Analytical and fem studies on buckling of sandwich beams

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ANALYTICAL AND FEM STUDIES ON BUCKLING OF SANDWICH BEAMS

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ABSTRACT

The paper is devoted to the analytical and numerical studies of buckling of the three-layer beams with core from aluminium foam. Mechanical properties of this core are variable on vertical direction. Young’s modulus is the highest in the middle of the core.

There are two schemes of displacement of faces and core of the beam: a broken line hypothesis and a nonlinear hypothesis. The mathematical models for both types of displacement are presented. The equations of equilibrium are written using the principle of stationary of total potential energy. Numerical analysis of sandwich beam is conducted in ANSYS environment. Finite elements analysis has been performed using a linear buckling model. The analysis is carried out for constant and variable Young’s modulus of the core of the beam. Modelling of variable Young’s modulus is difficult. To solve this problem the core is divided into several layers. Each layer is modelled with different mechanical properties. The load is applied symmetric to the faces. The values of the critical load obtained by the analytical and numerical (FEM) methods are compared.

KEYWORDS

Sandwich beams, metal foam core, variable mechanical properties of the core, analytical solution.

INTRODUCTION

Aerospace and automotive structures designed nowadays should be lighter, with better physical, mechanical, thermal, electrical and acoustical properties. They might need to also be high-class impact-energy absorbers. For these types of sandwich structures are popular. Contemporary sandwich structures are studied since the mid-twentieth century. The foundations of the theory of that kind of structures are discussed in the literature Libove and Butdorf (1948), Reissner (1948), Plantema (1966). Grygorowicz, Wasilewicz, (2013), conducted experimental investigation on three-layer beam with an aluminium metal foam core. Magnucki, Szyc, (2012) presented analytical, numerical and experimental investigations on strength and stability of sandwich beams and plates with aluminium foam core. Fatigue of foam core sandwich beams was presented by Burman, Zenkert (1997). Similar problem studied Harte et al (2001). He described fatigue strength of sandwich beams with an aluminium alloy foam core. Jasion, and Magnucki (2001) investigated the buckling wrinkling of faces of sandwich beam under pure bending. Strength and buckling of a sandwich beam with thin binding layers between faces and a metal foam core were described by Magnucki et al (2014). Influence of physical nonlinearity on local buckling in sandwich beams was described by Koissin et al (2010). Grygorowicz (2014) presented dynamic stability of sandwich beams. Magnucka-Blandzi (2009) presented non-linear hypotheses of deformation of flat cross sections of elastic sandwich beam.

The next part of the paper presents buckling of the three-layered beam with an aluminum foam core with physically variable mechanical properties. Two analytical models and numerical (FEM) model are studied.
Geometry and Loading

Fig. 1 shows the scheme of considered construction, its geometry, load and support conditions.

The faces of the beam are made from metal sheets while the core is made from metal alloy foam core. Density of the foam is ten times smaller than the faces. The length of the beam equals $L$. Total thickness $H=t_c+2t_f$, where $t_c$ – thickness of the core, $t_f$ – thickness of the faces. The width of the beam equals $b$. The beam carries a compressive axial force $F_0$.

ANALYTICAL STUDY

Two mathematical models of deformation are presented for the investigated beam. Both models are derived for linear mechanical properties of material of the faces. Linear mechanical properties of the core are taken under consideration in the first model. The field of displacement is described by classical “broken line” hypothesis (Fig. 2a). In the second model a physically non-linear material is considered for the core. The non-linear hypothesis describes the field of displacement taken to consideration the shift of a neutral line due to variable Young’s modulus (Fig. 2b) Magnucka-Blandzi (2009).

Classical Broken Line Hypothesis

The scheme of the beam before strain is shown on upper drawing on Fig. 2a. After applying critical load the beam bends. Based on the lower drawing on Fig. 2a displacement of each layer of the beam after strain is determined:

- **a) Faces:** $-\left(\frac{1}{2} + x_i\right) \leq \xi \leq -\frac{1}{2}, \quad \frac{1}{2} \leq \xi \leq \left(\frac{1}{2} + x_i\right)$:
  \[ u(x, \xi) = -t_c \left( \xi \left( \frac{dw}{dx} \pm \psi(x) \right) \right), \]  

- **b) Core:** $-\frac{1}{2} \leq \xi \leq \frac{1}{2}$:
  \[ u(x, \xi) = -t_c \left( \xi \left( \frac{dw}{dx} - 2\psi(x) \right) \right), \]  

where: $\zeta = \frac{z}{t_c}$ - dimensionless coordinate, $\psi(x) = \frac{u}{t_c}$ - dimensionless displacement, $x_i = \frac{t_f}{t_c}$ - dimensionless parameter.

Deformation of the beam:

\[ \varepsilon_x = \frac{\partial u}{\partial x}, \quad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{dw}{dx}. \]  

Stresses in the faces:

\[ \sigma_x(x, \xi) = -E_c t_c \left( \xi \left( \frac{d^2 w}{dx^2} \pm \frac{dy}{dx} \right) \right), \quad \tau'_{xz} = 0. \]  

Stresses in the core:
\[
\sigma_x(x, \zeta) = -E_c t_c \left\{ \frac{\zeta}{1 + (k_0 \zeta)^2} \left[ \frac{d^2 w}{dx^2} - 2 \frac{d \psi}{dx} \right] \right\}, \quad \tau^e_{x \zeta} = \frac{\psi(x)}{1 + v_c} E_c (\zeta),
\]

where: \( E_c \), \( E_c (\zeta) \) - elastic modulus of the faces and of the core, \( k_0 \geq 0 \) – parameter.

Figure 2. Scheme of deformation of the flat cross-section: a) classical “broken line” hypothesis, b) non-linear hypothesis Magnucka-Blandzi E. (2009)

Non-linear Hypothesis

The Young’s modulus of material of the core in the second model changes during the vertical direction of the core as is shown in Fig. 3.

Fig. 3. Physically variable Young modulus of the core of sandwich beam

The value of this parameter changes according to function bellow:

\[
E_c (\zeta) = f_{E_c(\zeta)} \ast E_{c0},
\]

where: \( f_{E_c(\zeta)} = \frac{1}{1 + (\zeta k_0)^2}, \quad 0 \leq k_0 \).

Fig. 4 presents several characteristics of Young’s modulus of the core calculated for different value of parameter \( k_0 \). If \( k_0 \) equals 0 elastic modulus is constant.

Similarly to classical “broken line” hypothesis, the strain of the beam after bending shows lower drawing on fig. 2b. Base on this scheme displacement of each layer of the beam is as follows:

a) Faces: \(- (1/2 + x_f) \leq \xi \leq -1/2, \quad 1/2 \leq \xi \leq (1/2 + x_f)\):

\[
u(x, z) = -t_c \left\{ \zeta \left[ \frac{d w}{dx} \pm \psi_1(x) \right] \right\},
\]

b) Core: \(- 1/2 \leq \xi \leq 1/2 \):

\[
u(x, z) = -t_c \left\{ \zeta \left[ \frac{d w}{dx} - 2 \psi_1(x) + \frac{1}{2\pi} \psi_2(x) \sin(2\pi \xi) \right] \right\},
\]

Deformation of the beam is determined according to Equation (3). Stresses in the faces:
\[
\sigma_z(x, \zeta) = -E_t t_c \left( \frac{d^2 w}{dx^2} + \frac{dy_1}{dx} \right) \quad \tau'_{xz} = 0. \tag{9}
\]

Stresses in the core:
\[
\sigma_z(x, \zeta) = -E_c t_c \left\{ \frac{\zeta}{1 + (k_0 \zeta)^2} \left[ \frac{d^2 w}{dx^2} - 2 \frac{dy_1}{dx} \right] + \frac{1}{2 \pi} \frac{dy_2}{dx} \sin(2\pi \zeta) \right\}, \tag{10}
\]
\[
\tau'_{xz} = G_c \gamma'_{xz}. \tag{11}
\]

where: \( E_p, E_f, G_c = E_c/[2(1+v_c)] \) – elastic moduli of the metal faces and metal foam core, \( v_c \) – Poisson ratio of the core, \( w(x) \) – deflection, \( \psi_1 = \psi_{a1} \cos(\pi x/L), \psi_2 = \psi_{a2} \cos(\pi x/L) \) – dimensionless displacement function.

Figure 4. Non-linear characteristic of Young Modulus according to different \( k_0 \) parameter

The equilibrium equations of beam were formulated from the stationary principle of total potential energy (\( \delta( U_e - W) = 0 \)):
\[
\begin{align*}
\alpha_{11} \frac{d^4 w}{dx^4} - \alpha_{12} \frac{d^3 w}{dx^3} + \alpha_{13} \frac{d^3 y_1}{dx^3} &= \frac{1}{E_t b t_c^3} \left( q - F_0 \frac{d^2 w}{dx^2} \right), \tag{12} \\
\alpha_{21} \frac{d^3 w}{dx^3} - \alpha_{22} \frac{d^2 y_1}{dx^2} + \alpha_{23} \frac{d^2 y_2}{dx^2} + \alpha_{24} \frac{\psi_1(x)}{t_c^4} - \alpha_{25} \frac{\psi_2(x)}{t_c^4} &= 0, \tag{13} \\
\frac{d^3 w}{dx^3} - 2 \frac{d^2 y_1}{dx^2} + \alpha_{33} \frac{d^2 y_2}{dx^2} + \alpha_{34} \frac{\psi_1(x)}{t_c^4} - \alpha_{35} \frac{\psi_2(x)}{t_c^4} &= 0, \tag{14}
\end{align*}
\]

where:
\[
\alpha_{11} = \frac{1}{6} \left( 3 + 6x_i + 4x_i^2 \right) x_i + J_1 \tilde{E}_{i0}, \quad \alpha_{12} = (1 + x_i) x_i + 2J_1 \tilde{E}_{i0}, \quad \alpha_{13} = \frac{1}{2\pi} J_2 \tilde{E}_{i0}, \\
\alpha_{21} = \alpha_{12}, \quad \alpha_{22} = 2 \left( x_i + 2J_1 \tilde{E}_{i0} \right), \quad \alpha_{23} = \frac{1}{\pi} J_2 \tilde{E}_{i0}, \quad \alpha_{24} = \frac{1}{1 + v_c} J_4 \tilde{E}_{i0}, \quad \alpha_{25} = \frac{1}{1 + v_c} J_5 \tilde{E}_{i0}, \\
\alpha_{33} = \frac{1}{1 + v_c} J_2, \quad \alpha_{34} = \frac{2\pi}{1 + v_c} J_2, \quad \alpha_{35} = \frac{1}{1 + v_c} J_6 \tilde{E}_{i0} - \frac{E_i}{E_y} - \frac{E_i}{E_f}, \quad \alpha_{35} = \frac{E_i}{E_f} \left( 1 + (k_0 \zeta)^2 \right), \quad 0 \leq k_0. \\
\]

\( v_c \) - Poisson ratio of the core.

The equation (12) is reduced to a second-order differential equation:
\[
\alpha_{11} \frac{d^2 w}{dx^2} - \alpha_{12} \frac{dy_1}{dx} + \alpha_{13} \frac{dy_2}{dx} = \frac{M_b(x)}{E_t b t_c^3}. \tag{15}
\]

Taking into account that each layer of the beam is made from elastic material the deflection shape of the beam is as follow:
\[
\begin{align*}
w(x) &= w_a \sin \frac{\pi x}{L}, \quad \psi_1(x) = \psi_{a1} \cos \frac{\pi x}{L}, \quad \psi_2(x) = \psi_{a2} \cos \frac{\pi x}{L}, \tag{16}
\end{align*}
\]
and the bending moment $M_b(x)$:

$$M_b(x) = F_0 w_a \sin \frac{\pi x}{L}$$

(17)

the critical force $F_{0,CR}$ is obtained:

$$F_{0,CR} = \frac{\pi^2 E_f b t_c^3}{L^2} f_0,$$

(18)

where:

$$f_0 = \alpha_{11} (\alpha_{22} \beta_{22} - \beta_{12}) - \beta_{13} (\alpha_{24} \beta_{23} - \beta_{11}),$$

$$\beta_{11} = \alpha_{22} + \left(\frac{\lambda}{\pi}\right)^2 \alpha_{24}, \quad \beta_{12} = \alpha_{23} + \left(\frac{\lambda}{\pi}\right)^2 \alpha_{25}, \quad \beta_{21} = 2 + \left(\frac{\lambda}{\pi}\right)^2 \alpha_{34}, \quad \beta_{22} = \alpha_{33} + \left(\frac{\lambda}{\pi}\right)^2 \alpha_{35},$$

$$\beta_{13} = \alpha_{13} + \left(\frac{\lambda}{\pi}\right)^2 \alpha_{14} + \left(\frac{\lambda}{\pi}\right)^2 \alpha_{15}.$$

The critical load $F_{0,CR}$ is function of geometrical parameters of investigated sandwich beam and mechanical parameters of materials used to make each layer of construction.

**NUMERICAL STUDY**

The numerical study of investigated beam with metal foam core is conducted in ANSYS software. Geometry parameters and mechanical properties used in analytical and numerical study were similar. Finite Element Analysis has been performed using a linear buckling model. Because of the symmetry of investigated problem only a half of the beam was modeled with proper boundary conditions on the symmetry planes. The faces of the beam were modeled using SHELL 181 element with four nodes and six degrees of freedom in each node. The element SOLID 186 was used to model the core of the beam. The element is defined by twenty nodes having three degrees of freedom per node.

For both models of deformation of flat cross-section the FEM analysis of critical load of the axially compressed beam was performed. In case of linear Young’s modulus the lowest buckling load equals $F_{0,CR} = 5.83$ kN. The shape of buckling mode similar to half-wave is shown on fig.5.

Figure 5. The buckling mode of the beam with linear core.

The core of the beam is divided into several layers to solve problem of non-linear mechanical properties. Each layer is modeling with different Young’s modulus calculated accordance with function (6). Calculations were performed for the following parameters of the beam: length $L=1000$ mm, width $b=50$ mm, thickness of the faces $t_f=1$ mm, thickness of the core $t_c=18$ mm, Young’ modulus of the faces $E_f = 65,600$ MPa, Poisson ratio of the faces $\nu_f = 0.33$, Young’s modulus of the core $E_c = 600$ MPa, Poisson ratio of the core $\nu_c = 0.3$. Taking to account above parameters the lowest critical load analytically calculated equals $F_{0,CR} = 5.84$ kN.
CONCLUSIONS

This article is part of a broader study on the three-layer beams with metal foam core. The results of these studies are in the process of publishing. Further studies relate, inter alia, to bending and dynamic stability.

Table 1. Comparison of results obtained with different models

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<th>Parameter</th>
<th>Analytical</th>
<th>FEM</th>
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<td>k₀</td>
<td>Critical load [kN]</td>
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<td>Linear model</td>
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REFERENCES


