and experience with using the interstate system. For children from an isolated farming community or another country, however, the interstate driving scenario might not be imaginable. It is hoped that teachers would be able to use the instructional sequences presented as a model for how they might 1) use contexts that are meaningful for their students and 2) provide scaffolding to differentiate instruction to meet the needs of all learners.
Chapter 7

Results 3: The Strategies that Emerged

As overviewed in chapter 3, several studies have posited theory to characterize the nature of children’s thinking with regard to early base-ten strategies. Steffe, Cobb, and von Glasersfeld (1988, pp. 232-233) identified five concepts of ten: ten as a numerical composite, ten as an abstract composite unit, ten more, ten as a repeatable unit, and ten as an iterable unit. A child who operated with ten as a numerical composite would understand the nested nature of ten. He or she might be able to identify twenty as two tens and would count-on by ones to add twenty and count-back by ones to subtract twenty, stopping once two sets of ten had been counted. However, the child would be unable to use ten as a composite unit to solve problems. A child who operated with an abstract composite unit concept of ten would be able to manipulate ten as a composite unit to count by tens or by ones in limited situations. The materials might be hidden behind a cover, but the child would be able to reflect on the use of the composite units and anticipate solving the problem with the units. A child with a ten more unit would make counts by one in sets of ten, keeping track of the number of sets counted. This child might even refer to counting ten-twenty-thirty when referring to the series of sets of ten counts by one. A child with ten as a repeatable unit would be able to substitute a re-presentation of counting by tens for an actual counting of objects. He or she might count by ten to solve problems but could not easily shift between counting by tens and ones within the same task. A child with an iterable unit concept of ten would be able to solve a bare number problem without the use of materials.

Cobb and Wheatley (1988, pp. 4-7) further refined these concepts of ten to three levels: “ten as a numerical composite” in which the child focuses on the individual units
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within the ten, “ten as an abstract composite unit” in which the child can coordinate tens and ones but must re-present the tens in advance before commencing the count, and “ten as an iterable unit” in which the child solves a problem by iterating units of ten without need of the materials before beginning the count. They also mentioned two other concepts of ten: “ten as an abstract collectible unit” which Beishuizen (1985) called 1010 and “ten as an abstract singleton” in which each digit is treated as what Fuson called a “concatenated single digit” (Fuson, 1990a, 1990b, 1992; Fuson & Smith, 1997). Wright, Martland, and Stafford (2006, p. 22) have incorporated Cobb and Wheatley’s first three concepts of ten into their Learning Framework in Number as “initial concept of ten,” “intermediate concept of ten,” and “facile concept of ten.” Jones et al.(Jones, Thornton, & Putt, 1994; Jones et al., 1996) appear to have drawn their framework in a similar fashion. The first three levels of their framework deal with place value up to 100. Their first three levels are “pre-place value,” “initial place value,” and “developing place value” as described below:

In particular, it was hypothesized that Level 1 (pre-place value) requires the use of single units; Level 2 (initial place value) is characterized by the movement from using only single units to using ten as a composite unit; Level 3 (developing place value) extends the use of two-digit numbers to mental addition; and Level 4 broadens the thrust to three-digit numbers. (1996, p. 312)

The purpose of this chapter is to propose an amplification of the theory of a framework for early base-ten strategies.

Explicated Strategies

Throughout the course of the teaching experiments, several interesting patterns of strategies for solving two-digit addition and subtraction emerged. Students developed the strategies in different orders. In the collections class the more common progression of strategies was: unitary counts to add such as counting-on by ones, followed by 1010,
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followed by facility with both 1010 and N10. In the linear class, the development of strategies was more typically: unitary counts to add such as counting-on by ones, followed by N10, followed by facility with both N10 and 1010. However, in both classrooms there were some children who never developed two advanced strategies. Some progressed from unitary counts to add to N10 only. Others progressed from unitary counts to 1010 only. Some of the children never developed an advanced strategy for solving two-digit addition and subtraction. These children would attempt to count-on to solve two-digit addition and subtraction with varying degrees of success.

Complete the Ten Construct

An interesting construct emerged in both classrooms. I have labeled this construct the complete the ten construct. For some children, it appears they interpret “add 10 more” to be “add enough more so that you will have an even number of tens.” The following protocols of “Keith” illustrate this construct. The setting involves bundles of ten craft sticks bound by rubber bands, loose craft sticks, and an opaque cylindrical container.

Protocol from March 8

T: How many do we have there? [Places a bundle of ten on the table.]
S: Ten.

T: Now how many? [Adds three single sticks beside the bundle.]
S: Thirteen. [Response without hesitation.]

T: OK. I’m gonna put these in here. [Places bundle and three sticks into container.] How many do we have in there?
S: Thirteen.

T: OK, and if I do this? [Places two more bundles into container one after the other.]

T: OK
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S: I knew that ‘cause I was counting.

T: So there are thirty in here? [Taps the container.]

S: Yeah, ‘cause I went thirteen, twenty, thirty [in sing-song count.]

T: OK. And if I put these in here? [Places four single sticks partially inside the container so they can clearly be seen.]

S: [Pause] Four – Thirty-four.

T: Alright. [Tilts container so the four sticks slide into the container and are no longer visible.] And if I do this? [Adds three more sticks so they are visible.]


T: Alright. [Tilts sticks into container.] And if I do this? [Adds a bundle of ten.]

S: Forty

T: And if I do this? [Places two single sticks so they can be seen.]

S: Forty-two.

T: OK. [Tilts container so they slide into container.] And if I do this? [Adds two bundles of ten one after the other.]

S: Fifty, sixty.

T: OK. So how many do I have in here?

S: Sixty!

Here is the progression of the first series of tasks with the student responses from above.

\[
10+3 \rightarrow 13; \quad 13+20 \rightarrow 30; \quad 30+4 \rightarrow 34; \quad 34+3 \rightarrow 37; \quad 37+10 \rightarrow 40; \quad 40+2 \rightarrow 42; \quad 42+20 \rightarrow 60
\]

The following is a protocol of the next tasks presented to the child.

T: [Dumps out container on to table.] Would you build thirty-two for me?

S: [Selects three bundles at one time and places them on the table. Then selects two singles and places them to the left of the bundles.] Easy.
T: Alright.

S: See look. ten, twenty, thirty, [slight pause] thirty-one, thirty-two, [while pointing from right to left as he counts each bundle or stick.]

T: How about that? [Clears table and places five bundles and three sticks to the left of the bundles.]

S: [Pause] Fifty. That was easy. See look. One, two, three, ten, twenty, thirty, forty, fifty [while pointing from left to right across singles and bundles.]

T: So all of this together gives you fifty?

S: Yeah, see look: one, two, three, ten, twenty, thirty, forty, fifty [while pointing from left to right across singles and bundles.]

T: So what if I had this? [Picks up the three singles and moves them to the right of the bundles.]


T: OK. So this is fifty-three? [Motions to collection.]

S: Uh huh.

T: And how many was this? [Picks up the three and moves to the left of the bundles.]

S: Fifty-three – I mean fifty [Motions to collection] [pauses] one, two, three, ten, twenty, thirty, forty, fifty [while pointing from left to right across singles and bundles] Fifty!

T: So that’s fifty?

S: Uh huh.

T: And how much was this? [Moves three singles back to right of bundles.]

S: Fifty-three [Holds up three fingers as he says three.]

The child appears to experience very little cognitive dissonance during the course of these tasks. He is seemingly unbothered by the fact that the quantity apparently changes depending on the placement of the single sticks.
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Steffe, Cobb, and von Glasersfeld (1988) noted this behavior in their teaching experiment. Rather than interpreting this behavior as indicative of a new construct, they interpreted it as the child had not yet developed a construct that would allow for “progressive integrations.” They speculated that the behavior could be attributed to the student having been exposed to pregrouped models of base ten materials in the regular classroom. The use of pregrouped models of base ten materials was not a factor for the children in the linear classroom because the classroom teaching experiment carefully controlled this variable. Therefore, there must be some other explanation.

**Complete the Ten Construct Fixating on Ones**

Another variation of this construct appears to fixate on the ones. Here is the progression above presented to another child.

\[
\begin{align*}
10+3 & \rightarrow 13; \\
13+20 & \rightarrow 50; \\
50+4 & \rightarrow 54; \\
54+3 & \rightarrow 57; \\
57+10 & \rightarrow 80; \\
80+2 & \rightarrow 82; \\
82+20 & \rightarrow 100
\end{align*}
\]

The pattern holds until the last task when attention appears to shift to the tens place. When less than ten is added, the child counts on that number of counts. However, when ten or multiples of ten are added, the child seemingly converts the amount in the ones place to ten and then adds the appropriate number of tens to that new place. This pattern breaks down for 82 + 20. One might speculate that the child has enough number sense to realize that “forty,” which would have been her answer if the pattern had continued, is smaller than the original quantity of “eighty-two” and is therefore not a reasonable answer. Another possible explanation might be that the child attends to the larger digit in the number and treats it as a ten. Presumably the child might be acting on a notion of “start with the larger ‘number’ and add on from there.”

This complete the ten construct was observed with bare number problems as well. Some might be tempted to explain this construct by saying the individual lacks the cognitive load capacity or working memory to keep up with all the parts of the
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decomposed number (Ashcraft & Kirk, 2001; Beishuizen, 1993; Boulton-Lewis, 1998; Boulton-Lewis & Tait, 1994; Deschuyteneer & Vandierendonck, 2005; DeStefano & LeFevre, 2004; Duverne, Lemaire, & Vandierendonck, 2008; Geary & Widaman, 1992; Imbo, Vandierendonck, & De Rammelaere, 2007; Imbo, Vandierendonck, & Vergauwe, 2007; Seitz & Schumann-Hengsteler, 2002; Sweller, 1988). This argument would suggest the following procedure for 13 + 20. The individual decomposes 13 into 10 and 3. The individual then adds 10 to the 20, but because the cognitive load is too great, the individual then fails to account for the 3 that was originally decomposed from the 13. Therefore, the individual does not add the 3 to the 30 and is therefore off by 3. However, from the protocol above it is more probable that the individual has a naive conception about what it means to add ten more. It is possible to predict the sum for each of the subtasks presented above when the individual is operating within the complete the ten construct. The sums are logical when operating within the construct. If the incorrect sums were the result of errors due to cognitive load capacity, one would not expect the level of consistency evident in both of the scenarios above. For children operating within the complete the ten construct, it appears that the statements “add another ten” and “add ten more” mean qualitatively different things. “Add another ten” seems to mean “place another of those things we call tens beside the collection.” “Add ten more” seems to mean “add enough more until you complete that partial ten.”

Further evidence in favor of the argument for the existence of the ‘complete the ten’ construct is the later performance of Keith from the first protocol. During the structured interview a little over one month later he no longer demonstrated evidence of this construct. Rather he incremented and decremented by ten off the decuple numbers. Interestingly, his weakness with number word sequences for incrementing by ten off the
decuple presents the opportunity to see a very different type of error for a very similar
series of tasks.

**Weakness in Number Word Sequences**

**Protocol from April 20**

T: How many do we have there? [Places six sticks over the lip of the cylinder.]  
S: [Subvocalizes counting] one, two, three, four, five, six…[aloud] six.  
T: [Tilts the cylinder so sticks slide inside the container.] So what do we have down in here?  
S: Six.  
T: [Adds a bundle to the cylinder] Now what?  
S: Twenty-six [Response without hesitation.]  
T: [Places two bundles on lip of container and looks at the child.]  
T: [Tilts cylinder to slide bundles into the container. Places one bundle and three sticks on the lip of the cylinder so they are visible. Looks at student.]  
S: Pause. Eighty-six…eighty-seven, eighty-eight, eighty-nine.  
… off task comment on the part of the student.  
T: [Tilts cylinder to slide bundles into the container. Places two bundles and four sticks on the lip of the cylinder so they are visible. Looks at student.]  
S: So I’m in eighty-six and ninety-six, one hundred six, one hundred seven, one hundred eight, one hundred nine, two hundred! [while pointing to the bundles and four sticks.]  
T: OK  
S: [Makes a face and says in a high-pitched voice] That’s a lot!  
T: Chuckles.  
T: [Dumps collection from cylinder on to tabletop. Places four bundles and three single sticks to the left of the bundles in front of student.]
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S: One, two, three, twenty-three, thirty-three, forty-three, fifty-three. [in sing-song count while pointing to the bundles and sticks.]

T: And I heard you say: “One, two, three, twenty-three, thirty-three, forty-three, fifty-three.”

T: What if I put these [Picks up the three single sticks and places then to right of the bundles.] here?

S: Fifty-three [Immediately.]

T: Can you show me how you figured that out?

S: ‘Cuz look. ten, twenty, thirty forty…[Looks at teacher with mouth agape.] Oh my gosh, that tr…. Oh [separates two bundles that are touching each other.] Stay in line. ten, twenty, thirty, F… [Counts each bundle and trails off when he realizes there are still only forty in tens.] Wait. ten, twenty, thirty, forty [while pointing again to each ten. Makes a face.]

T: Your faces are wonderful.

S: Gosh, I keep on getting forty. I thought it was fifty before. Did they disappear?

T: [Chuckling] No, I promise [hold palms up] none disappeared.

S: Ten, twenty, thirty, forty, forty-one, forty-two, forty-three. Then how did it disappear?

T: Alright, watch. [Picks up the singles and replaces them to the left of the bundles.]

S: [Points to each bundle while subvocalizing] one, two, three, four [as if to verify that the number of bundles was the same as before. Continues to subvocalize while pointing to the singles and then the bundles] one, two, three…one, two, three, four. [Aloud]

T: So count it the way you did before. [Points to three singles while saying] one, two, three [pauses]

S: Twenty-three, thirty-three, forty-three, [pauses] fifty-three.

T: Alright, let me do this. [Removes three of the bundles and holds them in sight]. Count this much [Gestures to the three singles and one bundle].

S: Twenty…twenty-three.

T: That’s twenty-three [Taps the table beside the collection.]

S: [Recounts silently while pointing to the singles and bundle] Twenty-three.

T: OK. And if I do this? [Moves singles to the right of the bundle.]
S: Now that’s twenty-three!

T: Prove it to me.

S: ‘Cuz it’s…[points to the bundle] …thirteen!

T: So how much is this? [Taps the collection.]

S: Ten … thirteen.

T: I heard you said ten [points to the bundle]…thirteen. What were you doing here? [Points to the three singles.]

S: One, two, three.

T: So that’s thirteen?

S: [Nods.]

T: [Moves the singles to the left of the bundle] So what is it now?

S: Thirteen.

T: [Adds another bundle back beside the bundle on the table and looks at the student.]

S: Twenty-three.

T: [Adds another bundle back beside the bundles on the table and looks at the student.]

S: Forty-three.

T: How much?

S: Forty-three.

T: Forty-three?

S: Yep.

T: [Moves the singles to the right of the bundle.] So what is that?

S: Thirty-three.

T: [Moves the singles back to the left of the bundle] What’s that?

S: Thirty-three.

T: [Adds another of the original bundles back beside the bundles on the table and looks at the student.]
S: Forty-three.

T: Alright. Count it for me so I can hear you count it one more time.

S: One, two, three, twenty-three, thirty-three.

T: [Teacher interrupts by removing last three bundles.] That’s twenty-three?

S: No.

T: What is it?

S: It’s thirteen. [Teacher replaces three bundles. Student points while counting] thirteen, twenty-three, thirty-three, forty-three.

T: So, how much is that? [Gestures to collection.]

S: 43.

T: Umm. So, when you originally got fifty-three, why did you get fifty-three? Can you figure out where you made your mistake?

S: Because I said twenty-three and thirty-three and forty-three and fifty-three [Points to the four bundles in turn.]

T: Ah, you skipped right over that thirteen, didn’t you?

S: Yeah.

T: So those sequences are pretty important, aren’t they?

S: [Sighs] Yeah.

**Analysis of April 20 Protocol**

In the original series of tasks, the student made several errors due to a lack of facility with the forward number word sequence needed to increment by tens. The errors were in the progressions of the decades rather than in the ones as in the original protocol.

\[6+10 \rightarrow 26; 26+20 \rightarrow 46; 46+13 \rightarrow 89; 89+24 \rightarrow 200\]

This weakness is further highlighted in the task in which he is asked to count the three singles and four bundles.
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One would expect cognitive load capacity to be more fixed within this relatively short period of time. Since the child was eventually able to begin incrementing by ten off the decuple, this supports the notion that the errors were in fact the logical outcome of a naïve construct for adding ten. The subsequent protocol illustrated inaccuracies due to weaknesses in producing the number word sequences not due to a naïve understanding of adding ten. Apparently he made an accommodation in his construct for adding ten between the two assessments.

It is interesting to note here that the lack of facility with the forward number words needed to increment by tens did not keep the child from attempting to use N10. Lack of facility obviously hindered his ability to increment successfully. With the scaffolding that was offered, he was able to use N10 successfully. Furthermore, during the second occasion, the scaffolding drew attention to the error in his number word sequence. In the first occasion when the child was operating under the complete the ten construct, the scaffolding offered did not draw attention to the inconsistencies because for him at that time, there were none. One can only speculate how long the faulty number word sequence may have persisted after the second occasion in the absence of the scaffolding.

Amplified Framework for Early Base-Ten Strategies

During the course of this experiment, it has been useful to amplify the framework for early base-ten strategies presented earlier. I have likened the progression of the development of two-digit addition and subtraction strategies to a river journey (see chapter 3 for initial discussion). Figure 7.1 illustrates this analogy.

As illustrated in Figure 7.1, the first construct in the amplified framework is the same as reported in Steffe, Cobb, and von Glasersfeld (1988). I have found it useful to
add the construct ten as a special numerical composite. Children operating under this construct are able to recognize and represent two-digit quantities using materials such as base-ten materials or two-color bead strings. They can add up to nine to decuple numbers without having to count-on by ones. However, when presented with a series of additive tasks in which multiples of ten are added to off-decuple quantities, they will display the “complete the ten” construct discussed earlier.

Figure 7.1. River Analogy
Because the materials to which children are exposed may predispose them either toward 1010 or N10, I have found it useful to think of having two branches in the construct of ten as an abstract composite unit. In both cases, the child would require materials or a re-presentation of those materials in order to solve two-digit addition and subtraction problems. Children initially exposed to collections would be more likely to construct ten as an abstract composite unit as evidenced by the 1010 strategy in the presence of collections. A possible troublesome eddy along the 1010 branch is becoming what has been called a pathological splitter (Rousham, 2003). A pathological splitter is one who will subtract the larger quantity from the smaller regardless of whether the larger quantity is in the minuend or the subtrahend. For example for 62-24, a pathological splitter would appear to reason, “twenty from sixty is forty and two from four is two so the difference is forty-two.” Alternately, one might say this arises from the concatenated single digits view of multi-digit numbers (Fuson & Smith, 1997).

The other branch in the development would be children who construct ten as an abstract composite unit as evidenced by N10 in the presence of a two-color bead string, a measurement context, or an empty number line. The branch has traditionally been more troublesome for American children presumably because inadequate attention has been given to the development of number word sequences for incrementing and decrementing by ten from any number (Cobb & Wheatley, 1988).

Eventually a child will no longer need materials or re-presentations of those materials in order to solve two-digit addition and subtraction problems. They will be able to solve bare number problems with 1010 or N10 but not both. In the amplified progression presented here, this construct would be labeled as ten as an initial iterable unit. What separates this construct from ten as a facile iterable unit is the ability to use more than one strategy to solve bare number problems. A child operating under the ten as
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a facile iterable unit construct would be able to choose from N10 or 1010 depending on the nature of the problem at hand. Children operating under this construct will also frequently combine strategies.

Throughout the course of this study, there were children operating under each of these different constructs of ten. Over the course of the study, most of the children moved from one construct to another. However, at the end of the study, not all of the children had developed ten as a facile iterable unit.

Synergy of Communal Thinking Process

A cautionary statement is perhaps in order here. There is a danger of looking at the behavior of an individual in an isolated occasion. One can misjudge an individual’s construct based on his or her contribution to class dialog. Within the structure of the group discussions, there appears to be a synergy of what might be called a collective or group thinking process (Brown & Renshaw, 1995). On occasion, a child would appear to have developed facility with a particular strategy within the context of the whole group dialog. However, when that same student was later individually assessed, the same strategy would not be observed. The discussion captured in the protocol included in Chapter 6 provides an example of this phenomenon. On May 17 Chase introduced the notion of incrementing by ten multiple times in order to add 20 to 28. This was the first time in the class discussions that the notion of adding multiple tens had been introduced. On subsequent days, this strategy was extended and notated many times. Chase frequently participated in these discussions. However, during the individual structured interview in June, Chase was unable to use the same process to solve problems independently.

Several features of the whole-class experience may have acted as a scaffold for Chase’s thinking. For example, the problem was introduced within the interstate driving context. This context made sense to Chase and he was able to imagine the problem. This
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provided a level of support that was not available during the structured interview. Furthermore, the empty number line model had been used in the classroom to notate various strategies. This visual representation also provided a form of scaffolding. Finally, within the culture of the classroom, a risk-free environment had been created. Students were encouraged to attempt to solve problems and communicate those solutions to the classroom community even if they were not completely certain of their solutions. Inaccurate solutions were seen as an opportunity for further analysis. It may be that Chase did not feel the same level of comfort in the one-on-one interview setting. Once some scaffolding was provided during the assessment, Chase was successful. The follow protocol illustrates the scaffolding that was provided during the dynamic assessment (Berman, 2001; Lidz, 2003, 1987; Lidz & Gindis, 2003; Tzuriel, 2001).

Dynamic Assessment

The following protocol exemplifies the type of scaffolding that can be provided in the course of a dynamic assessment (see chapter 3 for definition). The excerpt is taken from the second portion of the assessment. The student had already successfully solved additive tasks involving materials. Initially when the student indicated that he has no solution method, he was provided with an empty number line recording sheet. On the next problem the student was prompted to think about how he would solve the problem if he used the empty number line. On the third problem, the student was not given any prompt.

June Assessment Protocol

T: [Places a card with 38 + 24 in a horizontal format in front of student.] Read this to me.
S: OK. Thirty-eight plus twenty-four.
T: Do you have a way to work that out?
S: [Shaking head in negative]…[pauses for 2 seconds and whispers] I can’t do it.
T: OK, What if I gave you one of these? [Hands student a recording sheet with the line segment for the empty number line (ENL) and a place to record the equation and a pencil.] Could you do it then?

S: [Records 38 + 24 then proceeds to solve the problem using the ENL. He first draws a hash mark and labels it 38. He then draws an arc beginning at the 38. He labels the arc +10 and then, after a brief pause labels the other end of the arc 48. He then draws a second arc. Labels that arc +10 and then labels the end of the arc 58. He then drew four smaller arcs and labeled the end of each as 59, 60, 61, 62.] Sixty-two [with a big grin].

T: Alright. Would you record your answer right there for me? [Gestures to the number sentence he had begun earlier.]

S: [Records 62.]

T: Let me ask you something. Can you do this kind of thing in your head? [Points to the ENL.]

S: [Nods] ‘Cuz I knew there were two hops of ten and four little ones.

T: OK. [Removes pencil and recording sheet.] Well, let’s see about this one. [Places a card with 56-23 in a horizontal format in front of the student.] Read that one for me.

S: Fifty-six minus twenty-three. [Long pause.]

T: If you were going to do it on the ENL, how would you do it?

S: [After 10 second pause] Make two hops of ten and three hops of one.

T: OK. So if you made one hop of ten where would you be?

S: [After 34-second pause] Forty-six.

T: OK. Then what would you do?


T: So you can solve these without paper and pencil huh? Umm, that was a lot of thinking. Should we try another one?

S: [Nods.]

T: OK. How ‘bout this one? [Places a card with 43-15 in a horizontal format in front of the student.] Read this one for me.

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T: Wow, I’m impressed. So other than having to stop and think about the number that comes next as you count backward, you don’t have any problem with doing this at all, do you?

S: [Shakes head no and grins.] ‘Cause if you think in your head and like if it’s forty-three and you add ten more, oh I don’t know but if it’s fifteen and you add ten more it twenty-five and if you add ten more it’s thirty-five and if you add ten more its forty-five, then ten more it’s fifty-five and then ten more it’s sixty-five then ten more it’s seventy-five, then ten more it’s eighty-five, then ten more it’s ninety-five! [with another grin.]

Classroom Application

This kind of scaffolding can be provided during instruction in a whole class setting. Initially an appropriate scaffold can be given to enable a child to be successful. On successive problems, rather than automatically extend the same level of support, the teacher can prompt the child to reflect on the support that was given previously. Many times this level of support is adequate to enable the child to solve the problem successfully. In this way, the scaffold fades or is gradually removed. The same pedagogical progression can be used with most materials and contexts. Initially the students should have experiences with the materials and contexts. When they develop facility solving with the materials and contexts, they should be prompted to reflect on the use of these contexts. Within the context of bare number problems, if a child reaches an impasse (Wright, Martland, Stafford, & Stanger, 2006), the teacher might prompt with a suggestion to think about what he or she would do if the materials or context was available.

Summary

From the protocols presented, it seems reasonable to argue for the existence of the complete the ten construct. When given adequate experiences with carefully-scaffolded, contextual investigations involving units of ten, students can develop more sophisticated
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understandings of the base-ten strategies. Using the amplified framework for early base-
ten strategies can assist teachers in developing meaningful contexts and instructional sequences that allow for the emergence of more sophisticated strategies. The framework can assist teachers in evaluating the relative sophistication of student strategies and in thinking carefully about the scaffolding needed to facilitate the emerging strategies.
Chapter 8

Research Outcomes:

Discussion, Recommendations & Conclusions

Discussion of Research Questions

This study addressed eleven research questions as presented in chapter 4. Each of those questions will now be discussed.

1. Are children who have not used base-ten grouping materials (i.e., bundles and sticks, Unifix trains of 10, base-ten blocks) more likely to develop N10 than those who have used the base-ten grouping materials?

This study indicates that the answer to this question is no. Both groups were significantly more likely to use N10 at the time of the posttest than they were at the time of the first structured interview, but there was no significant difference between the groups. Therefore, the children who did not have exposure to grouping materials were no more or less likely to use N10 than the children who routinely used collections in the first two instructional sequences (see chapter 5 for a detailed statistical analysis).

2. Are children who have used base-ten grouping materials more likely to develop 1010 than those who have never used the materials?

3. If so, can the predisposition toward 1010 in those children who have experience using collections be overcome through subsequent instruction?
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4. Will the children with collections exposure develop N10, given the proper succeeding experiences?

This study indicated that children who had experienced collections materials were initially more likely to use 1010 than children in the linear classroom. However, this trend did not persist. By the end of the study, children in the collections classroom were no more likely to use 1010 than children in the linear classroom. It is argued here that the difference in the mathematical practices that emerged in the two classrooms is likely due, in some part, to the differences in instructional sequences. Since the collections class did not persist in the preference for 1010, it seems likely that the interstate driving context and the attention to building the number word sequences needed to increment and decrement by ten may have supported the development of N10 among children in both classes.

5. Will children who experience collections-based instruction still exhibit the smaller-from-larger bug if adequate attention is given to developing conceptual understanding and then connecting those experiences to symbolic representations?

Children still occasionally displayed the smaller-from-larger bug. Two boys from the linear classroom displayed it on the post assessment. However, when challenged to defend their solution, one immediately corrected the error. It is interesting to note, that children in both classrooms occasionally displayed this error during classroom dialog, and its incidence was higher during the last assessment than during the first two. This has led to the speculation that children with a firm conceptual understanding can begin to act in
procedural fashions. It seems plausible that when solving the problem has become routine, this bug will occasionally occur due to lack of attention.

6. Can *ten as an iterable unit* (Cobb & Wheatley, 1988) be established without the use of base-ten collections?

Table 8.1 presents the descriptive statistics for the development of ten as an iterable unit by class. Just under 32% of the children in each class developed ten as an iterable unit. Considering the fact that two-digit addition and subtraction was merely an exploration unit in the first grade curriculum, for nearly one in three children to be able to solve mentally two-digit addition and subtraction with regrouping without materials is an encouraging outcome. It seems likely that the numbers would be even greater if the proposed instructional sequence (discussed below) had been used. The proposed instructional sequence (see discussion below) combines the strongest features of both instructional sequences. For that reason, it seems logical to argue that children would have potentially made more growth if they had experienced the proposed sequence.

Based on the results of this study, it is possible for some children to develop ten as an iterable unit without using base-ten or collections materials. However, it takes longer and some children never developed ten as an iterable unit. This study found that the use of base-ten collections material does not permanently predispose an individual to use 10!0. Instead, 10!0 appeared to be a stage through which children progressed. Furthermore, there was indication that the use of collections contexts such as the candy factory supported the earlier development of ten as an iterable unit as indicated by the difference in performance between the two classes after the first instructional sequence.
7. Will the addition of mental calculation warm-up activities designed to build facility with number word sequences and partitions of ten eliminate some of the difficulties noted in previous studies such as difficulty developing ten as an iterable unit?

Initially, very few children in either the linear or the collections classroom demonstrated facility with the number word sequences needed to increment by ten from any number. This was assessed in the following manner. Children were told, “I am going to begin a number sequence. When I stop, continue the pattern until I tell you to stop. 4, 14, 24, 34.” If children could produce the number words needed to complete the sequence to 104, they were scored with facility. If they made no attempt, omitted a number in the sequence, or were unable to bridge the century, they were scored as not facile. Table 8.2 presents the descriptive statistics with respect to facility with forward number word sequences by tens.

<table>
<thead>
<tr>
<th>Class</th>
<th>Jan - Pre</th>
<th></th>
<th>June - Post</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Count</td>
<td>Percent</td>
<td>Count</td>
<td>Percent</td>
</tr>
<tr>
<td>Linear (n=22)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Not Facile</td>
<td>19</td>
<td>86.4%</td>
<td>7</td>
<td>31.8%</td>
</tr>
<tr>
<td>Facile</td>
<td>3</td>
<td>13.6%</td>
<td>15</td>
<td>68.2%</td>
</tr>
<tr>
<td>Collections (n=19)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Not Facile</td>
<td>18</td>
<td>94.7%</td>
<td>4</td>
<td>21.1%</td>
</tr>
<tr>
<td>Facile</td>
<td>1</td>
<td>5.3%</td>
<td>15</td>
<td>78.9%</td>
</tr>
</tbody>
</table>
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Children were also assessed for facility with combinations of ten. They were asked all the combinations in this manner, “What goes with 2 to make 10?” If the child answered immediately with the correct combination, they were coded as having facility with that particular combination. If they had to use a counting strategy, look at their fingers, or did not respond within a half second, they were coded as not demonstrating facility. In order to receive an overall coding of facility for combinations of ten, the child needed to demonstrate facility with each combination. Table 8.3 reports the descriptive statistics for facility with combinations of ten.

Table 8.3. Facility with Combinations of 10

<table>
<thead>
<tr>
<th>Class</th>
<th>Jan - Pre Count</th>
<th>Jan - Pre Percent</th>
<th>June - Post Count</th>
<th>June - Post Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear (n=22)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Not Facile</td>
<td>16</td>
<td>72.7%</td>
<td>8</td>
<td>36.4%</td>
</tr>
<tr>
<td>Facile</td>
<td>6</td>
<td>27.3%</td>
<td>14</td>
<td>63.6%</td>
</tr>
<tr>
<td>Collections (n=19)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Not Facile</td>
<td>18</td>
<td>94.7%</td>
<td>5</td>
<td>26.3%</td>
</tr>
<tr>
<td>Facile</td>
<td>1</td>
<td>5.3%</td>
<td>14</td>
<td>73.7%</td>
</tr>
</tbody>
</table>

Over the course of the teaching experiment, the majority of students in both classes developed facility with the combinations of ten and the number word sequences needed to increment by ten. It is reasonable to assume that the mental calculation activities supported this growth. This facility, particularly with respect to the number word sequences, in turn supported the use of N10 to solve two-digit addition. Facility with the number word sequences needed to decrement by ten was not assessed on the post assessment.

Saxton and Cakir (2006) recommended giving explicit instruction in the areas of “(1) counting-on, (2) trading, and (3) partitioning” (2006, p. 767) in order to promote procedural and conceptual knowledge in base-ten tasks. Those three skills all address
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building capacity with 1010 and the written standard algorithm. As others have suggested (Fuson, 1992; Wright, Martland, Stafford, & Stanger, 2006), this study found the ability to produce the number word sequences for incrementing and decrementing by ten was critical to engendering N10 for addition and subtraction. If proper instructional attention is given to developing facility with the number word sequences for incrementing and decrementing by ten from any number as in the Mental Calculation Activities in this study, students can progress from using 1010 to using N10 to choosing between the two strategies given the nature of the problem.

8. Will children who have demonstrated N10 in the context of materials or a contextual problem demonstrate 1010 when presented with bare two-digit addition and subtraction sentences?

When presented with bare number problems in a horizontal format, five of the children in the linear classroom and three children in the collections classroom used 1010 to attempt to solve the problem even though they had demonstrated N10 with materials (i.e., bead strings in the linear class and bundles of sticks in the collections class). There appears to be something in the nature of the bare number problems that engenders 1010, even when presented in a horizontal format.

9. Are there gender differences in the construction of specific strategies?

In the preliminary building of each multilevel model, gender was evaluated as a predictor variable for each of the different models. It was never a significant predictor of
strategy use. Neither were race nor socio-economic status a significant predictor of strategy usage.

10. Is it possible to use inquiry-based investigations while operating within the constraints of local curricular and pacing guidelines?

This study demonstrated that it is possible to use inquiry-based investigations while operating within the constraints of local curricular and pacing guidelines. See chapter six for details of the instructional sequences.

11. Can theory be developed to describe more accurately the range of strategies that emerge for solving two-digit addition and subtraction?

Theory for an amplified framework for early base-ten strategies was presented in chapter 7. This amplified framework allows for the complete the ten conceptual structure (see chapter 7 for details).

**Discussion of Research Design**

Several recommendations can be made to improve the design of the experiment. The quantitative analysis would have had more power if a complete battery of assessment interviews had been given as a pretest. This recommendation is made because of the three children who had already developed both 1010 and N10 by the time of the Interview 1. This created a ceiling effect. If data about two-digit addition and subtraction strategies were gathered before the teaching experiment began, these three children may not have already developed both strategies. Having a complete pre-assessment would have given
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another wave of data and would have reduced the standard error by increasing the variability of the measurement time. “A change as simple as adding another wave of data to your research design, far afield from the central set of observations, can reap dramatic improvements in the precision with which change can be measured” (Singer & Willett, 2003, p. 42). This was not possible in this study due to time constraints. However, any future study should plan for this feature.

Sensitivity of the Instrument

If the three children who remained static due to using both strategies during Interview 1 had been presented with two-digit addition and subtraction problems during the preassessment, their growth might have been documented. Quantitatively the nature of the subtle changes of their facility with the different strategies was not captured. However, qualitatively, there were changes that can be addressed. For example, one of the three children was using a very procedural level of understanding of 1010 (R1010) to solve the bare number problems during the first interview. When she initially solved the problems, she mentally used the traditional algorithm, solving from right to left and focusing on the digits rather than the value of those digits. For example, 3 in 32 would be treated as a 3 rather than 30. Overtime, her use of the 1010 strategy changed. She became much more flexible in its usage and was able to deviate her approach given the problem at hand.

Another area in which the assessment instrument lacked sensitivity was in its ability to detect growth in strategy that did not capitalize on ten as an iterable unit. For example, during the preassessment there were children who were unable to count-on to add. By the time of Interview 3 they were able to count-on to add. Therefore, they did experience growth in their mathematical ability. However, because their growth did not involve change in a base-tens strategy, it was not detectible by the coding scheme used to
code the solution strategies. Both counting from one to add two collections and counting-on to add by ones were given the same code because neither strategy took advantage of ten as a unit. Future studies might want instruments sensitive to change in strategy usage that does not exploit ten as a composite unit. However, that was outside the scope of this quantitative analysis. All children in the study grew in addition strategies. However, the measure was not designed to detect change in non-base-ten strategies.

**Measurement of Time**

Another issue related to the metric of time is the consideration of age as the measurement of time. This study was interested in the influence of particular units of study and so used each unit of study as the metric of time. However, age in months at the time of each assessment rather than clocking time by the units of study is an interesting possibility for investigation. This data would allow one to explore the research question, “Does age-related maturation play a part in predicting the readiness for developing a ten strategy?”

**Sample Size**

A larger sample may have been more sensitive to differences in 1010 usage between the two classes during the second administration and may have allowed for the inclusion of a variable to evaluate accuracy of solution methods (see chapter 4 for related discussion). How one handles attrition is a related issue that may bear scrutiny. In this study, children who withdrew before the conclusion of the study were eliminated from the study. However, with the multilevel models, children for whom the data are not complete can still be included in the model. This could have increased the overall sample size and may have enabled the models to be more sensitive to differences between the groups. However, since the main interest in this study was the development of the more
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sophisticated strategies which emerged later in the study, those individuals who moved before the end of the study would have been excluded from the analysis as were all individuals who were not initially counting-on to add and never developed a ten strategy. Had the individuals left later in the course of the study, this might not have been the case.

**Alpha Level**

The decision to select .05 as the level of significance is another consideration. The quantitative analysis reported in this study was an exploratory data analysis (Tukey, 1980; Velleman, 1998). That means that the intent of the analysis was to identify patterns in the data. By setting alpha at .05 with a relatively small sample size, the possibility exists that potentially real differences between the two classes could be overlooked due to the lack of power of the study. Because of this danger, I will comment on some of these potential differences that would have been considered significant if alpha had been set at .1 or if theory had predicted the direction of the difference so that a one-tailed test could have been used rather than a two-tailed test.

The ability to count-on in January would have been significantly associated with developing an advanced strategy for both classes and not just the collections class ($p = .10$). If, in fact, the ability to count-on is associated with later developing an advanced strategy, the inability to count-on might be an excellent indicator of need for early intervention. If this is the case, schools could easily screen all first grade children to determine which children are in need of intervention. This could be an important tool for practitioners in their attempts to devise a universal screening instrument. Therefore, it is recommended that future studies attend to the potential association between the ability to count-on in January of first grade with future development of more sophisticated multi-digit additive thinking.
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Another aspect of potential theoretical importance that might warrant further investigation is the fact that all four children who were initially counting-on but failed to develop an advanced strategy were in the linear class \((p = .08)\). Had the alpha been set at .1, this would have been considered significant. For that reason, it is inadvisable to conclude that there was no real underlying difference between the two groups based on this limited study. When this finding is combined with the findings from the qualitative analysis, it strengthens the notion that there may be something about the instructional sequence from the collections classroom that was more likely to engender the development of ten as a unit.

The final potential difference between the two classes is the ability to develop an advanced strategy. Initially the difference in predicted probability that children in the collections classroom would use an advanced strategy over children in the linear classroom had a significance level of \(p = .06\). Given the limitations of this study, it would be inadvisable to discount completely this potential difference. The proposed instructional sequence was designed with this potential difference in mind. Given the fact that this would have been a significant difference with a more generous alpha or if the hypothesis had proposed a direction for the difference, it is recommended that future studies explore this potential association.

**Measurement Error**

Any time one conducts a quantitative analysis, one must consider measurement error. In this study with regards to strategy usage, measurement error might not represent an inability to record accurately the observations of the strategy usage. It might represent, instead, an inability to measure the potential for strategy usage. Children develop preferences for particular strategies. They might, in fact, have more than one strategy with which they are capable of solving a given problem. These measurement tools might not
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reflect the differences of choice. Just because an individual has the ability to solve problems using more than one strategy does not mean they will not tend to favor one strategy over another even when the task is designed to elicit the non favored strategy. For example, a child might not demonstrate usage of both N10 and 1010 in final assessment because he simply finds 1010 more to his liking. In the evaluation of whether a child uses “Both,” strategy preference is a factor that needs to be addressed in future models. The routine use of the question, “Do you have another way to solve that?” might serve this purpose.

Unexpected Results

Previous research (Beishuizen, 1993; Beishuizen & Anghileri, 1998; Heirdsfield, 1995) has indicated that weaker children will tend to use 1010 while more adept children gravitate to N10. The findings from this study indicate instead that preference for 1010 could be a phase through which children progress rather than a static strategy preference for weaker children. Furthermore, in the linear classroom, less than half of the children displayed 1010 at all and of those children, most of them were children who developed 1010 in addition to N10. It is reasonable to assume that students become stronger in their computation ability over time. This would be consistent with prior research that suggests that weaker students tend to utilize 1010 over N10. The findings of this study indicate that as time progresses, students will be more likely to use N10 (see chapter 5). Since the interstate driving instructional sequence was at the end of the experiment, the students would be expected to be stronger in their ability to calculate. This suggests they would be more likely to use N10 when experiencing instruction designed to engender it merely because they are stronger calculators.

It should be noted that Blöte, Klein, and Beishuizen (2000) found the opposite to be true. In their study there was a change ($p < .001$) in N10 usage in solving addition
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problems over time. However, students were less likely to use N10 later in the year. This change corresponded with the introduction of 1010 by the teachers later in the year. These findings, in conjunction with the finding of this study, would seem to indicate that student strategy usage is closely related to the instructional sequences that the children experience.

The linear classroom experienced the art gallery and the tape storage problems, both of which were designed to engender N10. The collections classroom experienced the inventory problem and the candy factory scenario, both of which were designed to strengthen conceptions of ten as a unit and may predispose children to 1010. Both classrooms subsequently experienced the interstate driving scenario which was designed to engender N10. If one observes the change over time in N10 usage, there were differences between the classes in the way in which the N10 strategy developed. However, the differences were not statistically significant. If the sample had been larger, the test may have been more sensitive to differences between the classes. Since there is not a significant difference between the two classes with respect to N10 usage during the first two assessments, it would be reasonable, therefore, to assume that both classes would benefit from the interstate driving instruction and thereby increase the likelihood of N10 use.

Cobb, Gravemeijer, Yackel, McClain, and Whitenack (1997) found that once students developed 1010 they continued to use that strategy even when subsequent instructional sequences were introduced that were designed to engender N10 strategies. This was not the case in this study. There was a substantial drop in the estimated probability of the children in collections-based class using 1010. This drop corresponds to the time frame in which the children were experiencing the interstate driving context which was designed to elicit N10 strategies (see chapter 6 for details of the instructional
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sequence). Cobb et al. (1997) also reported a lack of success with combining collections with the empty number line. This study successfully combined collections with the empty number line and counting on the hundred chart. The collections class routinely used collections while incrementing and decrementing by tens from off-decuple numbers.

One possible explanation for the differences between the results of these studies is the attention that was given in this study to building capacity with forward and backward number word sequences for incrementing and decrementing on and off the decuple numbers as was recommended by Cobb and Wheatley (1988). If children do not have facility with those number word sequences, they will not use N10 or what Cobb et al. (1997) labeled counting-based strategies. Another difference between the two studies is the intentional use of collections to support the incrementing and decrementing sequences. In the Mental Calculation Activities in the collections classroom bundles of sticks, Unifix trains, and dimes (10 cent coins) and pennies (1 cent coins) were used in conjunction with hundred charts and empty number lines to help children conceptualize the sequences. For example, three individual sticks were displayed. Children placed a chip on 3 on the hundred chart. Next a bundle of ten sticks was added to the collection. The children identified the total number of sticks as 13, and a chip was added to the 13 on the hundred chart. The children continued to track the growing sum on the hundred chart up to 93. Next one bundle at a time was removed from the collection and the corresponding chip was removed from the hundred chart. This procedure continued until only three sticks remained. This type of activity was conducted as a warm-up before the investigation was introduced for that day. In this way, nearly every child developed the ability to produce all the number word sequences needed in order to use N10 to add and subtract two-digit numbers.
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Another potential explanation is the unexpected results point to a potential methodological weakness in the retrospective meta-analysis of identifying taken-as-shared understandings. In the methodology proposed by Cobb (2000a), no measure was taken at the individual level. Individual constructions of mathematical concepts were conjectured based on the classroom dialog that happens at a communal level. This does not take into account the possibility that one dominant student could potentially drive the dialog in a direction that is not representative of all the members of the community. It also does not account for students who were inhibited from contributing to the communal dialog for whatever reason. Therefore, it is recommended that attention to individual performance be a part of the design for classroom teaching experiments. In this way, the researcher is not forced merely to conjecture what is happening on the individual level. With this proposed methodological adjustment, there are data to support or refute such conjectures.

Collections Exposure in the Linear Class

The case could be argued that the linear class had a collections exposure when they experienced the district-generated unit on money since each different coin represents a collection of cents. If one follows this logic then one would have expected an increase in collections-based strategies after that exposure. Interview 2 was administered from April 20 – 26. Both classes began the money unit on April 9. The children in the linear class displayed their greatest use of 1010 during assessment 2. Children in both classes also increased ($p < 0.001$) their use of N10 since interview 1. Even though coins are not a proportional collections model, it could be argued that the use of the coins may have helped the linear classroom develop ten as an iterable unit. The study would have been greatly improved if interview 2 had been conducted before the money unit. An additional data collection could have been added after the money unit in order to analyze its effect
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on the development of ten as an iterable unit. With the design of this study, one can only speculate about the influence of the money unit on the linear classroom.

Other Strategies Used

Pathological Splitters

One of the major questions that inspired this study was what factors influence children to become pathological splitters? It was assumed that if children developed a conceptual understanding of the operations they would not make this type of senseless error. However, two children from the linear classroom correctly used both strategies during interview 2, but displayed pathological splitting with subtraction with regrouping on interview 3. In cases of pathological splitting with 1010, perhaps the child who previously demonstrated facility with both strategies has become cognitively “lazy.” Their solution strategies had moved from problem-solving to procedural application of strategies. When this happens, they may fall into the trap of mindlessly applying a procedure. When challenged to prove their solutions, they generally caught the error. Kamii and Dominick (1998) noted the danger of teaching the traditional algorithm. However, this study would seem to indicate that there is a danger of conceptual algorithms becoming mindless procedures as well. Sandrini, Miozzo, Cortelli and Cappa (2003) in a study of a brain damaged adult found that a brain damaged individual may maintain conceptual understanding but develop the smaller-from-larger procedural bug after experiencing brain injury. Their findings provide further evidence that procedural and conceptual knowledge can function somewhat separately.

A10 Strategy

Initially, this study was designed to analyze the emergence of A10 (Beishuizen, 1997; Rousham, 2003) or jump-via-ten (Treffers & Buys, 2001) strategy. Specific lessons
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were designed to engender this strategy. However, A10 as a strategy choice did not emerge as a taken-as-shared mathematical practice during the discourse in either class and A10 usage during assessment was extremely rare. This is consistent with the findings of Blöte, Klein, and Beishuizen (2000). Because of the rarity of usage, when a child used A10 during an assessment, it was coded with N10 because the procedure more closely resembled N10 than 1010 in that the first addend was kept in its entirety. The second addend was then decomposed in such a way to reach the decuple first. The remaining tens were then added to the decuple number. Finally, any remaining ones were added. For example, with $48 + 27$ the 7 from 27 would be decomposed into 2 and 5. The 2 would then be added to 48 to make 50. The 20 would then be added to the 50 to make 70. Finally the remaining 5 would be added. An alternative explanation to this procedure is that 27 could be decomposed into 2 and 25. The 2 would then be added to the 48 to form 50. Then 25 would then be further decomposed into 20 and 5. The 20 would be added to the 50 to make 70. Finally, the remaining 5 would be added for the sum of 75.

One can only speculate about why A10 was not used by more students. A lack of facility with the ability to decompose all numbers less than 10 may contribute to the difficulty in using this strategy. For example, in order to solve the problem above, one must know that 2 more are needed to make a ten. Furthermore, one must be able to decompose 7 into 2 and the remaining amount. If one cannot flexibly decompose 7 into all the possible whole number combinations, one cannot use A10. While most of the children in this study demonstrated facility with the combinations of 10, they did not demonstrate the same level of facility with all the sums less than 10.

Another possible explanation is that the children were so facile with incrementing and decrementing by 10 from any number that they did not see the advantage in getting to the decuple number. Since 20 can easily be added to 48, why would one need to get to
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It is interesting to note that many children who used N10 to solve $48 + 27$ eventually decomposed 7 into 5 and 2. However, they did so after the 20 had been added to the 48. For this reason, it is likely that a lack of facility in decomposing numbers less than 10 was not the reason children did not use A10. It seems more probable that the children did not perceive any advantage in A10 over N10. A future study many want to make a comparison of the two strategies an explicit topic of instruction in which the children evaluate the relative efficiency of the two strategies.

**Complete the Ten**

Further study of the complete the ten construct is needed. It has been hypothesized (see chapter 7) that for children operating within the complete the ten construct, it appears that the statements “add another ten” and “add ten more” mean qualitatively different things. “Add another ten” seems to mean “place another of those things we call tens beside the collection.” “Add ten more” seems to mean “add enough more until you make the left over ones into a complete ten.” A recommendation for modification of the structured interview is the addition of a dynamic branch that would be used if the child responds to the base-ten tasks with solutions consistent with the complete the ten construct. The tasks would ask the child to add specific amounts to and identify the total of a growing collection. The following series of tasks serve as an example of a string of tasks that might assess this construct. 1) The child would be asked to use the proportional base-ten materials to construct the number 24. 2) The child would then be asked to “add ten more.” If the child is operating within the complete a ten construct, we would expect him or her to add six units to complete the ten from the four. 3) The child would then be asked to add 2 more (i.e., singles) and identify the new sum (e.g. 32 from the complete the ten construct reckoning). 4) The children would then be asked to “add another ten.” From the hypothesis presented here, the child would be expected to add another collection
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of ten. It is further recommended that more than one series of tasks be piloted with children acting within the complete the ten construct. In the other series of tasks, the order of the tasks would be changed so that the child was first asked to add another ten and then asked to add ten more. It may be that the language of one task would provide a scaffold for the other task.

Need for a Preponderance of Evidence

Having just discussed the complete the ten construct as an example of how children operate from within their understandings in seemingly logical fashions, it might be prudent to offer a word of caution about using isolated episodes to judge children’s constructions about number. This hypothesis about the complete the ten construct was not based on a single isolated task. Rather, it emerged from repeated observations of the same or very similar behaviors from different children in both classrooms. It was evident with seven children in the collections class and four children in the linear class during at least one of the structured interview assessments. It was also noted during teaching episodes in both classrooms. For this reason, it seems a viable explanation or model for the thinking of those children in that phase of their development.

However, children will at times act in a manner that is not characteristic of their general mode of operation. Using a single isolated observation as a ‘proof text’ if you will, can lead to erroneous models of their thinking. For example, both “Cole” and “Kendrick” were extremely able children. They both demonstrated the ability to use both 1010 and N10 with facility in both the March and April assessments. However, they both made the smaller-from-larger bug “error” during the June assessment. If their performance on the June assessment on that single problem was viewed in isolation, it might lead one to believe that they had invented immature procedures for solving subtraction that did not account for the need to regroup as has been documented by
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numerous researchers (see Resnick, 1980 for a detailed discussion of the history of the research). However, from other assessments, from classroom dialog, and from the responses received when they were asked questions about their procedures, it seems instead, that they both mindlessly applied a procedure that had become so comfortable as to no longer require much thought. In doing so, they failed to think deeply about the meaning of the numbers. Rather, they used what Cobb and Wheatley (1988) called an abstract singleton strategy and Fuson labeled a strategy based on a concatenated single digit conceptual structure (Fuson, 1990a, 1992; Fuson et al., 1997) (see chapter 3 for details). Rather striking here, is the fact that this occurred after they had developed a profound understanding of both N10 and 1010 and not before. When confronted with the illogic of their responses, they were able to activate their iterable unit concept of ten (Cobb & Wheatley, 1988; Steffe, 2004) to amend their procedure and arrive at a more sensible (from the iterable unit concept) response.

These two examples highlight the need for teachers to confront even young children with the logic, or lack therein, of their responses. As demonstrated by this study, even children as young as six years of age can develop the metacognitive habit of explaining, and thus evaluating, their thinking. Although a formal proof was not expected, I frequently challenged children to “prove it” when they had posited a particular solution to a problem. This was done in instances in which their solutions might have been correct as well as on occasions of incorrect solutions. This required the child to be ready to explain his or her solution and allowed others to challenge solutions that seemed illogical to them. In this way, the norm of only offering considered responses (i.e., not wild guesses), the norm of routinely explaining strategies, and the norm of attempting to understand the solutions of others were established.
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**Recommendation for Early Intervention**

Several studies have found benefit in early intervention (Gersten, Jordan, & Flojo, 2005; Lindjord, 2002; MacLean, 2003; Ramey & Ramey, 2004). The fact that so many of the children who were not initially counting-on to add did not develop any strategy that capitalized on the ten structure of numbers for solving two-digit addition and subtraction may be evidence of the need for intervention to be offered early in first grade. The teaching experiment began in January. At that time, children who were not yet counting-on to solve simple addition problems were at a much greater risk for not developing any sophisticated strategy that uses ten as a unit to solve two-digit addition and subtraction problems. If data were collected at the end of second grade, it is likely that more children would have developed an advanced strategy based on the iterable unit of ten. Future studies would be strengthened by allowing for a follow-up assessment.

Nevertheless, the fact remains that the classroom instruction did not enable all children to develop an advanced strategy. Algebra is often labeled as a gateway for mathematics as demonstrated in future success in post secondary education (Atnada, 1999; Bond, 2003; Ellington, 2005; McCoy, 2005; Stacey & MacGregor, 1999). The same could be said for counting-on being a gateway for success in multi-digit addition and subtraction. This study indicates the instructional importance of giving children plenty of opportunities to develop facility with counting-on to add before proceeding with multi-digit addition and subtraction. However, one would be remiss not to mention that the ability to count-on to add can also be taught in a very procedural manner. Simply mimicking the strategies of others to count-on to add does not appear to be sufficient. In this study there were individuals who were able to count-on to add, but were not about to count-up-to in order to solve missing addend tasks. This indicates the importance of constructivist instruction. Children need opportunities to construct a meaningful
Discussion, Recommendations & Conclusions

understanding of addition from which the strategy of counting-on to add is a natural development. It is worth noting the fact that during the course of the experiment, all but four of the seventeen children who were not initially counting-on to add did develop this strategy. However, even though they made growth from their pre-assessment status, they continued to lag behind their peers. This would seem to indicate that intervention was needed during the first semester of first grade in order to close the gap. Of the four children who did not develop counting-on to add, they were evenly split between the two classes. The two from the linear class has IEPs in mathematics. The two from the non special education inclusion class were later identified with learning disabilities.

Discussion of a Proposed Instructional Sequence

The results of this study indicate that the collection classroom instructional sequences may be better in helping develop ten as a unit and thus build capacity for the use of 1010 and N10. The inventory and candy factory sequences may have engendered ten as a composite unit more effectively than the art gallery and tape measurement instructional sequences. Of those initially counting-on to add, there were four children in the linear classroom who did not develop an advanced strategy that uses ten to solve two-digit addition and subtraction. Because of these four individuals, there was not a significant difference in the development of N10. If these individuals had initially developed ten as a composite unit, they in all likelihood would have subsequently developed either 1010 or N10.

In the discussion that follows, the proposed instructional sequence is intended to combine the most effective segments of each of the actual realized instructional sequences. In the view of this researcher, this sequence is most likely to advance first graders’ number knowledge. Thus, the proposed instructional sequence (see Table 8.4) might initially contain contexts that involve collections such as the inventory and candy
factory scenarios as well as money. During those instructional sequences, however, attention should be given to developing the forward and backward number word sequences for incrementing and decrementing by ten both on and off the decuple. The Mental Computation Activities described in this study appear to be effective in building capacity with those sequences. A collection context should then be followed by a measurement context that capitalizes on ten such as the tape storage context. This would allow for the introduction of the empty number line. Finally, a linear context such as the interstate driving context that allows for both addition and subtraction would enable mathematizing away from the empty number line context. Chapter 6 gives detailed descriptions of all of these instructional sequences. Table 8.4 lists the instructional sequences in the order they would be presented in the proposed instructional sequence.

<table>
<thead>
<tr>
<th>Table 8.4. Proposed Instructional Sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Order of Instructional Contexts</strong></td>
</tr>
<tr>
<td>Collections-based Contexts</td>
</tr>
<tr>
<td>(With mental calculation activities embedded to build capacity with number word sequences for incrementing and decrementing by ten)</td>
</tr>
<tr>
<td>Inventory</td>
</tr>
<tr>
<td>Candy Factory</td>
</tr>
<tr>
<td>Money (Dimes and Pennies in American Coins)</td>
</tr>
<tr>
<td>Measurement-based Contexts</td>
</tr>
<tr>
<td>Art Gallery</td>
</tr>
<tr>
<td>Combined Tape Storage</td>
</tr>
<tr>
<td>Interstate Driving</td>
</tr>
</tbody>
</table>

Ultimately, the goal should be for children to choose flexibly from multiple strategies depending on the nature of the problem. Rittle-Johnson and Star (2007) conducted a randomized, pretest-intervention-posttest design experiment among highly able, middle school algebra students to evaluate the benefit of students comparing solution strategies from hypothetical students. The control group students analyzed the same solutions presented to the compare treatment but in a sequential order. They found, "[c]omparing and contrasting alternative solution methods led to greater gains in
Discussion, Recommendations & Conclusions

procedural knowledge and flexibility, and comparable gains in conceptual knowledge, compared to studying multiple methods sequentially” (2007, p. 571). The same principles can be used with lower socio-economic, ethnically-diverse, first grade children. They can compare their solutions that emerge in the contextual investigations during whole group discussions. Gravemeijer mentions student-initiated discussions of efficiency as evidence of “vertical power” (1998, p. 290). The notion of relative efficiency and sophistication of strategies should eventually be an explicit topic of discussion so that children begin to value elegant, efficient solutions. In a first grade classroom, efficiency of strategies might be defined in terms of which procedure was ‘easier’ or ‘had less jumps.’ Yackel and Cobb identified “mathematical sophistication” (1996, p. 461) as one of the sociomathematical norms that emerged in a first grade classroom. Although the teacher in the study never explicitly discussed the relative sophistication or efficiency of strategies, the notion was established implicitly though the reactions of the teacher to children’s explanations. The children also used the term ‘easier’ when explaining why they valued one strategy over another. Several studies have established the sociomathematical norms of sophisticated mathematical solutions and efficient solutions (Cobb, 1995b, 2000a, 2001; Gravemeijer & Cobb, 2006; Gravemeijer, Cobb, Bowers, & Whitenack, 2000; McClain & Cobb, 2001; McClain, Cobb, & Bowers, 1998). Others have specifically investigated the notion of efficiency of approach (Bednarz & Janvier, 1988; Heirdsfield, 1999; Lee, 1999; Threlfall, 2002; Torbeyns, Verschaffel, & Ghesquiere, 2002). In this study, the relative efficiency and sophistication of strategies, operationalized as ‘easier,’ became an explicit topic of discussion. The teacher frequently posed the questions “Which one is easier? Why?” during whole-group comparison of solutions strategies.
Recommendations for the Amplified Framework

A question for future research would be the possible ramifications for the amplified framework proposed earlier (see chapter 7) if the proposed instructional sequence were implemented. Would instruction that intentionally mixed collections and measurement contexts influence the manner in which strategies emerged? Another question with respect to the framework is the role of A10 in strategy development. What instructional sequences would encourage A10? Furthermore, is engendering A10 an important educational goal?

Conclusions and Summary of Key Recommendations

This study led to several major recommendations. The quantitative aspect of the study could be strengthened in the following ways. 1) The quantitative analysis would have had more power with the addition of another wave of assessment data collection. 2) A larger sample may have increased the level of sensitivity of the statistical tests and may have allowed for more strategy codes to be used in the analysis. This would have allowed the analysis of growth in strategies that did not use ten as a unit.

The design research aspect of the study also gave rise to several recommendations. 1) Careful attention should be given to developing facility with the number word sequences for incrementing and decrementing by ten from any number. 2) The improved instructional sequence should begin with collections contexts such as the candy factory until students demonstrate an ability to use 1010 to add and subtract. The instructional sequence should then move to a linear context such as the interstate driving context to engender N10 for addition and subtraction. Efficiency of strategies given the particular problem should then become an explicit topic of class dialog. 3) The amplified framework for early base-ten strategies should be reevaluated in light of the strategies that emerge from such an proposed instructional sequence as presented above. Additional
Discussion, Recommendations & Conclusions

study is needed with regard to the complete the ten construct. 4) Failure to develop counting-on to add by ones may provide an excellent indicator of the need for early intervention which should be provided before the introduction of contexts such as the candy factory. 5) Further study is recommended on the influence of other languages, particularly those that emphasize the structure of ten such as East Asian languages, on the development of base-ten strategies.

As evidenced by this study, a solid conceptual foundation with 1010 and N10 does not prevent the occurrence of buggy procedural algorithms such as the smaller-from-larger bug. It appears likely that even non-standard algorithms can become routine and thus are subject to buggy errors when students mindlessly apply those procedures. However, when confronted with a request to explain their buggy procedure, children with a firm conceptual foundation will frequently amend their procedure. This conclusion is consistent with the findings of Sandrini, Miozzo, Cortelli and Cappa (2003) who, based on the study of an individual with acquired brain damage, proposed that it is possible for procedural and conceptual knowledge to be disassociated from each other with respect to subtraction and the smaller-from-larger bug.

This study found that it is possible to use contextual, inquiry-based investigations and still fulfill local mandates for curriculum indicators and pacing. These investigations can be differentiated to meet the needs of special needs students. Socioeconomic status and gender were not good predictors of success in this educational environment as neither variable improved the estimates when building any of the multilevel models. Therefore, it seems plausible that children of either gender without respect of socioeconomic status benefited equally from these instructional sequences. Furthermore, when children were successful with these types of instructional sequences, there was a strong predicted probability that this success would translate into success on district-mandated
Discussion, Recommendations & Conclusions

assessments. Therefore, it seems likely that inquiry-based contextual investigations are a viable option for typical American primary classrooms.
References


References


References


References


References


References


References


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References


References


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References


References


References


References


References


Sfard, A. (2001). There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning. *Educational Studies in Mathematics, 46*(1-3), 13-57.


References


References


References


References


References


References


Appendix A: Assessment Interview Schedules

March Collections Schedule

Mar Assessment Schedule for Collections Classroom

<table>
<thead>
<tr>
<th>Name</th>
<th>Date</th>
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</table>

(1a) 38 + 24 =
If Successful:
(1b) 56 - 23 =
(1c) 43 - 15 =

If Unsuccessful:

(2) "What number goes with 9 to make 10?" (If necessary, "What number do you add to 9 to equal 10?"

The task will be repeated with "What number goes with ____?"

2 5 7 4 1 6 8 3

(3) Number Word Sequences:

(a) Say: "I am going to begin a growing number pattern and I want you to continue until I tell you to stop. 10, 20, 30."

(120)

If needed prompt with, "What is next?"

(b) "I am going to begin another pattern and I want you to continue until I tell you to stop. 4, 14, 24, 34." (to 124)

March Collections Assessment

<table>
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<th>Name</th>
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</tbody>
</table>

(4) bundles and sticks additive tasks:

(a) Ask, "What can you tell me about these?" If the child does not mention ten of each bundle, prompt with, "How many are here?" while pointing to a bundle.

(b) Covered Task One: Add sticks and bundles, one section at a time and ask, "What do we have there?" If the child does not understand prompt with "How many are there now?" (After sum is identified, slide that portion into the tube before adding the next group.

(5) If Unsuccessful on (d):

(1) Say, "Use the bundles and sticks to build:" 20 32

(2) Build and say, "What is this?" 41 19 53 (Place ones to the left of the tens for this one only.)

Page 2 of 3
Appended A

March Linear Assessment Schedule

March Assessment Schedule for Linear Classroom

Name ___________________________ Date ____________

(1a) 38 + 24 =

If Successful:

(b) 56 - 35 =

(3a) 43 - 15 =

If Unsuccessful:

(2) “What number goes with 9 to make 10?” (If necessary, “What number do you add to 9 to equal 10?)

The task will be repeated with “What number goes with ______?”

2 5 7 4 1 6 8 3

(c) Number Word Sequence:

(a) Say: “I am going to begin a growing number pattern and I want you to continue until I tell you to stop. 10, 20, 30.”

(0 - 129)

If needed prompt with “What is next?”

(b) “I am going to begin another pattern and I want you to continue until I tell you to stop. 4, 16, 24, 32.” (0 - 124)
Appendix A

March Linear Assessment Schedule

Name:

(4) Bead string additive task:

(a) Ask, “What do you notice about the bead string?” If the child does not mention ten of each color alternation, prompt with, “How many red ones are here?” while pointing to a group of ten red beads in the middle of the string. If necessary, prompt with “How many blue ones are here?” while indicating a group of ten blue beads.

(b1) Present ten-catcher and ask, “How many beads will it hold?”

(b2) ‘Catch’ and slide over 10 beads at a time, asking, “Now what do we have?” each time (10, 20, 30 …)

(b3) Slide 3 over. Then ‘catch’ 10 each time, asking, “Now what do we have?” each time (3, 13, 23, 33…)

(c) Covered Task One: Slide over the beads, one section at a time and ask, “What do we have there?” If the child does not understand, prompt with “How many are there now?” After sum is identified, slide that portion into the tube before sliding the next group.

(d) Covered Bead String Task Two:

(5) Horizontal Sentences:

“Do you have a way now to figure these out?”

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
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</thead>
<tbody>
<tr>
<td>(a)</td>
<td>16 + 10 =</td>
<td>(b)</td>
</tr>
<tr>
<td></td>
<td>so what is …? 16 + 9</td>
<td></td>
</tr>
<tr>
<td>(d)</td>
<td>39 + 53 =</td>
<td>(e)</td>
</tr>
</tbody>
</table>
April Collections Assessment Schedule

Appendix A

April Collections Assessment Schedule

Name _____________________________ Date __________

(1a)  49 + 23 = 
If Successful:
(1b)  68 - 25 = 
(1c)  32 - 14 =

If Unsuccessful:

(2) “What number goes with 2 to make 10?” (If necessary, “What number do you add to 9 to equal 10?”)
The task will be repeated with “What number goes with _____?”

<table>
<thead>
<tr>
<th>2</th>
<th>5</th>
<th>7</th>
<th>4</th>
<th>1</th>
<th>6</th>
<th>8</th>
<th>3</th>
</tr>
</thead>
</table>

(3) Number Word Sequences:

(a) Say: “I am going to begin a growing number pattern and I want you to continue until I tell you to stop. 10, 20, 30.” (to 120)

If needed prompt with, “What is next?”

(b) “I am going to begin another pattern and I want you to continue until I tell you to stop. 6, 16, 26, 36.” (to 126)

Page 1 of 3

April Collections Assessment

Name _____________________________

(4) Bundles and Sticks Additive Tasks:

(a) Ask, “What can you tell me about these?” If the child does not mention ten of each bundle, prompt with, “How many are here?” while pointing to a bundle.

(b) Covered Task One: Add sticks and bundles, one section at a time and ask, “How many are there now?” The child is identified, slide that portion into the tube before adding the next group.

(c) If Unsuccessful on (d):

(1) Say, “Use the bundles and sticks to build!”
(2) Build and say, “What is this?”

Page 2 of 3
April Linear Assessment Schedule

Appendix A

April Assessments Schedule for Linear Classroom

Name __________________________ Date ______________________

(1a) 49 + 23 =

If Successful:

(1b) 68 – 25 =

(1c) 52 – 14 =

If Unsuccessful:

(2) “What number goes with 9 to make 10?” (If necessary, “What number do you add to 9 to equal 10?”) The task will be repeated with “What number goes with ___?”

2 5 7 4 1 6 8 3

(3) Number Word Sequences:

(a) Say: “I am going to begin a growing number pattern and I want you to continue until I tell you to stop. 10, 20, 30.” (to 120)

If needed prompt with, “What is next?”

(b) “I am going to begin another pattern and I want you to continue until I tell you to stop. 6, 16, 26, 36.” (to 126)
April Linear Assessment Schedule

Name: ________________________________

(4) Bead string additive tasks:

(a) Ask, “What do you notice about the bead string?” If the child does not mention ten of each color alternation, prompt with “How many red ones are here?” while pointing to a group of ten red beads in the middle of the string. If necessary, prompt with “How many blue ones are here?” while indicating a group of ten blue beads.

(b1) Present ten-catcher and ask, “How many beads will it hold?”
(b2) Catch and slide over 10 beads at a time, asking, “Now what do we have?” each time (10, 20, 30 …)
(b3) Slide 3-over. Then catch 10 each time, asking, “Now what do we have?” each time (3, 13, 23, 33…)

(c) Covered Task One: Slide over the beads, one section at a time and ask, “What do you have there?” If the child does not understand prompt with “How many are there now?” After sum is identified, slide that portion into the tube before sliding the next group.

(c) If unsuccessful on (b3):
(1) Say, “Use the bead string to build.”
(2) Build and say, “What is this?”

<table>
<thead>
<tr>
<th>10</th>
<th>14</th>
<th>20</th>
<th>24</th>
<th>30</th>
<th>34</th>
<th>40</th>
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</table>

Page 2 of 3

(d) Covered Bead String Task Two:

(3) Horizontal Sentences:
‘Do you have a way to figure these out?’

<table>
<thead>
<tr>
<th>(a) 16 + 10 =</th>
<th>(b) 42 + 23 =</th>
<th>(c) 38 + 24 =</th>
</tr>
</thead>
<tbody>
<tr>
<td>So what is ...? 16 + 9</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(d) 39 + 53 =</th>
<th>(e) 56 + 23 =</th>
<th>(f) 43 + 15 =</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(6) Contextual Problem: “Let’s say that you and your friend visited the prize box and each chose stickers. The boy you chose had 27 stickers and your friend got 42 stickers. How many stickers would your friend need to put back in order for it to be fair?”

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Appendix A

June Collections Assessment Schedule

June Assessment Schedule for Collections Classroom

<table>
<thead>
<tr>
<th>Name</th>
<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(1a) 38 + 24 =
If successful:
(1b) 56 – 23 =
(1c) 43 – 15 =

If unsuccessful:

(2) “What number goes with 2 to make 10?” (If necessary, “What number do you add to 9 to equal 10?”)
The task will be repeated with “What number goes with _______?”

2 5 7 4 1 6 8 3

(3) Number Word Sequences:

(a) Say: “I am going to begin a growing number pattern and I want you to continue until I tell you to stop, 10, 20, 30.”

(30 to 120)
If needed prompt with, “What is next?”

(b) “I am going to begin another pattern and I want you to continue until I tell you to stop, 4, 14, 24, 34.” (to 124)

Page 1 of 3

June Collections Assessment Schedule

Name

(4) Bundles and sticks additive tasks:

(a) Ask, “What can you tell me about these?” If the child does not mention ten of each bundle, prompt with, “How many are here?” while pointing to a bundle.

(b) Covered Task One: Add sticks and bundles, one section at a time and ask, “What do we have there?” If the child does not understand prompt with “How many are there now?” After sum is identified, slide that portion into the tube before adding the next group.

(c) If unsuccessful on 4b:
(1) Say, “Use the bundles and sticks to build!” 20 32
(2) Build and say, “What is this?” 41 19
(5) (Place ones to the left of the tens for this one only.)

Page 2 of 3
Appendix A

June Linear Assessment Schedule

Name:

(d) Covered Bundles and Sticks Task Two:

(e) Horizontal Sentences:

*Do you have a way now to figure these out?

<p>| | |</p>
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>(a)</td>
<td>16 + 10 =</td>
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<td></td>
<td>so what is ...? 16 + 9</td>
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<td>(b)</td>
<td>42 + 23 =</td>
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</tr>
<tr>
<td>(f)</td>
<td>43 - 15 =</td>
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</tbody>
</table>

Page 3 of 3

June Linear Assessment Schedule

Post Assessment Schedule for Linear Classroom

Name _______________ Date _______________ June 2005

(1a) 38 + 24 =
   If Successful:
   (b) 56 - 23 =
   (c) 43 - 13 =

If Unsuccessful:

(2) “What number goes with 2 to make 10?” (If necessary, “What number do you add to 9 to equal 10?”
   The task will be repeated with “What number goes with 3/.”
   2 5 7 1 6 8 3

(3) Number Word Sequences:

(a) Say: “I am going to begin a growing number pattern and I want you to continue until I tell you to stop. 4, 14, 24, 34.”
   (to 124)
   If needed prompt with “What is next?”
(b) “I am going to begin another pattern and I want you to continue until I tell you to stop. 3, 13, 23, 33.” (to 134)

Page 1 of 3
Appendix A

June Linear Assessment Schedule

Name________________________

(4) Bead string additive tasks:

(a) Ask, “What do you notice about the bead string?” If the child does not mention ten of each color alteration, prompt with, “How many red ones are here?” while pointing to a group of ten red beads in the middle of the string. If necessary, prompt with “How many blue ones are here?” while indicating a group of ten blue beads.

(b1) Present ten-catcher and ask, “How many beads will it hold?”

(b2) “Catch” and slide over 10 beads at a time, asking, “Now what do we have?” each time (10, 20, 30 …)

(b3) Slide 3 over. Then “catch” 10 each time, asking, “Now what do we have?” each time (3, 13, 23, 33…)

(c) Covered Task One: Slide over the beads, one section at a time and ask, “What do we have there?” If the child does not understand prompt with “How many are there now?” After sum is identified, slide that portion into the tube before sliding the next group.

(4d) Covered Bead String Task Two:

(e) If unsuccessful on 4d:

(1) Say, “Use the bead string and ten catcher to build.”

(2) Build and say, “What is this?”

(3) Horizontal Sentences:

“Do you have a way now to figure these out?”

(a) 16 + 10 = 26

(b) 42 + 23 = 65

(c) 38 + 24 = 62

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Appendix B
Support Materials for Instruction
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Art Gallery Record Sheet 1

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<td>Pink</td>
<td></td>
</tr>
<tr>
<td>Gray</td>
<td></td>
</tr>
<tr>
<td>Green</td>
<td></td>
</tr>
<tr>
<td>White</td>
<td></td>
</tr>
<tr>
<td>Yellow</td>
<td></td>
</tr>
<tr>
<td>Light Blue</td>
<td></td>
</tr>
<tr>
<td>News Print</td>
<td></td>
</tr>
</tbody>
</table>
Numeral Tile Draw with Arrow Cards

Draw a numeral tile from the bag. Read the numeral aloud. Build the numeral with the arrow cards to verify that the numeral was read correctly. To verify, “expand” the arrow numeral to read each part. Put it back together to form the whole number. Repeat the process several times.

Parents: Please do this activity for about 10 minutes. Sign below after the activity has been completed.

__________________________________________________
Build the Number with Abacus

Draw a number tile. “Build” that number on your AL abacus. If you have difficulty reading the number, build it with the arrow cards to help you. Have your partner check your work. See the example below.
Build the Number with Bead String

Draw a number tile. “Build” that number on your bead string. If you have difficulty reading the number, build it with the arrow cards to help you. Have your partner check your work. See the example below.

This shows 43 built on a bead string with groups of 5.

This shows 43 built on a bead string with groups of 10.
Jack Rabbit Jump Spinners

+ 10

10 Less

10 More

-10

46

91 71

53 23

86

+ 10

10 Less

10 More

-10

46

91 71

53 23

86
Directions:
The object of the game is to jump your rabbit from cart to cart.
- Place your markers on start.
- Spin both spinners
- If your answer is in the first cart, jump to that cart.
- If your answer is not in that cart, stay put.
- Take turns trying to jump to the next cart.
- You may not move to the next cart if your answer is not in the cart.
- The first person to reach the last cart wins!
**20 minus**

<table>
<thead>
<tr>
<th>12</th>
<th>14</th>
<th>15</th>
<th>13</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>19</td>
<td>20</td>
<td>18</td>
<td>17</td>
<td>16</td>
</tr>
<tr>
<td>16</td>
<td>11</td>
<td>13</td>
<td>15</td>
<td>17</td>
</tr>
<tr>
<td>20</td>
<td>19</td>
<td>18</td>
<td>14</td>
<td>16</td>
</tr>
<tr>
<td>13</td>
<td>12</td>
<td>14</td>
<td>15</td>
<td>18</td>
</tr>
</tbody>
</table>

**Directions:**
1. Roll the 0-9 number generator or spin the 0-9 spinner.
2. Subtract the number you rolled from 20. For example, if you roll a 6, subtract 20-6 and place your marker on 14.
3. The winner is the first player to get 3 in a row, either horizontally, vertically, or diagonally.
Candy Inventory 1

Name _________________________

Directions: Roll the yellow decahedron decade die. Record the number in the square. Decide how many packages of candy can be made from the number of pieces that you rolled.

<table>
<thead>
<tr>
<th>Number Of Candies</th>
<th>=</th>
<th>Number of Packages</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td></td>
<td>3</td>
</tr>
</tbody>
</table>

|                          |                          |                          |
|                          |                          |                          |
|                          |                          |                          |
|                          |                          |                          |
|                          |                          |                          |
## Candy Inventory 2

Name _________________________

Directions: Roll the red die. Record the number in the square. Decide how many pieces of candy are in the number of packages that you rolled.

<table>
<thead>
<tr>
<th>Number Of Packages</th>
<th>=</th>
<th>Number of Pieces of Candy</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td></td>
<td>30</td>
</tr>
</tbody>
</table>

|                   |
|                   |
|                   |
|                   |
|                   |
|                   |
|                   |

236
# Candy Inventory 3

Name _______________________

Directions: Roll the 1-30 die. Record the number in the square. Decide how many packages of candy can be made from the number of pieces that you rolled. Then record the number of left over pieces.

<table>
<thead>
<tr>
<th>Number Of Candies</th>
<th>Number of Packages</th>
<th>Number of Left Overs</th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>
Candy Inventory 4
Name _________________________

Directions: Roll the yellow and red dice. Add the amounts on the dice to determine the inventory of that kind of candy. Decide how many more pieces of candy will be needed to make another package of candy without any left over pieces. Record that number in the square.

<table>
<thead>
<tr>
<th>Pieces of Candy (Yellow Die)</th>
<th>Pieces of Candy (Red Die)</th>
<th>Total Pieces</th>
<th>Number of Pieces Needed</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>9</td>
<td>29</td>
<td>1</td>
</tr>
</tbody>
</table>

+ + = 29

+ + =

+ + =

+ + =

+ + =

+ + =
Candy Inventory 5
Name _________________________

Directions: Roll the red die to determine the inventory of that kind of candy. Record the number on the blank. Decide how many more pieces of candy will be needed to make one package of candy without any let over pieces. Record that number in the square.

<table>
<thead>
<tr>
<th>Number of Pieces of Candy in Stock</th>
<th>Number of Pieces Needed</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
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<tr>
<td></td>
<td></td>
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<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

239
Candy Order 1

Name _________________________

Directions: Roll the 1-30 die to find the number of pieces in the box. Record the number in the square. Roll the yellow die to find the number of pieces in the second kind of candy. Add to find the total number of pieces in the order.

<table>
<thead>
<tr>
<th>Candy #1 Number</th>
<th>Candy #2 (Yellow Die)</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>26</td>
<td>20</td>
<td>46</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
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<td></td>
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<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
## Candy Order 2

Name __________________________

Directions: Roll the 1-30 die to find the number of pieces in the box. Record the number in the square. Roll the 0-9 die to find the number of pieces in the second kind of candy. Add to find the total number of pieces in the order.

<table>
<thead>
<tr>
<th>Candy #1</th>
<th>Candy #2</th>
<th>Total Number Of Pieces</th>
</tr>
</thead>
<tbody>
<tr>
<td>18</td>
<td>+ 4</td>
<td>= 22</td>
</tr>
<tr>
<td></td>
<td>+</td>
<td></td>
</tr>
<tr>
<td></td>
<td>+</td>
<td></td>
</tr>
<tr>
<td></td>
<td>+</td>
<td></td>
</tr>
<tr>
<td></td>
<td>+</td>
<td></td>
</tr>
</tbody>
</table>


# Candy Order 3

**Name _________________________**

Directions: Roll the 1-20 die to find the number of pieces in the box. Record the number in the square. Roll the 1-3 die to find the number of pieces in the second kind of candy. Add to find the total number of pieces in the order.

<table>
<thead>
<tr>
<th>Candy #1</th>
<th>Candy #2</th>
<th>Total Number Of Pieces</th>
</tr>
</thead>
<tbody>
<tr>
<td>19</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>+</td>
<td>+</td>
<td>21</td>
</tr>
<tr>
<td>__________</td>
<td>__________</td>
<td>______________________</td>
</tr>
<tr>
<td>+</td>
<td>+</td>
<td></td>
</tr>
<tr>
<td>__________</td>
<td>__________</td>
<td>______________________</td>
</tr>
<tr>
<td>+</td>
<td>+</td>
<td></td>
</tr>
<tr>
<td>__________</td>
<td>__________</td>
<td>______________________</td>
</tr>
<tr>
<td>+</td>
<td>+</td>
<td></td>
</tr>
<tr>
<td>__________</td>
<td>__________</td>
<td>______________________</td>
</tr>
</tbody>
</table>
## Tape Storage Problem Record Sheet

<table>
<thead>
<tr>
<th># of Cubes</th>
<th># of Tapes</th>
<th>Left Over Space (In Cubes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>22</td>
<td></td>
<td></td>
</tr>
<tr>
<td>24</td>
<td></td>
<td></td>
</tr>
<tr>
<td>36</td>
<td></td>
<td></td>
</tr>
<tr>
<td>48</td>
<td></td>
<td></td>
</tr>
<tr>
<td>54</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Names:

_________________________
## Tape Storage 1

Name _________________________

Directions: Roll the yellow decahedron decade die. Record the number in the square. Decide how many tapes can fit into a box the same size in cubes that you rolled.

<table>
<thead>
<tr>
<th>Number Of Cubes</th>
<th>=</th>
<th>Number of Tapes</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>=</td>
<td>3</td>
</tr>
</tbody>
</table>

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
## Tape Storage 2

Name _________________________

Directions: Roll the red decahedron die. Record the number in the square. Decide how many cubes long the box would need to be to store the number of tapes that you rolled.

<table>
<thead>
<tr>
<th>Number Of Tapes</th>
<th>=</th>
<th>Number of Cubes</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td></td>
<td>30</td>
</tr>
</tbody>
</table>
# Tape Storage 3

Name _________________________

Directions: Roll the 1-30 die. Record the number in the square. Decide how many tapes can fit inside a box the length of cubes that you rolled. Then record the amount of left over space.

<table>
<thead>
<tr>
<th>Box Length (In Cubes)</th>
<th>Number of Tapes</th>
<th>Left Over Space (In Cubes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

|                      |                  |                            |
|                      |                  |                            |
|                      |                  |                            |
|                      |                  |                            |
|                      |                  |                            |
|                      |                  |                            |
**Tape Storage 4**

Name _________________________

**Directions:** Roll the dice. Add the amounts on the dice to determine the length of the box. Decide how many more cubes long the box would need to be to fit exactly one more tape. Record that number in the square.

<table>
<thead>
<tr>
<th>Cubes (Yellow Die)</th>
<th>Cubes (Red Die)</th>
<th>Total Length (In Cubes)</th>
<th>Amount of Cubes Needed</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>9</td>
<td>29</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Tape Storage 5

Name _________________________

Directions: Roll the red die to determine the length of the box. Record the number in the square. Decide how longer the box would need to be for one tape to fit exactly.

<table>
<thead>
<tr>
<th>Length of Box (In Cubes)</th>
<th>Amount of Cubes Needed</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Combined Storage 1
Name _______________________

Directions: Roll the 1-30 die to find the length of the first item. Record the number in the square. Roll the yellow die to find the length of the tapes being stored. Add to find the total length of the box needed.

<table>
<thead>
<tr>
<th>Item #1 Needed</th>
<th>Item #2 (Yellow Die)</th>
<th>Total Length (In Cubes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>26</td>
<td>+ 20</td>
<td>= 46</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>+</td>
<td>=</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>+</td>
<td>=</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>+</td>
<td>=</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Name _______________________

Directions: Roll the 1-30 die to find the length of the first item in the box. Record the number in the square. Roll the 0-9 die to find the length of the second item. Add to find the total length of the box needed.

<table>
<thead>
<tr>
<th>Item #1</th>
<th>+</th>
<th>Item #2</th>
<th>=</th>
<th>Total Length Needed</th>
</tr>
</thead>
<tbody>
<tr>
<td>(In Cubes)</td>
<td></td>
<td>(In Cubes)</td>
<td></td>
<td>(In Cubes)</td>
</tr>
</tbody>
</table>

| 18 | + | 4 | = | 22 |

|   |   |   |   |   |

|   |   |   |   |   |

|   |   |   |   |   |

|   |   |   |   |   |

|   |   |   |   |   |

|   |   |   |   |   |
## Combined Storage 3

Name _______________________

Directions: Roll the 1-20 die to find the length of the first item in the box. Record the number in the square. Roll the 1-3 dot die to find the length of the second item. Add to find the total length of the box needed.

<table>
<thead>
<tr>
<th>Item #1 (In Cubes)</th>
<th>+</th>
<th>Item #2 (In Cubes)</th>
<th>=</th>
<th>Total Length Needed (In Cubes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>19</td>
<td>+</td>
<td>2</td>
<td>=</td>
<td>21</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>+</td>
<td></td>
<td>=</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>+</td>
<td></td>
<td>=</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>+</td>
<td></td>
<td>=</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>+</td>
<td></td>
<td>=</td>
<td></td>
</tr>
</tbody>
</table>

251
# Appendix C: District-Mandated Assessments

## District-Mandated Unit 7 Assessment

<table>
<thead>
<tr>
<th>Name</th>
<th>Date</th>
</tr>
</thead>
</table>

### Place Value and Patterns

#### N1.1.3
1. Count by 10s from 10 to 100. __________ [Score: 44]

2. Count backwards from 64 to 49. __________ [Score: 57]

3. Count backwards by 10s from 50 to 10. __________

4. Write the numbers that come just before, just after, and between: 40, 41, 96, 97, 49, 51

5. Write the numbers. 49
   - 1 more is ________.
   - 1 less is ________.

6. Complete the number line below.

7. Read the words. Write the number for each word.
   - eleven ________
   - twelve ________
   - fifteen ________
   - nineteen ________

8. Use connecting cubes to show 24. ________ [Score: 57]
   - Add one more cube. ________ [Score: 57]
   - How many cubes now? ________

9. Skip count by 10s. Write how many.
   - ____ bananas

10. Skip count by 2s. Write how many.
    - ____ popcicles

11. Continue skip counting by 10s three more times. Show your counts on the empty number line below.

12. Use connecting cubes to make 22. Add ten more. Write how many tens and ones you have now.
    - ____ tens  ____ ones

    Write how many in all. ____ in all

13. Write how many tens and ones. Then write the number.
    - ____ tens  ____ ones

14. Ring all the ways to make 43.
    - 40 + 3  ____ tens and 3 ones
    - 30 + 4  ____ tens and 3 ones
Appendix C

A.1.5 15. Ring 38 on the hundred chart. Draw an arrow to find 10 more.

10 more than 38 is ________.

A.1.6 16. Use the hundred chart. Start at 34. Write the pattern that goes ________.

24, 34, 44, ________, ________, ________, ________.

A.1.7 17. Using the hundred chart. Continue the decreasing pattern.

100, 90, 80, ________, ________, ________, ________, ________.

A.1.8 18. Compare. In the boxes below, ring is greater than, is less than, or is equal to.

<table>
<thead>
<tr>
<th>Ring 47 than</th>
<th>Ring 53 than</th>
</tr>
</thead>
<tbody>
<tr>
<td>42 ____________</td>
<td>53 ____________</td>
</tr>
</tbody>
</table>

5 tens and 3 ones is greater than, is less than, or is equal to

92 ____________ 29

92 is greater than, is less than, or is equal to 29

STOP

The next section of the unit assessment is timed. Please wait for directions from your teacher.

A.1.29 19. Name: 

\[
\begin{align*}
7 + 2 & \quad 6 + 2 & \quad 2 + 3 \\
4 + 2 & \quad 5 + 2 & \quad 2 + 7 \\
6 + 3 & \quad 4 + 3 & \quad 3 + 5 \\
7 + 3 & \quad 3 + 3 & \quad 3 + 2 \\
\end{align*}
\]
Appendix C

Modifications to District Unit 7 Used with Linear Classroom

8. Use the bead string to show 24. __________ bead
Add one more bead. __________ bead
How many beads now? __________

50 + 3
is greater than
is less than
is equal to 53

12. Use the bead string to make 22. __________ bead
Use the ten catcher to add ten more.

Write how many in all. __________ in all

13. Fill in the numbers on the arrow cards to match the bead string. Write how many beads are shown below. [Look at the picture that matches your bead string.]

This shows a number built on a bead string with groups of 10.

This shows a number built on a bead string with groups of 5.

Fill in the missing numbers on the arrow cards.
Write how many in all. __________

14. Think of the ways to make 43:

\[ 40 + 3 \]
\[ 40 + 3 \]
\[ 30 + 4 \]
\[ 33 + 10 \]
Appendix C

District-Mandated Unit 11 Assessment

1. Count by two's. Start with 34 and end with 70.
2. Count by ten's. Start with 43 and end with 93.
3. Complete the growing pattern.
4. Belinda has 20 stickers. She put 10 stickers on her picture. How many stickers does she have left?
5. Use connecting cubes to show 35. Ring the picture that shows how many are left.
6. Write how many tens and ones.
7. Then find the sum.

Subtract. Check by adding. You may use connecting cubes, the empty number line, or a 100 chart if you wish.
8. 50 + 30 =
9. 75 - 31 =
10. Ring the number fact that best helps you solve 30 + 40.

5 - 4
6 + 3

D. 39
C. 40

A. 30
B. 40

Write how many are left in the number sentence below.
35 - 12 = __________

Subtract, check by adding. You may use connecting cubes, the empty number line, or a 100 chart if you wish.
11. 37 - 5 = __________
12. Check: ______ + ______ = 37
13. 59 - 45 = __________
14. Check: ______ + ______ = 59
15. Colors of Candy in One Bag

<table>
<thead>
<tr>
<th>Colors</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
<th>90</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Orange</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lime</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cherry</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Number of Candies

Use the graph. Solve.

a. How many more cherry candies than lime?
   _____ more cherry candies

b. How many orange and lime candies?
   _____ orange and lime candies

16. Use the empty number line to solve. Label your work.
   \[41 + 23 = \]
   
   

17. Use the empty number line to solve. Label your work.
   \[83 - 40 = \]
   
   
26. Name:

\[
\begin{array}{ccc}
10 & -2 & 10 \\
10 & -7 & 10 \\
10 & -3 & 10 \\
10 & -8 & 10 \\
10 & -6 & 10 \\
6 & -3 & 8 \\
\end{array}
\]
5. Use the bead string to show 35.
   Subtract 12.
   Ring the picture that shows how many are left.

A.  
B.  
C.  
D.  

Write how many are left in the number sentence below.

\[ 35 - 12 = \underline{\quad} \]

6. Write each number from the arrow cards in expanded form.

\[ \begin{array}{c}
5 \times 10 + 4 \\
3 \times 10 + 5
\end{array} \]

7. Then find the sum.

\[ \underline{\quad} + \underline{\quad} + \underline{\quad} = \underline{\quad} \]
Appendix D: Glossary of Terms, Acronyms and Abbreviations

1010 – strategy for solving two-digit addition and subtraction problems in which each addend is decomposed into tens and ones. See chapter 3 for discussion.

1010c – 1010 variation, split with compensation, strategy for solving two-digit addition and subtraction. This strategy is also called ‘jump too far.’ In this case, 96 – 29 might be solved: 96 – 30 = 66, 66 + 1 = 67.

1010R – reduced 1010 is the 1010 strategy in which the individual treats the value of each place as the value of each digit. For 32 + 24 the individual would say 3+2 is 5 and 2+4 is 6 so the sum is 56.

1010s – stepwise 1010, splits into tens and then sequences the remaining unit groups. 57 + 32 would be solved 50 +30 = 80, 80 + 7 = 87, 87 + 2 = 89. Likewise, 86 – 32 would be solved 80 – 30 = 50, 50 + 6 = 56, 56 – 2 = 54.

A10 – adding through ten, is a variation of N10 in which the second operand is decomposed in such as way as to enable the use of the decuple numbers. For example, in solving 47 + 25, one might add 47 + 3 + 20 + 2. In this manner, the “friendly” decuple numbers are used.

BNWS – backward number word sequence

CI – confidence interval is a statistical terms indicating the range of scores that fall with the chosen alpha level. In this study, 95% of the true scores would be assumed to fall within the confidence intervals.

CO – count-on by ones to add

Constructivism – the psychological approach that focuses on the reorganizations that occur within the individual. Constructivism draws on Piagetian theory. See chapter 2 for more detail.

df – degree of freedom

Dynamic assessment – a form of assessment that developed from the desire to allow special education students to demonstrate the breadth of their ability. The assessment follows an interview schedule. The next task presented depends on the response the student gives to the previous task. The flow of the interview schedule contains a number of branches that afford the interviewer choices depending on the student responses. Potential scaffolds are included within the design of the interview schedule. The actual assessment may take various forms depending on the responses given, thus the name dynamic assessment (Berman, 2001; Lidz, 2003, 1987; Lidz & Gindis, 2003; Tzuriel, 2001). “Dynamic assessment is most frequently characterized by the inclusion of interaction for the purpose of optimizing the functioning of the learner during the course of the assessment” (Lidz, 2003, p. 113).
Emergent perspective – an approach that attempts to coordinate analysis of learning from both an individual, psychological perspective, as well as a cultural, sociological perspective. See Chapter 2 for more detail.

ENL – empty number line

FNWS – forward number word sequence

IEP – individualized educational program mandated by the U.S. federal legislation for all students receiving special education services. It is a legal document that contains the specifics of services that will be provided to the child, accommodations that will be made for the child, and measurable learning objectives specific to the child.

MQL – marginal quasi-likelihood (see chapter 4 for discussion)

N10 – strategy for solving two-digit addition and subtraction problems in which one quantity is kept in its entirety (see chapter 3 for discussion)

Neo-Piagetian – term refers to the renewed interest in the theory of Jean Piaget. Constructivism would fall into this category.

PQL – penalized (or predictive) quasi-likelihood (see chapter 4 for discussion)

Psychological Perspective – the approach that focuses on the individual. Constructivism is considered part of the psychological perspective. See Chapter 2 for more detail.

RIGLS – restricted iterative generalized least squares (see chapter 4 for discussion)

RME – Realistic Mathematics Education is a mathematics curriculum developed in The Netherlands by the Freudenthal Institute.

S – student – used in protocols

se – standard error

SE – special education

SES – socioeconomic status as measured by free and reduced lunch status

Socioculturalism – the anthropological approach that focuses on the acculturation that occurs within a society. Socioculturalism draws on Vygotskiian theory. See Chapter 2 for more detail.

T – teacher – used in protocols

TIMSS – Third International Mathematics and Science Study


VHS – Analog video tape format most commonly in the US in the 1990s
Appendix E: Equations for Statistical Models

For each of the equations below, labels have been used in place of traditional statistical variable notation. The following table gives an explanation for each variable or notation contained in the equations.

<table>
<thead>
<tr>
<th>Variable Label</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Logit(π_{ij})</td>
<td>Logit (natural log odds) estimate for the model</td>
</tr>
<tr>
<td>β_{0j}</td>
<td>Constant</td>
</tr>
<tr>
<td>i</td>
<td>Within individual level</td>
</tr>
<tr>
<td>j</td>
<td>Between individual level</td>
</tr>
<tr>
<td>CLASS3_1_j</td>
<td>Collections class – (compared to) linear class</td>
</tr>
<tr>
<td>TIME_2</td>
<td>Assessment time 2 – (compared to) reference time 1</td>
</tr>
<tr>
<td>TIME_3</td>
<td>Assessment time 3 – (compared to) reference time 1</td>
</tr>
<tr>
<td>CLASS3_1. TIME_2_{ij}</td>
<td>Cross level interaction (product) of linear class by time 2</td>
</tr>
<tr>
<td>CLASS3_1. TIME_3_{ij}</td>
<td>Cross level interaction (product) of linear class by time 3</td>
</tr>
<tr>
<td>β_{0j}</td>
<td>Residual component: within individual</td>
</tr>
<tr>
<td>u_{0j}</td>
<td>Residual component: interindividual</td>
</tr>
<tr>
<td>~N(0, Ω_u)</td>
<td>Binomial distribution</td>
</tr>
<tr>
<td>STRA1010_{ij}</td>
<td>1010 strategy variable</td>
</tr>
<tr>
<td>Var(STRA1010_{ij}</td>
<td>π_{ij})</td>
</tr>
<tr>
<td>STRAN10_{ij}</td>
<td>N10 strategy variable</td>
</tr>
<tr>
<td>Var(STRAN10_{ij}</td>
<td>π_{ij})</td>
</tr>
<tr>
<td>advstrat_{ij}</td>
<td>Advanced strategy variable</td>
</tr>
<tr>
<td>Var(advstrat_{ij}</td>
<td>π_{ij})</td>
</tr>
</tbody>
</table>
Appendix E

Equation 1. 1st Order Taylor Expansion Extra Binomial Variation (RIGLS) PQL Model for All Participants Excluding Pre0Nev10

\[
\text{STRA10}_{ij} \sim \text{Binomial}(\text{cons}_{ij}, \pi_{ij})
\]

\[
\logit(\pi_{ij}) = \beta_0 \text{cons} + 5.895(1.794)\text{CLASS3}_{ij} + 1.905(0.849)\text{TIME}_2 + 0.972(0.813)\text{TIME}_3 + 2.284(1.178)\text{CLASS3}_1 \text{TIME}_2 + 5.206(1.398)\text{CLASS3}_1 \text{TIME}_3 + \nu_{ij}
\]

\[
\beta_0 = -2.267(2.232) + \nu_{ij}
\]

\[
[k_{ij}] \sim \text{N}(0, \Omega) : \Omega = \begin{bmatrix} 14.215(4.568) \end{bmatrix}
\]

\[
\text{var}(\text{STRA10}_{ij}|\pi_{ij}) = 0.325(0.061)\pi_{ij}(1 - \pi_{ij})/\text{cons}_{ij}
\]

Equation 2. 1st Order Taylor Expansion Extra Binomial Variation (RIGLS) PQL Model for N10 for All Participants Excluding Pre0Nev10

\[
\text{STRA10}_{ij} \sim \text{Binomial}(\text{cons}_{ij}, \pi_{ij})
\]

\[
\logit(\pi_{ij}) = \beta_0 \text{cons} + 0.865(1.187)\text{CLASS3}_{ij} + 2.531(0.626)\text{TIME}_2 + 5.362(0.785)\text{TIME}_3 + \nu_{ij}
\]

\[
\beta_0 = -1.907(0.902) + \nu_{ij}
\]

\[
[k_{ij}] \sim \text{N}(0, \Omega) : \Omega = \begin{bmatrix} 8.360(2.644) \end{bmatrix}
\]

\[
\text{var}(\text{STRA10}_{ij}|\pi_{ij}) = 0.452(0.084)\pi_{ij}(1 - \pi_{ij})/\text{cons}_{ij}
\]

Equation 3. 1st Order Taylor Expansion PQL with Extra Binomial Variation (RIGLS) Model with Non-Significant Interactions for N10 Excluding Pre0Nev10

\[
\text{STRA10}_{ij} \sim \text{Binomial}(\text{cons}_{ij}, \pi_{ij})
\]

\[
\logit(\pi_{ij}) = \beta_0 \text{cons} + 0.856(1.295)\text{CLASS3}_{ij} + 2.948(0.919)\text{TIME}_2 + 3.707(0.994)\text{TIME}_3 + 1.183(1.261)\text{CLASS3}_1 \text{TIME}_2 + 1.846(1.505)\text{CLASS3}_1 \text{TIME}_3 + \nu_{ij}
\]

\[
\beta_0 = -1.590(0.859) + \nu_{ij}
\]

\[
[k_{ij}] \sim \text{N}(0, \Omega) : \Omega = \begin{bmatrix} 5.246(2.952) \end{bmatrix}
\]

\[
\text{var}(\text{STRA10}_{ij}|\pi_{ij}) = 0.524(0.097)\pi_{ij}(1 - \pi_{ij})/\text{cons}_{ij}
\]
Equation 4. 1st Order Taylor Expansion PQL with Extra Binomial (RIGLS) for Both Classes Excluding Pre0Nev10

\[ \text{STRAN10}_i \sim \text{Binomial}(\text{cons}_i, \pi_{ij}) \]
\[ \logit(\pi_{ij}) = \beta_{0j} \text{cons} + 2.852(0.582)\text{TIME}_2 + 5.671(0.711)\text{TIME}_3 \]
\[ \beta_{0j} = -4.298(0.812) + \nu_{0j} \]
\[ \begin{bmatrix} \nu_{0j} \\ \nu_{ij} \end{bmatrix} \sim \mathcal{N}(0, \Omega_{\nu}) \quad : \quad \Omega_{\nu} = \begin{bmatrix} 15.906(4.032) \end{bmatrix} \]
\[ \text{var}(\text{STRAN10}_i | \pi_{ij}) = 0.320(0.050)\pi_{ij}(1 - \pi_{ij})/\text{cons}_i \]

Equation 5. 1st Order Taylor Expansion PQL with Extra Binomial Variation (RIGLS) for Advanced Strategy for Both Classes Excluding Pre0Nev10

\[ \text{advstrat}_i \sim \text{Binomial}(\text{cons}_i, \pi_{ij}) \]
\[ \logit(\pi_{ij}) = \beta_{0j} \text{cons} + 2.946(1.455)\text{class3}_1 \]
\[ \beta_{0j} = 0.543(0.882) + \nu_{0j} \]
\[ \begin{bmatrix} \nu_{0j} \\ \nu_{ij} \end{bmatrix} \sim \mathcal{N}(0, \Omega_{\nu}) \quad : \quad \Omega_{\nu} = \begin{bmatrix} 8.118(3.326) \end{bmatrix} \]
\[ \text{var}(\text{advstrat}_i | \pi_{ij}) = 0.350(0.070)\pi_{ij}(1 - \pi_{ij})/\text{cons}_i \]

Equation 6. 2nd Order Taylor Expansion MQL with Extra Binomial Variation (RIGLS) for Advanced Strategy for Both Classes Excluding Pre0Nev10

\[ \text{advstrat}_i \sim \text{Binomial}(\text{cons}_i, \pi_{ij}) \]
\[ \logit(\pi_{ij}) = \beta_{0j} \text{cons} + 2.794(1.425)\text{class3}_1 + 1.865(1.050)\text{time}_2 + 2.840(1.338)\text{time}_3 \]
\[ \beta_{0j} = -0.751(0.847) + \nu_{0j} \]
\[ \begin{bmatrix} \nu_{0j} \\ \nu_{ij} \end{bmatrix} \sim \mathcal{N}(0, \Omega_{\nu}) \quad : \quad \Omega_{\nu} = \begin{bmatrix} 3.967(2.564) \end{bmatrix} \]
\[ \text{var}(\text{advstrat}_i | \pi_{ij}) = 1.471(0.292)\pi_{ij}(1 - \pi_{ij})/\text{cons}_i \]