Three-dimensional transient fracture analysis using scaled boundary finite elements and octree mesh

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THREE-DIMENSIONAL TRANSIENT FRACTURE ANALYSIS USING SCALED BOUNDARY FINITE ELEMENTS AND OCTREE MESH

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ABSTRACT

The scaled boundary finite element method (SBFEM) is applied to three-dimensional (3D) transient fracture analysis. A systematic octree meshing process is used to divide the problem domain into smaller subdomains to model complicated geometries and stress singularities accurately. To model a subdomain, the SBFEM only requires discretization of the boundary (using 2D elements) while solution in the interior of the subdomain is described analytically. The meshing process is therefore simpler than that required by the conventional finite element method (FEM) with 3D elements. The octree mesh also increases the computational efficiency as the stiffness and mass matrices of subdomains of the same type only need to be computed once. The SBFEM is shown to solve fracture problems efficiently as the analytical solution in the scaling direction allows simple extraction of the stress intensity factors without excessive mesh refinement in the vicinity of stress singularities.

KEYWORDS

Scaled boundary finite element method, octree mesh, transient analysis, 3D crack, dynamic stress intensity factors.

INTRODUCTION

Cracks are present in various structures. If a crack propagates under external loading, it may lead to catastrophic failure. In situations that exhibit small-scale yielding at the crack tips, the theory of Linear Elastic Fracture Mechanics (LEFM) can be applied and the stress intensity factors (SIFs) are commonly employed in a fracture criterion to measure the severity of stress concentration around a crack. Computations of 3D SIFs are challenging due to the presence of singular stresses along the crack front. Some numerical methods that have been used for this purpose include the finite element method (FEM) (e.g. Shivakumar and Raju 1990), boundary element method (BEM) (e.g. Zhou et al. 2005) and meshless methods (e.g. Brighenti 2005). These methods usually require information on the asymptotic solutions of the stress field around the crack front to accurately compute the SIFs.
In this paper, 3D crack is modelled using the SBFEM and an octree mesh to evaluate the dynamic SIFs. The octree mesh permits an efficient mesh transition for the regions of high stress gradients (e.g. where complex geometries or cracks exist). The advantages of using the SBFEM in computing the dynamic SIFs include the accurate representation of stress singularities at the scaling centre, simple extraction of SIFs based on their definitions and the fact that no asymptotic stress solutions are required. The SBFEM does not require a very fine mesh in the vicinity of the crack front. Therefore, the computational effort is reduced compared to the FEM.

**METHODOLOGY**

**Octree Mesh**

The octree is a hierarchical tree-like data structure. In this technique, a cubic parent volume which covers the entire domain is first constructed. At the regions where discontinuities or complex geometry exist, the parent element is divided recursively into 8 octants (children) until the desired mesh density is achieved. In this study, a balanced octree mesh is generated. This is achieved by enforcing that the length ratio between neighbouring octants does not exceed 2:1. For 2 neighbouring octants of different sizes, hanging nodes are introduced to connect the smaller octants to the larger one as illustrated in Figure 1a. Octants that lie along the domain’s boundary may not be strictly cubic as they are modified to fit the problem geometry.

A subdomain is typically an octant comprising of 6 square faces. Usually, eight octants around the crack front are merged to form a subdomain for the purpose of SIFs computations. The hanging nodes are handled by dividing the boundary of the subdomains into a combination of triangular and quadrilateral elements. There are a total of 6 possible hanging nodes configurations on each face of an octant as shown in Figure 1b.

![Figure 1](image.png)

Figure 1. Octree meshing

Due to the limited number of patterns in the octree decomposition, the stiffness and mass matrices of these cubic octants can be pre-computed and stored for quick retrieval. The stiffness and mass matrices of two similar cubic octants with different sizes are simply scalar multiples.
Scaled Boundary Finite Element Method

The SBFEM combines some of the advantages found in FEM and BEM. In this paper, the 3D SBFEM formulation will be reviewed. Only the important equations will be described and detailed explanations can be found in Song and Wolf (1997).

In each subdomain, only the boundary is discretised using surface elements as shown in Figure 1b for a typical subdomain and in Figure 2 for a subdomain surrounding the crack front. The geometry of the boundary is scaled with respect to a scaling centre (O) located at \((\hat{x}_0, \hat{y}_0, \hat{z}_0)\) to represent a subdomain. For a subdomain containing a crack, the scaling centre is located at a position along the crack front. The Cartesian coordinates are transformed to the local radial coordinate \((\xi)\) and the two local coordinates \((\eta, \zeta)\). The coordinate \(\xi\) is 0 at the scaling centre and 1 at the boundary of the domain while the other two local coordinates are defined from -1 to 1.

![Figure 2. Three-dimensional cracked subdomain model using the SBFEM](image)

A point inside a subdomain \((\hat{x}, \hat{y}, \hat{z})\) is described by interpolating the vectors of all the Cartesian nodal coordinates at the boundary \(x, y, z\) using the shape functions \(N(\eta, \zeta)\) and the scaling coordinate \((\xi)\) as

\[
\begin{align*}
\hat{x}(\xi, \eta, \zeta) &= \hat{x}_0 + \xi N(\eta, \zeta)x, \\
\hat{y}(\xi, \eta, \zeta) &= \hat{y}_0 + \xi N(\eta, \zeta)y, \\
\hat{z}(\xi, \eta, \zeta) &= \hat{z}_0 + \xi N(\eta, \zeta)z. 
\end{align*}
\]

High-order elements can be used to improve the accuracy of the results. For the quadrilateral elements on a surface, the nodes coincide with the 2D Gauss-Lobatto quadrature points. The quadrature points in each dimension \((\eta, \zeta)\) satisfy the following equation,

\[
\begin{align*}
\frac{d}{d\eta} P_p(\eta) &= 0, \quad (2a) \\
\frac{d}{d\zeta} P_p(\zeta) &= 0, \quad (2b)
\end{align*}
\]

where \(P_p\) is the Legendre polynomial of order \(p\). The shape functions are formulated by multiplying together, the Lagrange interpolation polynomials in the \(\eta\) and \(\zeta\) directions.

In triangular elements, the integration points are determined by the method referred to as “Lobatto grid over the triangle” explained in Blyth and Pozrikidis (2006). The nodes at the sides of the triangle coincide with the Gauss-Lobatto quadrature points which ensure compatibility with the quadrilateral elements.

The displacements of any point inside the domain \(u(\xi, \eta, \zeta)\) are defined as

\[
\begin{align*}
u(\xi, \eta, \zeta) &= N(\eta, \zeta)u(\xi). \quad (3)
\end{align*}
\]

Assuming linear elastic material behaviour, the relation between strains \(\epsilon = [\epsilon_x \ \epsilon_y \ \epsilon_z \ \gamma_{yz} \ \gamma_{xz} \ \gamma_{xy}]^T\) and stresses \(\sigma = [\sigma_x \ \sigma_y \ \sigma_z \ \tau_{yz} \ \tau_{xz} \ \tau_{xy}]^T\) is defined as

\[
\begin{align*}
\epsilon &= Lu, \quad (4a) \\
\sigma &= D\epsilon, \quad (4b)
\end{align*}
\]

where \(L\) is the differential operator of 3D elasticity and \(D\) is the 3D elasticity matrix.
The differential equation of motion in the frequency domain can be expressed as

\[ L^T \sigma + \omega^2 \rho u = 0, \]  

where \( \rho \) is the mass density and \( \omega \) is the frequency.

The dynamic stiffness matrix \( S(\omega, \xi) \) relates the amplitudes of the nodal forces \( q(\xi) \) to those of the nodal displacements \( u(\xi) \) as

\[ q(\xi) = S(\omega, \xi) u(\xi). \]

Based on Galerkin’s weighted residual method applied to the equation of motion formulated in 3D scaled boundary finite element coordinates, the SBFEM equation in dynamic stiffness is obtained at the boundary (\( \xi = 1 \)) as

\[ (S(\omega) - E_1 E_0^{-1}(S(\omega) - E_1^T) - E_2 + S(\omega) + \omega S(\omega), \omega + \omega^2 M_0 = 0. \]

An improved continued fraction solution based on Chen et al. (2014) is used to construct a high-order stiffness matrix \( K_h \) and a high-order mass matrix \( M_h \) so that the equation of motion in the time domain can be expressed as

\[ K_h z(t) + M_h \ddot{z}(t) = f(t), \]

where

\[ K_h = \text{diag} \left( K_{s0}^{(1)} , S_{0}^{(1)} , \cdots , S_{0}^{(M_{cf})} \right), M_h = \begin{bmatrix} M & -X^{(1)} & 0 & \cdots & 0 \\ -X^{(1)T} & S_{1}^{(1)} & -X^{(2)} & \cdots & 0 \\ 0 & -X^{(2)T} & S_{1}^{(2)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & S_{1}^{(M_{cf})} \end{bmatrix}, \]

\[ z(t) = \begin{bmatrix} u(t) \\ u^{(1)}(t) \\ u^{(2)}(t) \\ \vdots \\ u^{(M_{cf})}(t) \end{bmatrix}, f(t) = \begin{bmatrix} q(t) \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \]

\( K \) and \( M \) are static stiffness and mass matrix, respectively. The matrices \( X^{(i)} \) denote the scaling factors in the expansion, \( S_{0}^{(i)} \) and \( S_{1}^{(i)} \) are coefficient matrices, \( u^{(i)}(t) \) are the internal variables while \( M_{cf} \) is the highest term of expansion. Typically, 3 to 4 expansion terms are required per wavelength.

**Singular Stresses for 3D Dynamic Stress Intensity Factors**

At each time step, the singular stresses \( \sigma^S \) of each cracked subdomain are used to compute the generalised stress intensity factors \( K \) at the scaling centre. These parameters are computed at the Gauss integration points on each element which corresponds to \( \theta \) and \( \phi \) in spherical coordinates. At a characteristic length \( L \) they are written as (Song and Vrcelj 2008)

\[ \sigma^S(\hat{r}, \theta, \phi) = \frac{1}{L^{2\pi}} \left( \frac{\hat{r}}{L} \right)^{-S^*(\theta, \phi)} K(\theta, \phi), \]

where the singular stress modes at a characteristic length \( L \) are denoted as \( \Psi_{\alpha L}^S \) and the matrix of orders of singularity, \( S^* \) are defined as

\[ S^* \geq 0 \text{ with } S^* = \Psi_{\alpha L}^S(\theta, \phi) \]

The terms \( c^S \) in Eq. 11 are integration constants (related to the singular stress term) which are computed at each time step. They are determined from the boundary conditions. The SIFs \( (K_I, K_{II}, K_{III}) \) of isotropic materials are extracted from their definitions. The results at the Gauss points are interpolated to obtain the SIFs at the crack front \( (\theta_f, \phi_f) \) as
\[
\begin{bmatrix}
K_t \\
K_{tt} \\
K_{tt}
\end{bmatrix}
= \sqrt{2\pi E} \begin{bmatrix}
\Psi^{s}_{xx}(\theta_f, \phi_f) \\
\tilde{\Psi}^{s}_{xx}(\theta_f, \phi_f) \\
\tilde{\Psi}^{s}_{xy}(\theta_f, \phi_f)
\end{bmatrix} c^s.
\]  

(14)

RESULTS AND DISCUSSIONS

A 3D homogeneous isotropic plate with height \(2h = 40\) mm, width \(2w = 20\) mm and thickness \(t = 2.4\) mm containing a planar central crack \((2a = 4.8\) mm) is considered. The geometry and mesh of the plate are shown in Figure 3a. The material properties of the plate are Young’s modulus \(E = 20\) GPa, Poisson’s ratio \(\nu = 0.3\) and density \(\rho = 5 \times 10^{-6}\) kg/mm\(^3\). The plate is constrained so that it is under the conditions of plane strain, i.e. the displacements in the \(y\)-direction are restrained. A uniform tension \(\sigma_0(t) = \sigma H(t)\) where \(H(t)\) is the Heaviside function, is applied on the \(xy\)-faces of the plate.

![Plate with a central crack](a) Problem geometry

Figure 3. Plate with a central crack

![Octree mesh](b) Octree mesh

Considering the plate is symmetric at \(x = 0\), only half of the plate is modelled. The octree mesh is shown in Figure 3b. It can be seen that the mesh is refined around the region where the crack exists. 8 octants around the crack plane are combined into one subdomain. Two different analyses are conducted with different element orders, namely order 3 and 5. The computed SIFs are compared with those reported by Song and Paulino (2006).

The dilatational wave velocity of the material is \(c_p = 7.338\) mm/\(\mu s\). The largest element length in the mesh is 5.6 mm. If high-order elements are used, around 6 nodes per wavelength are required. Therefore, the shortest wavelength that can be modelled using 3rd order elements is approximately 8.4 mm. A time step size \(\Delta t = 0.1\) \(\mu s\) is chosen. This is equivalent to approximately 10 time steps for one period of the shortest wave. For a time history of 14 \(\mu s\), 280 time steps are required. The simulation requires at least 3 terms of continued fraction per wavelength to obtain accurate results. The number of required continued fraction terms is related to the longest distance from a scaling centre to a subdomain’s boundary which is approximately 4.5 mm for this mesh. Hence, the order of continued fraction \(M_{cf}\) chosen is 2.

The analysis is repeated for elements of order 5 using the same approach as outlined previously. For this simulation, \(M_{cf} = 3\) is used and \(\Delta t = 0.07\) \(\mu s\) is selected giving a total of 200 time steps.

For this problem, only the opening mode SIF, \(K_t\), is not zero. Figure 4 shows that the SBFEM results obtained from the two orders of elements investigated are in good agreement with the FEM solution of Song and Paulino (2006).
Figure 4. Normalised dynamic $K_I$ at the $y$ mid-plane ($y = 0$) of the plate

CONCLUSIONS

The results obtained in this paper have shown the capability of the SBFEM to simulate transient problem in cracked three-dimensional domains accurately. The straightforward extraction of dynamic stress intensity factors based on their definition has been shown without the need of any special treatment.

REFERENCES


