Threads through a labyrinth: characterising intervention instruction for multiplicative strategies as an interweaving of five dimensions of progression

David Leslie Ellemor-Collins

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Threads Through a Labyrinth

Characterising intervention instruction for multiplicative strategies as an interweaving of five dimensions of progression

David Leslie Ellemor-Collins

Bachelor of Arts (cum laude in Mathematics and Philosophy), Harvard University
Graduate Diploma in Education, The University of Melbourne

School of Education
Southern Cross University

Presented in fulfilment of the requirements for the degree of Doctor of Philosophy

April 2018
I certify that the work presented in this thesis is, to the best of my knowledge and belief, original, except as acknowledged in the text, and that the material has not been submitted, either in whole or in part, for a degree at this or any other university.

I acknowledge that I have read and understood the University's rules, requirements, procedures and policy relating to my higher degree research award and to my thesis. I certify that I have complied with the rules, requirements, procedures and policy of the University (as they may be from time to time).

Signed: ……

Date: 9 April 2018
Abstract

This thesis seeks to characterise how instruction progresses from task to task in highly responsive, one-to-one arithmetic instruction. Within a broader design research program developing arithmetic instruction for low-attaining students, the study addresses the content domain of basic multiplication and division, moving students from counting-based strategies to more advanced multiplicative strategies. The study seeks to characterise the instructional progressions involved in this domain.

Several mathematical dimensions of instructional progression are investigated, such as progressions from lower to higher multiples, and progressions from visible to screened materials. The aims of the study are: to identify key dimensions of instructional progression, to describe how each dimension progresses, and to characterise how responsive instruction progresses in terms of the dimensions. The study contributes to three areas of significance: instruction for multiplicative thinking; instruction for low-attaining students; and the design of instructional frameworks.

Using a teaching experiment methodology, data was drawn from an experimental arithmetic intervention program in which 10- and 11-year-old students participated over 20 weeks, with all assessments and lessons recorded on video. One student’s instruction in one topic—basic facts involving 5s—was selected as a case of progression from counting-based to multiplicative strategies. The case comprised segments from eight lessons. Task-by-task analysis tracked adjustments on dimensions of instruction, alongside progressions in the student’s thinking, revealing a rich tapestry in the instructional progression.

Five key dimensions are identified. The range dimension adjusts between lower and higher multiples, and also between even and odd multiples. The orientation dimension adjusts which parts of a multiplicative task are posed as unknowns. The setting dimension adjusts between having visible, screened, and absent materials. The notation dimension adjusts between informal and increasingly formal notating. Finally, the structuring and
strategies dimension includes a range of comments drawing students’ attention to mathematical relationships.

The instructional progression overall is characterised as a strategic, interwoven calibration of the five dimensions. Features of instructional progressions include: progressions are typically multidimensional; progressions are highly recursive along each dimension; different dimensions share distinctive relationships; and different lesson segments can be characterised by distinctive calibrations of the dimensions. The study informs an instructional framework organised in terms of the five dimensions, and reflects on the potential of such a framework as a form of instructional design.
I am grateful for the opportunity of this doctorate: the opportunity to devote myself to a close study of teaching and learning, and to learn all I have learned from this work. I am also grateful to have finished this doctoral thesis, and I wish to thank all who have supported me along the way.

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May this thesis be a celebration of the hope we all invest in teaching the next generation.
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Abbreviations

CPV  Conceptual Place Value
MISP  Mathematics Intervention Specialist Program, a program of specialist intervention in number and arithmetic across years K–4.
MR  Mathematics Recovery, a program of specialist intervention in early number learning.
RME  Realistic Mathematics Education

Codes

$F$  The number of 5s, in a multiple of 5. E.g. in $8 \times 5 = 40$, $F$ is 8.
High 5s  Higher multiples of 5: 6×5, 7×5, 8×5, 9×5, 10×5 and all turn-arounds and inversions.
Low 5s  Lower multiples of 5: 1×5, 2×5, 3×5, 4×5, 5×5 and all turn-arounds and inversions.
$N$  The total number, in a multiple of 5. E.g. in $8 \times 5 = 40$, $N$ is 40.
$N-F-T$  The network of multiplicative relationships between $N$, $F$, and $T$ for even multiples of 5, such as that half of $F$ gives $T$.
NTN  Dimension of notation: developing notations to pose or record or otherwise accompany tasks.
ORN  Dimension of orientation: for a task about an arithmetic relationship, the choice of which values are given, and which are to be determined.
RNG  Dimension of range: the range of numbers involved in tasks, such as low 5s, high 5s, even multiples.
SET  Dimension of setting: the materials in which a task is set, such as screened 5-tiles, or if no materials are used, a setting of bare numbers.
STR  Dimension of attention to structuring and strategies: words or actions by the teacher to draw attention to structuring of number relationships, or to draw attention to computation strategies.
$T$  The number of 10s in a multiple of 5. E.g. in $8 \times 5 = 40$, $T$ is 4.

Note: A more extensive list of the terms and codes used in the analysis is in Table 4.2.
Chapter 1 – Introduction

Besides macro-levels in the learning process, one can distinguish finer meshes and a stepwise structure of mathematising at the micro-level … The passage from one mental level to the next is not a neat lattice path but rather a patch-work of paths, if not a labyrinth, although one where the teacher, knowing the right path, can help the pupils to find their way up to the top. (Treffers, 1978/1987, pp. 248-249)

Ah, just wait, let me write this down. (Blair, Year 6 student)

This thesis is a close study of a sequence of instruction with one primary school student, in one aspect of multiplication and division. The study draws on a teaching experiment in which I was the teacher, aiming to develop instructional design in the domain of multiplication and division, for intervention with low-attaining students. The teaching experiment, in turn, contributes to a broader design research project, the Mathematics Intervention Specialist Program (MISP), which aims to build teacher expertise in one-to-one intervention for number and arithmetic.

I begin this introductory chapter with a sample transcript of the instruction, to illustrate the fascinating subtleties in the learning and teaching which motivate the aims of this research. The sample also illustrates the challenges of the research analysis, which helps explain the research methodology, and the organisation of the thesis.

1.1 A Sample of Responsive Instruction
Consider a sequence of three brief tasks with my student, Blair. These come from a longer sequence of similar tasks in a one-to-one intervention lesson on multiplication and division. The tasks used a setting of 5-tiles—small cards each of which has a row of 5 dots—and we had just begun recording results with arrow sentences in the form:

\[ 90 \rightarrow 18 \times 5s \rightarrow 9 \times 10s \]

For the first task, I arranged eight pairs of 5-tiles behind a screen, and asked Blair to write down 80.
Teacher: OK, I’ve got 80 here. How many 10s do I have?
Blair: [Pauses, looking at his page. Points to 18 × 5s in the previous line.]
They were the 10s for the 90 weren’t they?
T: [Pointing to 18 × 5s] Oh, well, that’s the 5s.
Blair: 5s, oh yeah.
T: [Pointing to 9 × 10s] That’s—
Blair: That’s the—oh, OK, OK.
T: How many 10s?
Blair: How many 10s? Ah, eight 10s.

I explained to Blair how to write 8 × 10s in place in the notation, then moved to check the answer, lifting the screen to reveal the 5-tiles arranged in two rows of four pairs.

T: This is what a 10 looks like [placing a hand on nearest pair of tiles.]
Blair: [Laughs.]
T: There’s a 10. Are there eight of them?
Blair: [In coordination with pointing to each pair of tiles in the first row]
One, two, three, four, [then pointing across to the second row] yep, there’s eight.
T: All right [replaces screen].
Blair: Eight 10s.
T: Now, how many 5s are there?
Blair: There’d be sixteen 5s.
T: Ah. [Unscreens the tiles.] Where were the sixteen 5s?
Blair: Ah. [In coordination with pointing to each of the 16 tiles in succession] One, two, three, … sixteen. [Writes 16 × 5s in place.]
T: Very good.

For the second task, I arranged six pairs of 5-tiles behind a screen.

T: I have—write this number down—I have 60.
Blair: [Writes 60 in place.]
T: How many 10s do I have?
Blair: Six 10s. [Writes 6 × 10s in place.]
T: How many 5s do I have?
Blair: There’d be twelve 5s. [Writes 12 × 5s in place, completing the arrow sentence: 60 → 12 × 5s → 6 × 10s.]

For the third task, I did not set up any 5-tiles.

T: OK, here’s a different kind of question. This time I’m gonna tell you:
I’ve got four 10s. [Writes 4 × 10s in position on the next line.]
Blair: Four 10s.
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T: Four 10s. How many 5s will I have?
Blair: If you had four 10s – [8 seconds] – you’d have eight 5s.
T: Eight 5s?
Blair: Just wait – [6 seconds] – yeah, you’d have eight 5s.
T: How did you just check that?
Blair: How did I just check that? I looked here [pointing to the previous line: 60 → 12 × 5s → 6 × 10s]. Six times six, ah, six times two [pointing to the 6] is twelve, right, there [pointing to the 12]. I thought I had to put a one on them. Like, four times four, ah, four, ah, two times four, or four times two, two times four equals eight, not eighteen. [Continues sentence: 8 × 5s → 4 × 10s.]
T: Very nice. Let’s just show this relationship—times two, double.
[Annotates the previous sentence with an arrow from the 6 to the 12 labelled ×2, and annotates the current sentence with an equivalent arrow from the 4 to the 8.]
And how many would I have altogether in this?
Blair: 40. [Completes sentence: 40 → 8 × 5s → 4 × 10s.]

The first fascination of these three tasks is Blair’s development of multiplicative reasoning. Prior to this lesson, he had only used strategies based on counting by 5s, to solve such tasks. In this lesson, he began to find more sophisticated multiplicative strategies. In each task described here, he appeared to find the number of 5s, by reasoning with the multiplicative relationship between the number of 5s and the number of 10s. However, his new insight was prone to uncertainty. In the first task, he revealed confusion about what part of the notation referred to the 10s. In the second task, he answered fluently. In the third task, he was uncertain again. He used a valid doubling relation, but also considered a false conjecture involving finding the number of 5s by “putting a one on them”, presumably as the previous numbers of 5s, 16 and 12, have a 1 in front. Observing the new insight, and the uncertainty, I think we are watching Blair in the process of learning multiplicative reasoning with multiples of 5.

The learning process is quickly revealed to be subtle, and circuitous. Blair is hesitant, then fluent, then hesitant again. He appears to find the notation helpful in recognising the multiplicative relationship; however, the notation is also the source of the confusions described above. Likewise, the 5-tile setting
could be a source of both clarity and confusion. In an earlier task, Blair had become unexpectedly confused about what counted as 10 in the 5-tile setting. By contrast, when I indicated in the first task above “this is what a 10 looks like”, Blair laughed, enjoying how obvious this fact now appeared, and he subsequently counted the 10s and 5s in the 5-tiles with ease.

The second fascination of these tasks is the responsiveness of the instruction. Each task was slightly different, and each answer was followed up in a different way. In the first task, I unscreened the tiles, asking Blair to check his answers. In the second task, I did not unscreen the tiles. In the third task, I did not actually set up the tiles, and I also changed the task, giving not the total number, but the number of 10s: “I’ve got four 10s”. After Blair’s answer, I asked him to explain his thinking—which I had not done for the previous tasks—and followed up, not by checking the tiles, but by annotating the arrow sentences.

There are further subtleties in the progression of instruction here. For example, at first I told Blair what to write and where to write it, then progressively left him to take initiative with the notation; but for the third task, when I changed to giving the number of 10s, I chose to notate the 4 × 10s myself. Given the role the notation came to play in Blair’s learning, this process of initiation into the notation seems significant.

I am fascinated by how one-to-one intervention instruction can make these subtle adjustments over a sequence of tasks, and can continue making adjustments over a sequence of lessons. I am convinced such responsive instruction supports learning. The question arises: How can such responsive instruction be described to better understand it, and indeed to design for it?

We know from research the significance in mathematics instruction of structured materials (e.g. Anghileri, 2000; Gravemeijer, 1997), and of notations (e.g. Meira, 2003; Tolchinsky, 2007). Varying the given values in tasks can be constructive (e.g. Chick, 2007; Mason & Johnston-Wilder, 2006); teachers drawing attention to students’ reasoning can be effective (e.g. Davis, 1997; Mason, 1998). But how does a teacher weave these aspects into a coherent responsive progression of instruction, moment by moment, task by task, lesson by lesson?
In his landmark investigation of instructional design for classroom mathematics, Treffers (1978/1987) observed:

Besides macro-levels in the learning process, one can distinguish finer meshes and a stepwise structure of mathematising at the micro-level … The passage from one mental level to the next is not a neat lattice path but rather a patch-work of paths, if not a labyrinth, although one where the teacher, knowing the right path, can help the pupils to find their way up to the top. (pp. 248-249)

This thesis can be a read as an attempt to study, in detail, one case of the learning and teaching process Treffers suggests here. I trace the passage of a single student’s thinking from one mental level—using counting strategies for multiples of 5—to the next—using multiplicative strategies. In keeping with Treffers’ suggestion, the passage is revealed as not a neat lattice path, but a labyrinth, involving several simultaneous threads of “mathematising at the micro-level”, with discernable moments of confusion, of insight, of loss of insight, of repetition, yet clearly overall a progression of increasing sophistication and coherence of thinking. In turn, I trace what a responsive teacher does along this passage, which appears to help the student “up to the top”. Given the student’s passage is labyrinthine and idiosyncratic, my particular interest is to find a way to describe the progression of the teacher’s instruction. How can we describe what the teacher is doing, in a way that is coherent and clarifying, that could inform instructional design for multiplication and division? In what sense can a teacher “know the right path” and guide the student’s way?

1.2 Significance of the Study

The significance of the study can be understood in relation to three areas of research:

1. designing instruction for intervention,

2. designing instruction in multiplication and division, and

3. developing forms of instructional framework.

1.2.1 The need to develop instruction for intervention

This study arises from the pressing need for well-developed intervention materials in primary school numeracy. A significant proportion of students have difficulties learning basic arithmetic (COAG Reform Council, 2013;
Australian Academy of Science, 2016), yet there are few instructional programs to address numeracy difficulties and very few Australian schools systematically address this problem (Milton, 2000b; *National numeracy review report*, 2008). Hence there are calls to develop intervention programs to support low-attaining students’ learning (Bryant, Bryant, & Hammill, 2000; Meiers, Reid, McKenzie, & Mellor, 2013; *National numeracy review report*, 2008). For example, Ann Dowker’s (2009) report on *What Works for Children with Mathematical Difficulties* concluded that “Individually targeted interventions appear worthwhile and should be continued and further developed” (p. 16).

Researchers have developed some programs of intervention in early number learning focusing on topics such as counting and early addition and subtraction (Dowker, 2003; Gervasoni, 2005; Pearn & Hunting, 1995; Wright, 2008). There is a need to extend these programs to address later number topics. The Mathematics Intervention Specialist Program (MISP) has been developing a program of intervention to extend across the first five years of school (Wright, Ellemor-Collins, & Lewis, 2011). The present study arose from the work of MISP, aiming to contribute instructional design for the domain of multiplication and division.

One-to-one intervention creates the opportunity for what I call *highly responsive instruction*. Well established principles of classroom mathematics instruction include the teacher guiding the students responsively, and attending to students’ activity and reasoning (Mason & Johnston-Wilder, 2006; Van de Walle, 2012; van den Heuvel-Panhuizen, 2001a). Building on such general principles, one-to-one intervention can aspire to principles of more highly responsive instruction (Wright, Martland, Stafford, & Stanger, 2006). With highly responsive intervention instruction, the teacher pays close attention to the student’s ongoing activity and finely adjusts tasks and questions both to encourage the student’s own problem-solving and to guide the student toward more sophisticated mathematical thinking. The sample of instruction described above offers a brief illustration. The present study aims to investigate such highly responsive intervention instruction.
1.2.2 Designing instruction for multiplication and division

Instructional design for multiplication and division can draw on a significant body of research into children’s learning. Foundational research described the multiplicative conceptual field (Vergnaud, 1994), intuitive models for multiplication (Fischbein, Deri, Nello, & Marino, 1985; Mulligan & Mitchelmore, 1997), and students’ constructions of multiplicative units (Mulligan & Mitchelmore, 2009; Steffe, 1992). Researchers have documented the development of students’ mental strategies for multiplication and division (Anghileri, 2006; Downton, 2010; Sherin & Fuson, 2005). Learning frameworks for multiplication and division have been developed, such as in Mathematics Recovery (Wright, Martland, & Stafford, 2006), and in the Scaffolding Numeracy in the Middle Years project (Siemon, Izard, Breed, & Virgona, 2006).

Much of this research highlights the significance of a student shifting from additive to multiplicative reasoning (Clark & Kamii, 1996; Jacob & Willis, 2003; Van Dooren, De Bock, & Verschaffel, 2010). Siemon, Breed, and Virgona (2006) alerted us to the challenge for low-attaining students, declaring “the transition from additive to multiplicative thinking is one of the major barriers to learning mathematics in the middle years” (p. 1). This study aims to develop a detailed description of intervention instruction to support this critical transition from additive to multiplicative reasoning.

1.2.3 Designing an instructional framework in terms of dimensions

Researchers in mathematics education have been seeking forms of instructional design that can support effective teaching. A prominent form has been the learning trajectory, in various guises. A learning trajectory is an instructional design that describes a sequence of instruction toward a learning goal, combined with an account of the potential learning in that instruction (Battista, 2011; Simon, 1995). Research has promoted the potential of designing learning trajectories (Empson, 2011; Sarama & Clements, 2009). For example, in a report by the Consortium for Policy Research in Education, Daro, Mosher, and Corcoran (2011) concluded that:

learning trajectories hold great promise as tools for improving instruction in mathematics, and they hold promise for guiding the development of better curriculum and assessments as well. We are
agreed that it is important to advance the development of learning trajectories to provide new tools for teachers who are under increasing pressure to bring every child to high levels of proficiency. (p. 13)

The present study is, in part, a response to such calls to develop learning trajectories.

MISP uses a form of design related to learning trajectories, which I call an *instructional framework*. Like a learning trajectory, an instructional framework offers an elaborated guide to instruction, including descriptions of how the instruction interacts with potential learning. However, an instructional framework is not described in terms of a sequence of instruction; instead, the framework seeks to coordinate several aspects of instructional progression. In particular, within MISP we have been developing a notion of mathematical *dimensions of progression*, or dimensions of mathematisation (Ellemor-Collins & Wright, 2011b) to help describe the instructional frameworks. An example of a mathematical dimension of progression for instruction in addition is the range of numbers. Tasks can begin in the range 1 to 20, then be extended to the range 1 to 100, and later to 1000 and beyond; and students’ reasoning can progress from being limited to 1 to 20, to extend through the same ranges. A second example of a dimension is the use of material settings in tasks, such as using groups of counters or ten frames. Instruction can progress from tasks with visible settings, through screening settings, to tasks presented as bare numbers. Instruction involves progression on more than one dimension, and can involve moving back and forth along dimensions: advancing to bare number tasks, then retreating to visible settings, before advancing to bare number again, for example. As an example of using dimensions in instructional frameworks, the MISP design for instruction in a domain called Conceptual Place Value is described as a coordination of three dimensions: extending the range, distancing the setting, and making the increments and decrements more complex (Ellemor-Collins & Wright, 2011a). To design such an instructional framework for a domain of mathematics, a central task becomes identifying and describing key dimensions of progression in the domain.
So, the present study seeks to investigate the mathematical dimensions of progression involved in highly responsive intervention instruction, for the transition from additive to multiplicative reasoning.

1.3 Teaching Experiment

The study is a form of design research, an appropriate methodology for contributing to instructional design (P. Cobb, Confrey, diSessa, Lehrer, & Schaubule, 2003; Gravemeijer & Cobb, 2006). A strength of design research is the combined development of both instructional design and local instructional theory (Gravemeijer & van Eerde, 2009). Thus, alongside identifying the key dimensions for the design of the instructional framework, the study can develop local instructional theory on how the dimensions interact with potential learning.

The study followed a design research cycle of three phases:

1. initial design;
2. teaching experiment;
3. retrospective analysis.

The initial instructional framework drew on existing MISP materials. The teaching experiment involved a small experimental intervention program in a local primary school. I taught multiplication and division with low-attaining 10- and 11-year-olds one-to-one over two terms. Video recordings of all the teaching sessions form the principal data. The main work of the present study is the retrospective analysis of one instructional case drawn from this teaching experiment.

1.4 Focus of the Study

For the present study, I selected a case of successful transition from additive to multiplicative strategies. The case is the instruction with one student, Blair, to develop multiplicative strategies for multiples of 5. I selected a coherent sequence of segments from eight lessons, each addressing this topic, which I labelled the High 5s Sequence. Further, I selected segments of one lesson within that sequence for closer analysis, which I labelled the Lesson 10 Sequence. The study became an investigation of these two nested cases.
The study has three aims:

1. Identify key mathematical dimensions of instructional progression.
2. Describe the progressions in each dimension.
3. Characterise the instructional progression, using the dimensions.

By identifying and describing the dimensions involved, the study contributes to the development of an instructional framework for multiplication and division. By characterising the instructional progression using the dimensions, the study contributes to the development of local instructional theory, elaborating how the dimensions can progress, how they can interact, and how they are woven together into coherent responsive instruction.

The sample transcript at the beginning of this chapter is drawn from the *Lesson 10 Sequence*. After presenting that transcript, I asked how the subtle responsive progression of the instruction could be described. My answer is to describe each adjustment made in each task as an adjustment along one dimension. For example, within the first task of the sample, the adjustment to unscreen the 5-tiles for checking can be understood as a retreat on the setting dimension. Screening the 5-tiles for the second task was an advance along the setting dimension, while posing the third task without using the 5-tiles was a further advance on the setting dimension. At the same time, the third task made a retreat on the range dimension, from numbers beyond ten 5s, to numbers up to ten 5s.

These adjustments interacted with the student’s activity. For example, the advance in setting can be seen as a response to Blair’s apparent fluency in the previous task. The shift in range leads to the student challenging a misleading pattern of “putting a 1 in front” that arose in the earlier range.

The instructional progression overall can then be described as an interweaving of adjustments on a few key dimensions. Over the three tasks of the sample, the instructional progression involved a small retreat in range, alongside a retreat and advance in setting. At the same time, the instruction maintained the new notation—a third dimension. The shift in the third task, from giving the total to giving the number of 10s, was an adjustment on what I call the orientation dimension. The move to ask “How did you just check
that?” is an example of a fifth dimension: attention to structuring and strategies.

My proposal is that the instruction can be understood to work along these five dimensions of progression:

- range;
- orientation;
- setting;
- notation;
- structuring and strategies.

The progression along any one dimension is not difficult to describe. The complexity in the instructional progression arises from the recursion back and forth along each dimension, and the interplay between the dimensions. The five dimensions can serve as a set of guiding threads through the labyrinth, a map which emerges out of the complexity to reveal the coherence of the instruction. Treffers’ (1978/1987) ideal of “knowing the right path” amounts to the teacher’s awareness of the progressions required along each of the five dimensions, to work toward formal multiplicative reasoning. Helping the student find their way involves responsively pitching tasks at the shifting cutting edge of the student’s learning, which becomes an interwoven calibration across these five dimensions. The five dimensions can form an instructional framework, to describe and inform teaching.

The main work of this thesis is to present an analysis that substantiates and illuminates this characterisation of the instructional progression.

1.5 Approach to Data Analysis

One feature of the approach to data analysis is the analysis of extended sequences of instruction. To understand the progression in any one dimension, I need to study a sequence of tasks. In the sample instruction, to appreciate the progression in the setting, we need to observe the task preceding the three tasks in the sample, when Blair showed significant confusion with reasoning about 10s in the setting, which reveals the significance of the use of the setting in the sample. To appreciate the progression in the orientation, we need to observe the instruction from six
tasks earlier, when the orientation was different and Blair had an insight about multiplicative relations by turning the orientation to the one used here. To appreciate the progression in the notation, we need to observe the next two lessons, as the notation evolved. Furthermore, no brief sequence illuminates all the relevant dimensions. Most tasks are only revealing of one or two dimensions—this is one of the characteristics of the texture of the instruction that I wish to demonstrate. To develop an account that illuminates all the dimensions, I need to analyse extended sequences. This is why I decided to structure the study as an investigation of two nested cases: The Lesson 10 Sequence tracks all the instruction on the topic in one long lesson, and the High 5s Sequence tracks all the instruction on the topic over eight lessons.

A second feature of the approach to data analysis is the organisation of the analysis in layers. I need to build up from interpreting activity in each task, to describing progressions through whole sequences of tasks. I need to analyse both the activity of the student, and the activity of the teacher, and how these interact. Further, I need to analyse individual dimensions, then describe how these dimensions interact. I have labelled six layers in the analysis:

Layer A: Observation
Layer B: Local interpretation
Layer C: Progressions in student activity
Layer D: Progressions in each dimension of instruction
Layer E: Interactions between dimensions
Layer F: Multidimensional progressions in instruction

These six layers are described further in Chapter 3 on Methodology. I use these layers to organise the presentation of the analysis in Chapters 5, 6, 7, and 8.

1.6 Structure of the Thesis
The thesis is organised in 10 chapters.

Chapter 2 “Literature review” reviews the research literature informing the study, introducing key notions of responsive instruction, multiplicative reasoning, and mathematical dimensions of progression. The chapter concludes by framing the research aims of the study.
Chapter 3 “Method” explains the design research methodology, reports the procedures of the teaching experiment, describes the initial instructional design, and describes the method of data analysis.

Chapter 4 “Cases, layers, and key terms for analysis” makes all the preparations for the main analyses: framing the two nested cases of instruction, explaining key terms such as the multiplicative strategies involved in the cases, defining the dimensions of instructional progression identified in the cases, and laying out the layers of analysis.

Chapters 5 to 8 present the analysis, the main work of the thesis. Chapters 5 and 6 address the first case, the Lesson 10 Sequence, with Chapter 5 analysing Layers A, B, C, and D and Chapter 6 analysing Layers D, E, and F. Chapters 7 and 8 address the second, longer case, the High 5s Sequence, with Chapter 7 analysing Layers A, B, and C and Chapter 8 analysing Layers D, E, and F. These chapters are founded on extensive descriptions of instruction, layering up a richly elaborated account of the learning and teaching in terms of dimensions of instructional progression.

Chapter 9 “Discussion” draws on the rich analyses of the previous four chapters to argue my responses to each of the three research aims, and to discuss further reflections on characterising the instructional progression in terms of dimensions.

Chapter 10 “Conclusion” summarises what has been achieved in the study, offers implications for instructional design, and suggests directions for further research.
This study arose in the context of a design research project developing intervention in arithmetic. The significance of the study can be understood in relation to three areas of research: instruction for intervention, instruction in multiplication and division, and forms of instructional design. In this chapter, I review each of these research bases in turn, leading toward an explanation of the aims of the research.

2.1 Instructional Design for Intervention

This study arose out of efforts to develop materials for intervention with low-attaining primary school students. For two decades now, there have been calls to develop intervention programs in primary mathematics.

2.1.1 Need for intervention

A significant proportion of students have difficulties doing basic arithmetic (COAG Reform Council, 2013; Milton, 2000b; National numeracy review report, 2008; Sullivan, 2011). This limits their development of numeracy (Anghileri, 2006; McIntosh, Reys, & Reys, 1992; Yackel, 2001). Reporting of national numeracy testing in Australia over the years 2008–2013 (Australian Curriculum, Assessment and Reporting Authority, 2013) shows the percentage of students failing to meet minimum benchmarks is 4–6% in Year 3, and 5.5–7.5% in Year 5. In Mapping the Territory (Milton, 2000b), a national survey of 377 Australian schools, a majority of school principals identified the percentage of their students having difficulty learning mathematics as between 10% and 30%. The National Numeracy Review (National numeracy review report, 2008) emphasised the concern that, for low-attaining students, the gap between their knowledge and the knowledge of other students increases over time (Ginsburg, 1998; Jordan, Kaplan, Ramineni, & Locuniak, 2009). Low attainment is of particular concern in the context of the curriculum emphasis on numeracy, both nationally and internationally (Ministerial Council on Education, Employment, Training and Development, 2006; National numeracy review report, 2008).
Youth Affairs, 2008; Office for Standards in Education, 1998; *National numeracy review report*, 2008; *Department of Education, Training, and Youth Affairs*, 2000; National Council of Teachers of Mathematics, 2000). For example, in its decadal plan for the mathematical sciences in Australia, the Australian Academy of Science (2016) drew on data from the Trends in International Mathematics and Science Study (Mullis, Martin, Foy, & Arora, 2012) to show that “more than a third of Australian Year 8 students do not have the skills to apply basic mathematical knowledge in straightforward situations”, (p. 19) and listed low attainment in numeracy as a key challenge for Australian education.

While the significance of low attainment in arithmetic is recognised, there are few research-based programs to address numeracy difficulties and few Australian schools systematically address this problem (Milton, 2000b). Hence, there have been calls to develop intervention programs to support low-attaining students’ learning (Bryant et al., 2000; Dowker, 2009; Milton, 2000a; *National numeracy review report*, 2008; Sullivan, 2011). Researchers have developed some programs of intervention in early number learning focusing on topics such as counting and early addition and subtraction. Some Australian state education departments have developed early intervention programs, such as Targeted Early Numeracy (TEN) in NSW, and Getting Ready in Numeracy (GRIN) in Victoria (Meiers et al., 2013). Pearn and Hunting initiated the *Mathematics Intervention* program for the second year of school (Pearn & Hunting, 1995). Gervasoni has developed the *Extending Mathematical Understanding* (EMU) program drawing on the successful classroom-based *Early Numeracy Research Project* (Gervasoni, 2001, 2004). Dowker in the UK has also contributed research on an intervention program (Dowker, 2003; Dowker & Sigley, 2010). However, more work is needed to improve the research base and extend the scope of these programs to later topics, including multiplication and division (Breed & Virgona, 2006; Ginsburg, 1998; Meiers et al., 2013). As Dowker’s (2009) report on *What Works for Children with Mathematical Difficulties* concluded: “Individually targeted interventions appear worthwhile and should be continued and further developed” (p. 16).
2.1.2 Mathematics Recovery and Mathematics Intervention Specialist Program

Bob Wright has led a long-running program of research and development in number intervention since the 1990s. Mathematics Recovery (MR) has been a successful program in early number, developed by Wright and colleagues in the 1990s (Dowker, 2008; Wright, 2003; Wright, Cowper, Stafford, Stanger, & Stewart, 1994; Wright, Martland, Stafford, et al., 2006; Wright, Stanger, Cowper, & Dyson, 1996). MR focuses on intensive intervention for low-attaining students in the second year of school. The program is now used extensively in various forms in Australia, the US, the UK, Ireland, and elsewhere (Willey, Holliday, & Martland, 2007; Wright, 2008). A review of the effectiveness of the program in the US reported that approximately 75% of Math Recovery students reach the average level of performance within 10 to 15 weeks (J. Cobb, 2005).

Building on MR, the Numeracy Intervention Research Project (Ellemor-Collins & Wright, 2009; Wright, Ellemor-Collins, & Lewis, 2007; Wright, Ellemor-Collins, & Tabor, 2012) focused on developing intervention for low-attaining students in the fourth and fifth years of school. A recent project, the Mathematics Intervention Specialist Program (MISP), has developed from these two earlier projects, running in a large school system in Victoria since 2009 (Tran, 2016; Wright & Ellemor-Collins, 2018; Wright et al., 2011). MISP has been developing an integrated program of intervention in number and arithmetic learning across the first five years of school.

The MISP materials are organised using a framework of key domains of number knowledge (Wright & Ellemor-Collins, 2018; Wright et al., 2007), listed in Table 2.1. For each domain, MISP aims to develop a suite of what are called pedagogical tools, including learning frameworks, instructional frameworks, schedules of assessment tasks, and teaching procedures (Wright & Ellemor-Collins, 2018; Wright et al., 2011). The development of these pedagogical tools is a form of design research involving cycles of design, experimentation, analysis, and refinement.
Pedagogical tools for the early domains have been well established through MR research (Wright, 1994a, 1998; Wright, Stanger, Cowper, & Dyson, 1996; Wright, Martland, & Stafford, 2006; Wright, Martland, Stafford, et al., 2006). Among the later domains, studies have been conducted developing materials in Structuring Numbers 1 to 20 (Ellemor-Collins & Wright, 2008b, 2009; Ellemor-Collins, Wright, & McEvoy, 2013); in Conceptual Place Value (Ellemor-Collins & Wright, 2007, 2011a); and in Addition and Subtraction to 100 (Ellemor-Collins & Wright, 2008a; Ellemor-Collins, Wright, & Lewis, 2007). The tools for these domains are now well established in the MISP practice.

However, research in the project has not yet addressed instructional design for the last domain: Multiplicative Basic Facts. The present study arose from the need to pursue design research on the instructional framework for intervention in multiplication and division.

### 2.2 Highly Responsive Instruction

For specialist, one-to-one intervention, we can aspire to what I call a *highly responsive instruction*. By this, I mean an instruction that pays close attention to the student’s ongoing activity, and finely adjusts tasks and questions to encourage the student’s own problem-solving, and to guide the student toward more sophisticated mathematical thinking (Wright, 1990). This aspiration is central to the aims of the study. Below, I explain further the notion of a highly responsive instruction.

#### 2.2.1 Principles of mathematics instruction

As a starting point, the stance taken here is that intervention instruction in mathematics can be developed from principles of general classroom

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<th>Table 2.1</th>
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instruction in mathematics. Rivera (1998) and Mastropieri (1998) expressed concern that intervention is impoverished when treated as a matter of special education or of educational psychology. Instead, they recommended the development of intervention programs informed by research-based reforms in general mathematics education. In similar vein, Gross (2007) has appealed for the development of “effective teaching strategies that work for all, rather than distinct routes based on diagnostic categories” (p. 153). The mathematics education community has generated a significant body of work over the last five decades. Each of the researchers in intervention cited earlier has drawn on this knowledge in general mathematics education. For example, MR grew out of a distinctive program of research in children’s early number learning in the 1980s and ’90s (Steffe & Cobb, 1988; Wright, 1994a, 2008). Thus, a sound approach is to use what we know about how children learn mathematics to develop quality instruction for low-attaining students.

2.2.2 Principles of Realistic Mathematics Education

Across many programs of design research in classroom mathematics, a broad consensus has emerged for principles of instruction. Following Verschaffel, Greer, and Torbeyns (2006), I use the list of principles proposed in Realistic Mathematics Education (RME) as a useful articulation of this consensus. These are broadly consistent with, for example, Sullivan’s (2011) six key principles, and Askew and colleague’s recommendations (Askew, Brown, Rhodes, Wiliam, & Johnson, 1997). Sources for the RME list include Gravemeijer and Kindt (2001), van den Heuvel-Panhuizen (2001a), Freudenthal (1991), and Treffers (1978/1987). The six RME principles are:

1. Activity principle: Students need to be actively solving problems, developing and inventing their understanding.

2. Context principle: Learning needs to be anchored in a context that students can imagine and make sense of.

3. Level principle: Learning involves progressing through levels of increasing sophistication.

4. Interaction principle: Social interaction brings new ideas, and promotes reflection on students’ own activity.
5. Intertwinement principle: The various strands of learning need to be intertwined.

6. Guidance principle: The teacher steers the learning process responsively, providing tasks with awareness of both the students’ current level, and the direction of the learning-teaching trajectory.

### 2.2.3 Principles of Mathematics Recovery teaching

For the context of specialist, one-to-one intervention, Wright and colleagues composed a list of nine guiding principles for teaching, which are in close accord with the RME principles above (Wright, Martland, Stafford, et al., 2006, p.26). These are summarised as follows:

1. Problem-based/inquiry-based teaching.
2. Initial and on-going assessment.
3. Teaching just beyond the cutting edge of the student’s current knowledge.
4. Selecting from a bank of teaching procedures.
5. Engendering more sophisticated strategies.
6. Observing the child and fine-tuning teaching.
7. Incorporating symbolizing and notating.
8. Sustained thinking and reflection.

The RME principles describe a responsive instruction: an instruction that recognises students’ current ways of thinking and encourages students’ own problem-solving. In the context of one-to-one intervention, the MR principles describe a *highly* responsive instruction: the assessment of student’s current ways of thinking can be fine-grained and continual, tasks can be individually tailored to encourage a student’s problem-solving, and the teacher can adjust sensitively to new insights or confusions as they emerge. The aspiration to highly responsive instruction is in accord with recommendations that intervention be individually tailored (Gervasoni & Sullivan, 2007; Ginsburg, 1998; Rivera, 1998). As Dowker and Sigley (2010) have advised, “Individually targeted interventions are effective for children with mathematical difficulties, and may work better than similar amounts of individual attention in mathematics that are not targeted to a child’s specific strengths and weaknesses” (p. 77). Below, I describe three features of highly responsive instruction emerging from these principles, which particularly motivate the present study.
2.2.4 Teaching at the cutting edge

MR Principle 3 recommends teaching be focused just beyond the cutting edge of the student’s current knowledge. This principle suggests posing tasks close to what the student can readily achieve, but making them sufficiently challenging to extend the student’s problem-solving. Wright, Martland, Stafford and Stanger (2006) associate the cutting edge with the *zone of proximal development* from activity theory (Hedegaard, 1990; Vygotsky, 1978; Wertsch, 1981), which can be understood as the knowledge that a learner is capable of learning next, with the support of appropriate teaching. Mason and Johnston-Wilder (2006), drawing on activity theory, emphasise the importance of teaching beyond the cutting edge: “For a task to be appropriately challenging and afford possibilities for learners to encounter something significant, there must be a gap between the learners’ current state of knowledge and the goals that they adopt” (p. 73).

In their review of research on learning, Bransford, Brown, and Cocking (1999) describe responsive instruction in these terms:

> There is a good deal of evidence that learning is enhanced when teachers pay attention to the knowledge and beliefs that learners bring to a learning task, use this knowledge as a starting point for new instruction, and monitor students' changing conceptions as instruction proceeds. (p. 11)

I understand this proposal, to start from students’ current knowledge and to monitor their changing conceptions, as a broad form of teaching at the cutting edge. While monitoring changing conceptions in a classroom may be envisaged over several lessons, or once in a lesson, in one-to-one intervention the monitoring can be fine-grained and continual.

Writing specifically of mathematics teaching, van den Heuvel-Panhuizen (2012) asserts that teachers

> should steer the learning process, not in a fixed way by demonstrating what has to be learnt, but rather by asking questions and providing learners with problems that provoke the need or provide the room for them to construct mathematical insights and tools. Therefore, teachers must be able to anticipate learners’ understanding and skills that are just coming into view. (p. 7)
Again, the sense here of posing tasks that anticipate what is “just coming into view” is in accord with teaching just beyond the cutting edge. And again, while van den Heuvel-Panhuizen might imagine the classroom teacher anticipating what is coming into view in a lesson, or over a series of lessons, in one-to-one intervention the anticipation can inform every task and question posed, moment-to-moment.

The fine-grained monitoring and anticipation in teaching at the cutting edge is emphasised in MR Principle 6. With this principle, Wright and colleagues highlight the teacher’s “moment by moment monitoring of the child’s responses and changing of the task or issue of current focus”, calling this responsiveness “a key characteristic of successful individualised teaching” (Wright, Martland, Stafford, et al., 2006, p. 29). They refer to this responsive fine-tuning of tasks as micro-adjusting. The present study is, in part, a close study of micro-adjusting.

In summary, I understand teaching at the cutting edge to be associated with several principles of instruction. The context principle is observed by staying close to the student’s current knowledge, and potentially connecting to contexts that the student can make sense of. At the same time, the activity principle is invoked by pitching beyond what the student knows to encourage active construction of insights, strategies, and solutions. Individual targeting, close monitoring, and micro-adjusting are also involved. In these ways, teaching at the student’s cutting edge is at the heart of what I mean by a highly responsive instruction.

2.2.5 Teaching with attention

Highly responsive instruction involves close monitoring. Furthermore, we can emphasise the quality of that monitoring. This is a second feature of highly responsive instruction: a quality of attention.

To illuminate different qualities of attention, Davis (1997) distinguished three forms of listening in teaching, which he termed evaluative, interpretive, and hermeneutic. With evaluative listening, I have a planned direction for the student, and listen for what I expect. With interpretive listening, I seek to understand what the student says. With hermeneutic listening, I engage with what the student says, perhaps checking what I do not find clear, probing
further, or challenging. I am willing to engage in “the messy process of negotiation of meaning and understanding” (Crespo, 2000, p. 156).

Distinctive in this third type, hermeneutic listening, is my interest in adapting the direction of the instruction, in response to what I hear. Thus, hermeneutic listening is important in highly responsive instruction. Mason and Johnston-Wilder (2006) suggest that hermeneutic listening can become, at its best, a conversation…there is a genuine meeting of minds as two or more people engage with each other, mediated by a common topic or issue. A conversation can assist the transformation of awareness that is associated with learning. (p. 119)

The quality of the teacher’s attention can in turn direct the student’s attention. In her transformative approach to attentive teaching, Montessori emphasised the teacher’s role in directing the students’ attention: “The teacher teaches little and observes much, and, above all, it is her function to direct the psychic activity of the children” (1912/1964, p. 173). Wheeler (1970) expressed similar convictions in describing the role of the responsive mathematics teacher: “Watching him, we are conscious that everything he does directs the children to attend to the problem and to their own actions in tackling it” (p. 27). Investigating this role of the teacher supporting the students’ attention, Mason (1998, 2002) recognised the importance of the teacher’s noticing, in order to support the student noticing significant moments in their own activity. The quality of teacher noticing matters (Jacobs, Lamb, & Philipp, 2010; Schoenfeld, 2011; Yackel, 2003).

In summary, highly responsive instruction involves a quality of attention that can engage with the student’s activity, and support the student’s own attention to their activity.

### 2.2.6 Teaching towards progressive mathematisation

A third feature of highly responsive instruction is to teach toward a progression in mathematical sophistication. RME Principle 3 asserts that learning mathematics involves progressing through levels of sophistication, and Principle 6 asks that teachers guide learning toward these levels. The MR principles are more specific about forms of mathematical sophistication in the context of number instruction. MR Principle 5 asks the teacher to engender
the development of more sophisticated computation strategies. MR Principle 7, to involve notation in instruction, asks the teacher to “build on the child’s intuitive, verbally-based strategies … as a basis for the development of written forms of arithmetic which accord with the child’s verbally based strategies” (Wright, Martland, Stafford, et al., 2006, p. 29). Following Freudenthal and the RME community, I refer to this kind of progression in mathematical sophistication as progressive mathematisation (Beishuizen & Anghileri, 1998; Freudenthal, 1991; Gravemeijer, Cobb, Bowers, & Whitenack, 2000).

Important here is the sense of progression through increasing sophistication, as distinct from just jumping to a different kind of mathematics. A student’s use of notation can begin by trying to record his verbal strategy. In turn, his use of more formal written notation can arise as an abbreviation and standardisation of the earlier informal notation. Likewise, his insight into a new efficient computation strategy can build on his earlier less sophisticated strategy, and he can return to the earlier strategy to check the newer one.

Also important here is the mathematics. Mathematising as a term seeks to suggest the work of mathematical activity: generalising, structuring, formalising, and modelling, for example. “I insist on including in this one term the entire organising activity of the mathematician”, was Freudenthal’s summary (1991, p. 31). Progressive mathematisation involves a student’s activity shifting in mathematically significant ways, or bringing more mathematical insight to the situation.

Progressive mathematisation has a strong sense of level-raising, that is, of recurring vertical reorganisation (Freudenthal, 1991). Early content is organised into form, but then that form is treated as content that gets re-organised into a more structured or more generalised form. Freudenthal, and Treffers (1978/1987) after him, emphasised this recursive level-raising as characteristic of mathematical activity, in accord with other characterisations such as Sfard’s reification (1991) and Pirie and Kieren’s folding back (Pirie & Kieren, 1994). Freudenthal (1991) counselled that:

Mathematics is learned differently and therefore should be taught differently, that is, neither as form nor as content but while maintaining respect for the interplay between them, acted out in
the teaching/learning process! ...Their interplay…is a change of viewpoint from content to form, and conversely, leading to ever higher levels, by jumps as high as the learner can perform, and guided but not lifted by the teacher. (p. 11)

Highly responsive instruction requires a respect for the mathematising shifts between content and form, and an awareness of the directions these shifts may lead.

2.2.7 **Instructional design for highly responsive instruction**

So, in developing instructional design for intervention, I seek design that supports a highly responsive instruction. I have described such instruction in terms of general principles of RME and MR, and of three particular features:

- teaching at the cutting edge;
- teaching with attention;
- teaching towards progressive mathematisation.

Aspiring to highly responsive instruction has significant implications for instructional design. Consider designing to support teaching at the cutting edge. The zone that is beyond a student’s current knowledge, yet within what she can next achieve, may be narrow, especially in the context of working with low-attaining students. Battista (2011) suggests that a design needs “indications of jumps in sophistication that are small enough to fall within students’ ‘zones of construction’” (p. 530). The design needs ways to adjust tasks in these small jumps, so the teacher can pitch tasks just beyond a cutting edge, and can keep adjusting tasks along a changing cutting edge, at the micro-level.

To support attentive instruction, the design needs to create space to observe students’ problem-solving, and to engage in dialogue with them about their mathematical activity. The design cannot set the teacher on a pre-determined sequence; rather, the design needs to create space for the teacher to respond to the dialogue by going sideways and backwards.

To support progressive mathematisation, the design needs to connect with informal, context-bound mathematics, yet have ways to build on and organise the initial activity, to lead toward more formal, context-independent mathematics. Treffers (1978/1987) suggests that “besides macro-levels in the
learning process, one can distinguish finer meshes and a stepwise structure of mathematising at the micro-level” (p. 248–9). The design needs to offer the teacher potential directions for mathematisation in these finer meshes.

This study seeks to inform these needs of instructional design, by investigating the micro-adjustments in tasks and the potential directions for mathematisation that arise in responsive instruction.

2.3 **Aims For Instruction in Arithmetic**

There has been excellent work on developing instructional approaches for arithmetic. Valuable accounts compatible with the general RME principles include van den Heuvel-Panhuizen (2001b), Fosnot and Dolk (2001b), Anghileri (2006), and Yackel (2001). Below, I describe four instructional aims for arithmetic, supported in the research literature, to incorporate in an instructional design for multiplication and division.

2.3.1 **Refining mental computation strategies**

Instruction can emphasise mental computation as the foundation for learning arithmetic. Mental computation is an important form of computation in its own right. It is also the basis for developing more formal written approaches to computation (Beishuizen & Anghileri, 1998; Treffers & Beishuizen, 1999; Wright et al., 2012). At the same time, mental computation is foundational for developing numerical reasoning and number sense (Anghileri, 2006; McIntosh et al., 1992; Yackel, 2001).

Instruction in mental computation begins with students’ own informal, context-dependent strategies. Students then need to progressively mathematise through increasingly flexible, efficient, and mathematically sophisticated strategies. Also, their reasoning needs to progress to increasing independence from contexts such as fingers or counters. Instruction can draw students’ attention to their own strategies, and prompt reflection on their own strategies. Reflection on strategies can be supported by developing labels for strategies—such as “Jack’s strategy” or “doubling-plus-1”—and by developing notations for strategies—such as jumps on an empty number line. Instruction can ask students to explain their reasoning, and clarify their reasoning. Through the activity of solving challenging computation problems, reflecting on strategies, and explaining reasoning, students can develop

2.3.2 Structuring numbers

Instruction can also aim to support students’ structuring of numbers. By this, I mean their progressive mathematising of numbers into richer and more densely organised networks of relations (Ellemor-Collins & Wright, 2009; Freudenthal, 1991; Treffers, 2001b; Wright et al., 2012). For example, a student could relate:

\[ 4 \times 5 \text{ to } 2 \times 10. \]

With further structuring, he could come to regard all even multiples of 5 as systematically related to multiples of 10:

\[ 4 \times 5 \rightarrow 2 \times 10, \quad 6 \times 5 \rightarrow 3 \times 10, \quad 8 \times 5 \rightarrow 4 \times 10. \]

At the same time, he could also situate the even multiples of 5 in a sequence of neighbouring multiples of 5:

\[ 4 \times 5 \rightarrow 3 \times 5 + 5, \quad 6 \times 5 \rightarrow 5 \times 5 + 5, \quad 8 \times 5 \rightarrow 7 \times 5 + 5. \]

In this sense, numbers can increasingly become nodes in networks of relations.

Van Hiele (1973) suggested that learning number requires a shift in broad levels of thinking, from a ground level where numbers are tied to observable quantities and physical actions, to a first level where relations between numbers are established and a relational framework is being constructed. For example, “on the first level, [four] is a junction in a relational framework. It might be two plus two, or two times two, or possibly five minus one” (Van Hiele, 1973, cited in Gravemeijer, 1994b, p. 23). Freudenthal, in his *Didactical Phenomenology of Mathematical Structures* (1983) laid out in some detail the kinds of networks of number relations students could structure, at different levels. Compatible with this notion of knowing several ways to connect numbers in a domain, Greeno (1991) proposed a metaphor of number sense as knowing one’s way around an environment. Drawing on these suggestions, Gravemeijer, Cobb, Bowers, and Whitenack (2000) developed an instructional design for early addition and subtraction with the aim for students to “come to act in a quantitative environment structured by relationships between numbers up to 20” (p. 244). I think this is a powerful
general aim in arithmetic instruction, for students to know their way around a richly structured domain of number relations. The instructional design in the present study aims for students to structure an increasingly layered network of multiplicative relations.

It is important to distinguish structuring from learning about structures. Structuring is an activity that begins with content, experienced as realistic or common sense, and organises it into more formal structures (Freudenthal, 1991). Mathematical structures are not considered inherent in the content of tasks, to simply be learnt about. Until a student has structured the content, the structure is not there for the student; a student will only reason with the structures they have organised. As Gravemeijer (1991) asserts, “For the pupil who does not yet have this mathematical knowledge, there is nothing to see!” (p. 65). In this regard, P. Cobb (1991) has emphasised vigilance in distinguishing the perspective of the student from the perspective of the adult teacher or researcher.

2.3.3 Formalising numbers

In accord with the RME context principle, instruction in an arithmetic topic typically begins in an informal setting such as a play shop with prices or a pile of counters. Bare numbers—numbers spoken of without further context or setting—can also be a suitably commonsense setting for initial instruction with experienced students. Meanwhile, the curriculum typically aims for knowledge of more formal mathematics: written number sentences, written algorithms, and multiplication tables. Thus, instruction will need to progress from informal settings and notations, toward bare numbers and conventional notations.

To support progressive formalisation or abstraction, instruction can introduce intermediate contexts (Beishuizen, Gravemeijer, & Van Lieshout, 1997; Gravemeijer, 1997). For example, in early addition and subtraction, instruction might shift from work with loose counters, to work with ten-frames, which have more prescribed form. In turn, instruction can begin to screen the ten-frames, challenging students to visualise and describe relationships in the dots (Bobis, 1996; Treffers, 2001a; Wright et al., 2012). For addition and subtraction in the range to 100, instruction beginning in an
oral context might introduce an informal notation using empty number lines, and later transition to a more formal arrow sentence notation (Beishuizen, 1999; Gravemeijer, Bowers, & Stephan, 2003; Treffers & Buys, 2001).

Such intermediate contexts do not function simply as stepping stones that students can be lead across. Each transition involves mathematisation for the students, such as modelling actions with new notations, structuring new material settings, or generalising number relations. An instructional design for a particular domain of arithmetic needs to offer insight into how to support students with this progressive formalisation (Anghileri et al., 2002; Gravemeijer, 1999; Sullivan, Clarke, Cheeseman, & Mulligan, 2001).

2.3.4 Automatising basic facts

A fourth aim in arithmetic instruction is for students to learn basic facts. Basic facts refer to the smallest arithmetic combinations: the sums of pairs of numbers 1–10, and likewise the products of pairs of numbers 1–10. These may also be referred to as the addition tables and multiplication tables. Developing fluency in answering basic facts tasks is valuable for students, as these answers become the basis for most other calculations (Anghileri, 1995; Treffers, 2001a; Van de Walle, 2012).

However, mere memorisation of basic facts is not productive for many students (Anghileri, 2006; Boaler & Williams, 2015; Kamii, 2000). As Treffers and Buys (2001) explain, at worst, the students don’t learn the basic facts, and meanwhile acquire “a false image of mathematics as a subject made up of the rote learning of arithmetic facts and calculation tricks” (p. 75). At best, they do learn the tables, but without developing awareness of arithmetical relationships and strategies, to enable intelligent use of their knowledge.

Instead, learning basic facts is approached as a process of automatisation of efficient mental computation strategies (Askew, Bibby, & Brown, 1997; Fosnot & Dolk, 2001a; Heege, 1985; Woodward, 2006). Instruction in number domains works first on developing mental computation strategies and the structuring of number relations; then on the mental strategies becoming more efficient and familiar. Automatisation of tables becomes “a process in which smart calculation takes increasingly shorter routes to the answer, with
complete assimilation as the final step” (Treffers & Buys, 2001, p. 82). The instruction can shift toward rehearsing these basic facts tasks, such that the computations are increasingly automatised: that is, the calculation with the strategy becomes quick and routine, reaching a solution without a sense of solving a problem. With automatisation, students’ awareness of their own strategy at work “often fades into the background” (van den Heuvel-Panhuizen, 2001b, p. 89). I seek an instructional design for multiplication and division with this approach, developing mental strategies first, then rehearsing towards automaticity.

2.3.5 Strong arithmetic with low-attainers

I must emphasise that, with instructional design for low-attaining students, these aims of increasing sophistication, formalisation, and automatisation are just as important. Low-attaining students’ arithmetic knowledge is often characterised by their use of unsophisticated computation strategies, involving long counts, and by their dependence on perceptual materials such as counters or fingers. Progression to more efficient, independent computation is essential for their mathematical development (Bryant et al., 2000; Gervasoni & Sullivan, 2007; Gray, 1991; Gray, Pitta, & Tall, 2000; Menne, 2001; Wright, 2001). The MR program has always aimed for increasing sophistication in its intervention with low-attaining students (Wright, Martland, Stafford, et al., 2006). Watson and Geest (2005) encourage the value of aiming for more sophisticated thinking with low-attaining students. Likewise, when the National Numeracy Review in Australia recommended increased resources for intervention for students at risk, it emphasised “enabling every student to develop the in-depth conceptual knowledge needed to become a proficient and sustained learner and user of mathematics” (National numeracy review report, 2008, p. xiii).

However, for low-attaining students, the levels may be in finer steps (Sullivan, Mousley, & Zevenbergen, 2006; Wright & Ellemor-Collins, 2018). The instructional design in MR and in MISP has laid out finer, more detailed progressions than are available in such texts as van den Heuvel-Panhuizen (2001b), Fosnot and Dolk (2001b), and Van de Walle (2012). The present study aims to design fine-grained instruction with the four aims described above, namely:
structuring numbers in a rich network of multiplicative relations;
refining efficient, sophisticated mental computation strategies;
formalising from informal settings and notations to conventional mathematics; and
developing fluency with basic facts by rehearsing efficient mental computation towards automaticity.

2.4 Instruction in Multiplication and Division
There is not an established fine-grained instructional design for intervention in multiplication and division. The design of such instruction can draw on significant research into children’s learning in this domain.

2.4.1 Multiplicative conceptual field
Foundational to the study of instruction in multiplication and division is the notion of the multiplicative conceptual field (Vergnaud, 1994). Vergnaud, and others since, have established that the operation of multiplication is not learned in isolation; rather, it is developed as part of a closely interconnected field of concepts including: division, ratio, proportion, fractions, decimals, and rational number (Empson & Turner, 2006; Greer, 1994; Hackenberg & Tillema, 2009; Lamon, 1994; Siegler, Thompson, & Schneider, 2011; Vergnaud, 1988). Confrey and colleagues have developed a broad map for the development of rational number reasoning (Confrey, 2009). They cast the map in the form of an interconnected web of multiple learning trajectories, including trajectories for ratio, division–multiplication, equipartitioning, similarity–scaling, area–volume, decimals–percentages, and fraction-as-number. Nodes of each trajectory link with related nodes of other trajectories, and the whole web indicates a progression from earlier to later learning. In developing instruction for multiplication, we need to be aware of the development of such associated knowledge, including division, proportional reasoning, and equipartitioning.

2.4.2 Multiplicative reasoning
Another foundational notion is multiplicative thinking or multiplicative reasoning. Multiplicative reasoning involves coordinating composite units (Anghileri, 1989; Clark & Kamii, 1996; Fosnot & Dolk, 2001a; Jacob &
Willis, 2003; Norton et al., 2015). Steffe’s (1992) formulation has become standard:

For a situation to be established as multiplicative, it is necessary to at least coordinate two composite units in such a way that one of the composite units is distributed over the elements of the other composite unit. (p. 264)

A composite unit is a mental construction of a number, where the individual items composing the number can be regarded as units, and at the same time the number itself can be regarded as a unit. For example, consider a student Abby who can calculate how many 5s in 15 by counting “5, 10, 15” while keeping track of the number of counts, answering “three”. Abby has regarded 5 as a unit, to count how many 5s, while at the same time tracking the count of individual items up to 15. Abby has operated with 5 as a composite unit.

In the example of Abby, the “three 5s” are constructed as a result of Abby’s counting. In the terms of Steffe’s formulation, the composite unit of 5 is distributed over the elements of the composite three through the act of counting. So Abby’s construction could be considered an enactive coordination of composite units (Steffe, 1992).

This enactive coordination can be distinguished from abstract multiplicative reasoning. Abstract multiplicative reasoning involves anticipating a multiplicative structure such as three 5s, without needing to create the structure through action (Olive, 2001; Steffe, 1992). Consider the example of Beth, who calculates seven 5s by reasoning that five 5s is 25, so seven 5s is two more 5s, which is 35. Beth has anticipated a structure of seven 5s, allowing her to partition the seven into five and two more.

### 2.4.3 Levels of sophistication in multiplicative reasoning

Researchers have described different levels of sophistication in the development of multiplicative reasoning. Mulligan and Wright built on Steffe’s work on children’s construction of composite units (Killion & Steffe, 1989; Steffe, 1988, 1994; Steffe & Cobb, 1998), to develop a learning progression for multiplication and division, which is incorporated into the learning framework for the MR program (Mulligan, 1998; Mulligan & Watson, 1998; Wright, Martland, & Stafford, 2006). The progression comprises five levels: (1) Initial grouping, (2) Perceptual counting in
multiples, (3) Figurative counting in multiples, (4) Repeated abstract composite grouping, (5) Multiplication and division as operations. The early levels involve the construction of composite units in multiplicative situations, laying the foundations for multiplicative reasoning. Level 4 includes Abby’s example activity above of repeated addition of three 5s, in the absence of perceptual materials, which required some kind of double count to keep track of the number of 5s (three), and the total of 15. Level 5 includes Beth’s example activity of finding seven 5s as 25 and two more 5s. At this level, both the number in each group (5) and the number of groups (7) are regarded as abstract composite units. Demonstrating Level 5 often involves solving a multiplication or division task by deriving from a known product, using a multiplicative relationship. Level 5 can be regarded as abstract multiplicative reasoning (Wright, Martland, & Stafford, 2006).

Siemon and colleagues have developed a differently constructed learning progression, which they called the Learning and Assessment Framework for Multiplicative Thinking, within the large-scale initiative Scaffolding Numeracy in the Middle Years (Siemon, Izard, et al., 2006). This learning progression was based on an analysis of the responses of over 3000 students to experimental assessment tasks, using Rasch (1980) modelling to rank and group assessment items on the basis of student performance. The framework consists of eight levels: (1) Primitive modelling, (2) Intuitive modelling, (3) Sensing, (4) Strategy exploring, (5) Strategy refining, (6) Strategy extending, (7) Connecting, and (8) Reflective knowing. Each level is described in terms of students’ typical strategies for a range of different task types. For example, Level 3 includes: with more familiar numbers using strategies such as doubling and halving; but with larger whole numbers tending to rely on counting or additive strategies; demonstrating intuitive sense of proportion on certain tasks; and in simple Cartesian product situations listing all options without being able to explain solutions. Whereas Wright and Mulligan’s learning progression focuses on the increasing sophistication of the handling of composite units, Siemon and colleagues’ learning progression also incorporates progressions in: the range of factors known about, the refinement of computation strategies with different factors, and the range of situations
addressed with multiplicative strategies. Students can develop multiplicative reasoning with different sets of multiples and different task types, over time.

Furthermore, as Van Dooren, De Bock and Verschaffel (2010) observed, students’ progression towards multiplicative reasoning involves back and forth movement between additive and multiplicative reasoning. We should not expect that multiplicative reasoning is simply achieved once and for all, as a singular conceptual accomplishment.

2.4.4 Mental computation strategies for multiplication and division
Researchers have documented a range of strategies students develop for mental computation with multiplicative tasks (Downton, 2010; Foxman & Beishuizen, 2002; Heege, 1985; Heirdsfield, Cooper, Mulligan, & Irons, 1999; Kouba, 1989; Mulligan & Mitchelmore, 1997). Less sophisticated strategies can be described as counting-based strategies. The total of three groups of 5 counters is found by counting the items one at a time. Seven 5s is found by saying the sequence 5, 10, 15 … 35, keeping track of seven counts using a finger pattern, a strategy also known as skip-counting. Other strategies are described as additive strategies. Five 7s is found by adding 7 and 7, another 7, another 7, and another 7, again keeping track of how many 7s using fingers, say. In this case, the student needs to calculate the sum of each additional 7, using some additive reasoning, whereas in the skip-counting strategy, the sequence of multiples of 5 is automatised. The additive strategy aligns with Level 4, repeated abstract composite grouping, in the MR levels described above.

The most sophisticated strategies generally involve multiplicative reasoning, and can be labelled multiplicative strategies. Five 7s is reasoned to be equal to seven 5s, which is five 5s and two more 5s, which is 25 and 10, which is 35. This strategy involves multiplicative reasoning in commuting the multiplier and multiplicand, and in treating the multiplier seven as a composite unit that can be partitioned into parts five and two which can be multiplied separately. The reasoning appears as an implicit use of the commutative and distributive properties of multiplication.

An alternative multiplicative strategy reasons five 7s to be half of ten 7s, which is half of 70, which is 35. Again, the strategy treats the multiplier five
as a composite unit within the composite unit ten, appearing as an implicit use of the associative property of multiplication:

\[ 5 \times 7 = (\frac{1}{2} \times 10) \times 7 = \frac{1}{2} \times (10 \times 7) = \frac{1}{2} \times 70 \]

Thus, the development of multiplicative reasoning is closely linked with the development of multiplicative strategies.

Professional texts on instruction in multiplication and division describe multiplicative computation strategies for students to develop, such as the lists offered in Van de Walle and Lovin (2006, pp. 88–93) and in Bobis, Mulligan, and Lowrie (2004, pp. 163–168). As well as development in the handling of composite units, instruction for these multiplicative strategies involves developing some known facts (Sherin & Fuson, 2005) and developing reasoning beyond physical models (Sullivan et al., 2001). Wright, Ellemor-Collins and Tabor (2012) demonstrate how multiplicative strategies generally rely on facility with mental two-digit addition, as in the example above of finding seven 5s via 25 + 10. Thus, when seeking to develop knowledge of multiplicative strategies, instruction needs to be aware of additive facility as well.

2.4.5 Importance of development of multiplicative strategies

The development of multiplicative reasoning and multiplicative strategies is critical in students’ mathematics learning. Fluent multiplication and division is central to basic numeracy (McIntosh, 1996; National numeracy review report, 2008; Sullivan, 2011). In turn, multiplicative reasoning is foundational to the several strands of the multiplicative conceptual field, extending beyond fractions (Greer, 1994; Hackenberg & Tillema, 2009) and ratios (Lamon, 1994; Misailidou & Williams, 2003) into rational number (Baturo, 1997) and algebra (Confrey & Harel, 1994; Gray & Tall, 1994).

Yet many students struggle to attain robust multiplicative mental strategies (Anghileri, 1999; Denvir & Brown, 1986a; Mulligan & Mitchelmore, 1997; Sullivan et al., 2001). For example, Siemon, Breed, and Virgona (2006), drawing on large-scale assessments of multiplicative reasoning of students from years 5 to 9, found that approximately 20% of students relied on weak, counting-based strategies. They concluded that:
the transition from additive to multiplicative thinking is nowhere near as smooth or as straightforward as most curriculum documents seem to imply, and access to multiplicative thinking...represents a real and persistent barrier to many students’ mathematical progress in the middle years of schooling. (Siemon, Izard, et al., 2006, p. 113)

Based on these findings, Breed and Virgona (2006) argued that intervention in the middle years of schooling needs to focus on multiplicative reasoning and strategies.

Mulligan and colleagues, based on a suite of studies (Mulligan, 2011; Mulligan & Mitchelmore, 2009), argued that low-achieving students show a lack of a critical awareness of mathematical pattern and structure, with multiplicative reasoning a major aspect of that awareness. They recommended intervention to support low-achieving students’ development of unitising and multiplicative structuring.

Low-attaining students can become dependent on unsophisticated, counting-based strategies for their arithmetic (Gray, 1991). Wright (2001) documented the elaborate counting-based strategies that students can develop to solve multiplicative tasks. For example, a third-grade student attempted to solve the written task $8 \times 4$ by counting from one, in coordination with raising five fingers of her right hand and three of her left hand, four times. During the count, she raised two fingers in coordination with the one word “twelve”, and so made 31 counts only, giving the incorrect answer of 31. By contrast, a multiplicative strategy could involve knowing double 8 is 16, and finding double 16, via $6 + 6 = 12$, makes 32. The multiplicative strategy is more efficient, and reveals number sense and reasoning. The counting-based strategy is inefficient, fraught with potential error, and unaware of valuable relationships. As Gray (1991) remarked, such counting-based arithmetic is a dead end for these students.

Given the importance of students’ development of multiplicative reasoning and strategies, and the challenges in succeeding, there is a strong need to design intervention instruction to support low-attaining students with this development.
2.5 **Forms of Instructional Design**

Gravemeijer (2004) has remarked that, in early experiments with inquiry-based instruction, instructional design was out of fashion, as it was associated with practices of ignoring or overriding students’ own problem-solving. In contrast, he now asserts that, to harness students’ own problem-solving for long-term mathematical development, “a well-founded plan is needed” (p. 105). I seek a well-founded instructional design that can support teachers’ responsive instruction. What supports can the design offer?

Consider Treffers’ (1978/1987) description of working responsively with an individual boy: “The pupil indicates by his activity at each moment his location in the learning field and his progress in the process of mathematising” (p. 249). An instructional design can offer support by mapping that “learning field” and providing tools to monitor the student’s changing location in that field. Further, the design can chart the potential “processes of mathematising” in that field to help locate the student’s level of progress, and to suggest directions the student might advance in next. This is central to what an instructional design can offer to support highly responsive instruction: a map of the learning field, and a chart of the potential progressions for mathematisation.

Instructional designers have sought forms to present such maps of the learning field, and charts of potential progressions. Several forms have been developed over the last 30 years, including *learning progressions*, *learning trajectories*, and *learning frameworks*. The form of design I am developing in this study draws on these various forms. I call it an *instructional framework*. Below, I review these forms to illuminate the form of instructional design I seek, and to highlight the value of developing such a design.

### 2.5.1 Learning progressions and learning trajectories

Forms of design, and terms for them, have not been standardised across the literature. I want to distinguish at least between forms which only describe a progression in learning, and forms which also incorporate a progression in instruction. Following Battista (2011), I refer to the former as learning progressions, and the latter as learning trajectories.
I use learning progression to mean a description of the potential progression in students’ levels of knowledge or conceptual development in a topic (2007). An example of a learning progression is the list of five levels given in Section 2.4.3 above, used in MR as a model of the development of sophistication in multiplication and division.

I use learning trajectory to mean a description of a sequence of instruction toward a learning goal, combined with an account of the learning in that instruction. Important in this notion is that the learning can be described in the context of the instruction: how students might respond to particular tasks, and how tasks might be adjusted to respond to students’ activity (Battista, 2011). Since learning and instructional tasks are interdependent, a learning trajectory offers a more complete description (Simon & Tzur, 2004). A learning progression, by contrast, provides no context for the learning.

The term learning trajectory was coined by Simon (1995) to describe a teacher’s local instructional planning for a class of students. For Simon, a hypothetical learning trajectory is a teacher’s conjecture regarding the sequence of tasks she might pose, along with what responses she anticipates, what conceptual challenges the students will be working with, and how they might reach her learning goal. In turn, an actual learning trajectory can refer to a description, after a series of lessons, of how the instruction actually played out, and how the students’ learning progressed.

As well as these uses for local instructional planning, designers have come to use the term learning trajectory to refer to a more general instructional design for a topic. For example, Clements and Sarama (2004) conceptualise learning trajectories as:

- descriptions of children’s thinking and learning in a specific mathematical domain and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children’s achievement of specific goals in that mathematical domain. (p. 83)

Clements and Sarama have developed general learning trajectories in this sense for over 10 domains of early mathematical learning, including: verbal
and object counting; early addition and subtraction strategies; and measurement of length (Clements & Sarama, 2009). Dutch teams have described their instructional designs as learning–teaching trajectories, and they have developed trajectories for several domains of primary school mathematics, following the principles of RME. The trajectory for whole number calculation (van den Heuvel-Panhuizen, 2001b) is conceived as an ambitious map for teachers, with a progression of learning targets, levels within the targets, and characterisations of the teaching and learning through each level. It is in this sense of a general instructional design that maps progressions of instruction and learning, that I seek a design for intervention in multiplication and division.

Note that a learning trajectory can also refer to a researcher’s account of an actual sequence of instruction. For example, for his research purposes Steffe (2004) defines a learning trajectory as:

a model of [children's] initial concepts and operations, an account of the observable changes in those concepts and operations as a result of the children's interactive mathematical activity in the situations of learning, and an account of the mathematical interactions that were involved in the changes. (p. 131)

The research methodology of the present study involves documenting and analysing a sequence of instruction between one teacher and student, which could be considered a learning trajectory. While Steffe does not focus on documenting the instructional tasks and instructional decisions, the present study does attempt to account for progressions in the instructional tasks, as well as in the student’s activity. This methodology is described in Chapter 3.

2.5.2 Learning frameworks and instructional frameworks

A related form of design is a learning framework. An influential early example was the Learning Framework in Number developed in MR (Wright, 1998; Wright, Martland, & Stafford, 2006). The Learning Framework in Number effectively brings together learning progressions for several different aspects of early number knowledge into a single framework. For example, in the framework there are six stages of Early Arithmetical Learning, three levels of Base-Ten Arithmetical Strategies, six levels of knowledge of Forward Number Word Sequences, and so on. In the MR approach, bringing the aspects and their progressions together into a coherent framework is seen
to bring further benefits in supporting teachers (Wright & Ellemor-Collins, 2018).

Learning frameworks have been used as central organisers for three large-scale programs in our region: Count Me In Too in New South Wales (Bobis & Gould, 1999; NSW Department of Education and Training, 2001), the Early Numeracy Research Project in Victoria (Clarke, McDonough, & Sullivan, 2002; D. M. Clarke et al., 2002), and the Numeracy Development Project in New Zealand (Higgins, Parsons, & Hyland, 2003; Thomas, Tagg, & Ward, 2003; Young-Loveridge & Wright, 2002). In a review of the three programs, Bobis and colleagues (2005) remarked on the importance of the frameworks on several counts: profiling diagnostic assessments, organising instruction, and anchoring professional learning.

MISP uses a learning framework to lay out the key domains and learning progressions, extended and elaborated from the Learning Framework in Number of MR. In turn, the MISP tools developed specifically for guiding instruction in each domain, I refer to as instructional frameworks. I see these as a form of design similar to a general learning trajectory that maps potential progressions of instruction and learning, and that can coordinate with the larger learning framework. Using the term instructional framework recognises the general, schematic form of these designs. The term also respects the concern of some educators (e.g. Empson, 2011) to reserve the term learning trajectory for the local planning of teachers. The present study seeks to contribute to developing an instructional framework for multiplication and division.

### 2.5.3 Value of learning progressions, trajectories, and frameworks

There is increasing interest in the potential of learning trajectories and learning progressions to serve teachers (Clements & Sarama, 2004; Sztajn, Confrey, Wilson, & Edgington, 2012; Wilson, Mojica, & Confrey, 2013). Empson (2011) notes the groundswell in related research literature since 2004. In turn, Battista (2011) summarises the strong encouragement in the literature to further develop these forms of design. For example, in a report by the Consortium for Policy Research in Education, Daro, Mosher and Corcoran (2011) concluded that:
learning trajectories hold great promise as tools for improving instruction in mathematics, and they hold promise for guiding the development of better curriculum and assessments as well. We are agreed that it is important to advance the development of learning trajectories to provide new tools for teachers who are under increasing pressure to bring every child to high levels of proficiency. (p. 13)

The present study is, in part, a response to such calls to develop learning trajectories.

Learning progressions and trajectories can support teachers in answering the basic planning questions: Where are the students now? Where do I want them to be next? How will they get there? (Wright, Martland, Stafford, et al., 2006) For example, in a review of the New Zealand Numeracy Development Project, Higgins, Parsons, and Hyland (2003) emphasised the importance of the project’s learning framework giving teachers “direction for responding effectively to children’s learning needs” (p. 166). Battista (2011) reports on the feedback of one teacher using his learning progression on measurement, called the Cognitively-Based Assessment (CBA):

In much of [his] discussion of CBA, he described how important it was for him to be able to say to himself, ‘Well, they’re here and this is where I need to take them,’ a major affordance of CBA learning progression. This is practical, decision-making information needed for everyday mathematics teaching. (p. 552)

Further, learning trajectories can be developed to support teachers with moment-to-moment teaching. Simon’s (1995) initial account of the hypothetical learning trajectory was intended “to emphasize aspects of teacher thinking … that are common to both advanced planning and spontaneous decision making” (p. 135). Battista (2011) asks that we extend this vision to learning progressions and trajectories generally:

From the constructivist perspective, learning progressions and learning trajectories should ideally help teachers not only plan instruction, but understand students’ learning on a moment-to-moment basis and appropriately and continuously adjust instruction to meet students’ evolving learning needs. (p. 513)

The teacher needs to have agency and responsibility for the local, moment-to-moment decisions (Steffe, 2004). Empson (2011) emphasises the role of the instructional design in supporting, rather than undermining, that agency and
responsiveness of teachers during instruction. For instructional design for intervention, this aspiration stands out: an instructional framework that supports teachers with their own agency in highly responsive instruction.

2.5.4 Presenting a non-linear design
Learning in a domain is not linear. Attempts to record a student’s learning path reveal manifold convolutions and recursions, as in Battista’s (2011) zig-zagging diagram of one student’s sequence of responses to measurement tasks, or Pirie and Kieren’s (1994) analysis of an individual’s circuitous progression through their proposed sequence of levels.

Consequently, developing a general learning trajectory is fraught. In Denvir and Brown’s (1986a, 1986b) analysis of a hierarchy of number tasks in children’s learning, they emphasised the complexity of the implied learning progression. In Empson’s (2011) research using learning progressions, she found that deviations from the planned progression were “consistent and numerous” (p. 580), and concluded “trying to represent research on learning in terms of trajectories quickly gets complicated” (p. 577). Other researchers caution that representing learning too sequentially could lead teachers to direct students through the sequences, rather than respond to students’ idiosyncratic thinking (Lesh & Yoon, 2004; Sikorski & Hammer, 2010).

Researchers have typically emphasised that their learning progressions and trajectories are complex, not linear. For example, Clements and Sarama (2004) defined learning trajectories as including “the simultaneous consideration of mathematics goals, models of children’s thinking, teachers’ and researchers’ models of children’s thinking, sequences of instructional tasks, and the interaction of these at a detailed level of analysis of processes” (p. 87). Confrey and colleagues (2009) defined a learning trajectory as a “researcher-conjectured, empirically-supported description of the ordered network of experiences a student encounters through instruction ... in order to move from informal ideas ... towards increasingly complex concepts over time” (p. 2).

Fosnot and Dolk (2001a) opt for a non-linear metaphor, calling their design a “landscape of learning”, showing the big ideas, strategies, models and other important landmarks for the teacher as she journeys with her students:
We have stopped calling it a learning line—the term seems too linear. Learning—real learning—is messy … We prefer instead the metaphor of a landscape. The paths…are not necessarily linear, and there are many such paths, not just one. As in a real landscape, the paths twist and turn; they cross one another, are often indirect … It is not up to us, as teachers, to decide which pathways our students will use…What is important, though, is that we help all our students reach the horizon. (pp. 17–18)

Reckoning with this same variation in paths, Battista (2011) emphasises that the design needs to support students’ variations at the micro-level. He presents his instructional progression using a metaphor of the terrain of a mountain that students are climbing.

To meet individual students’ learning needs, often we must zoom in on individual deviations from the path to more precisely determine the next steps that students can make successfully. Critical to aiding a student's moment-to-moment climb is flexibly and reactively choosing tasks that provide them with successful hand- and foot-holds in this cognitive terrain. (p. 514)

As referred to in the introductory chapter, Treffers (1978/1987) acknowledges that actual learning paths are messy, and sequences of levels are a contrivance, but is reassuring that the contrivance can still be useful for teachers.

This schematizing of the process is a bit artificial: the passage from one mental level to the next is not a neat lattice path but rather a patch-work of paths, if not a labyrinth, although one where the teacher, knowing the right path, can help the pupils to find their way up to the top. (p. 249)

2.5.5 A multidimensional instructional framework

Responding to these aspirations, the MISP program has developed a multidimensional form of instructional framework (Wright & Ellemor-Collins, 2018; Wright et al., 2011). A framework for a given domain is organised around a few key mathematical dimensions of progression. The framework suggests a fine-grained progression of levels along each dimension. The framework does not direct a sequence of tasks; rather, it lays out a map of potential directions for adjustment, all of which lead toward significant mathematisation. The intention is to enable the teacher to make small adjustments in tasks along different dimensions, with different
combinations of dimensions, both advancing and retreating levels of challenge, and in this way attend to the convolutions in students’ paths.

For example, for the domain of Conceptual Place Value (CPV), an instructional framework was developed incorporating three dimensions of progression: extending the range of numbers, distancing the setting, and making the increments more complex (Ellemor-Collins & Wright, 2011a; Wright et al., 2012). As a second example, for the domain of addition and subtraction in the range 1 to 20, we developed an instructional framework incorporating five dimensions: extending the range of numbers, distancing the setting, making the arithmetic relations more complex, formalising, and organising (Ellemor-Collins & Wright, 2009; Ellemor-Collins et al., 2013).

This is the kind of instructional framework I am aiming to develop for the domain of multiplication and division, which the present study is intended to inform. Below, I will further describe the framework for CPV to illustrate this form of instructional framework.

2.5.6 Example instructional framework: Conceptual Place Value (CPV)

The three dimensions for CPV are shown schematically in Figure 2.1. The potential calibrations along each of these dimensions can be summarised as follows. The dimension of extending the range can be calibrated along the ranges to 100, to 1000, across 1000, and beyond 1000. The dimension of distancing the setting can be calibrated between visible base-ten materials, screened materials, and bare number tasks. The dimension of making the increments more complex can be calibrated between incrementing by multiples of a unit; switching amongst units of ones, tens, and hundreds; and incrementing by mixed units. A distinctive form of teaching chart is used to lay out potential tasks for the whole matrix of different calibrations on these three dimensions. Figure 2.2 shows a portion of the chart, as an example. A more detailed description of the instructional design is also written, which explains task types, the use of settings, fine-tuning along the dimensions, and relationships between the dimensions (see Wright et al., 2012, pp. 78–83).
Figure 2.1 Three dimensions of instruction in Conceptual Place Value

![Three dimensions of instruction in Conceptual Place Value](image)

Teaching chart from Wright and Ellemor-Collins (2018).

The teaching chart can illustrate the idea of this multidimensional instructional framework. In the partial example chart for CPV in Figure 2.2, the dimension of *extending the range* is laid out in big steps running vertically down the chart, progressing from Range I, 0 to 130, through Range II, 0 to
1000, to Range III, 0 to 1000 and beyond. The dimension of *making the increments more complex* is laid out in the vertical progression within each of those ranges: from incrementing by 10s only, to incrementing by 1s and 10s, and as indicated within the cells, extending to switched, multiple, and mixed units. The dimension of *distancing the setting* is laid out horizontally along each row, as in the second topic row progressing from bundling sticks shown, to bundling sticks screened, to bundling sticks and arrow cards. With the three dimensions thus spanning the rows and columns of the chart, each cell of the chart indicates a particular combination of calibrations on those three dimensions. The intention is not that instruction will progress through every cell, nor that it will progress systematically through each row and column. Rather, the chart provides a schematic map for organising instruction that can micro-adjust forwards or backwards along any of these dimensions at any point of instruction (Ellemor-Collins & Wright, 2011a).

This form of multidimensional instructional framework is a promising innovation of instructional design. The multiplicity of possible learning paths can be held within the handful of framework dimensions. Adjustments on the dimensions create the space and flexibility for highly responsive instruction. The present study seeks to inform the development of such a multidimensional instructional framework for the domain of multiplication and division.

### 2.5.7 Dimensions of progression

The notion of a mathematical dimension of progression in a domain is still being developed. Ellemor-Collins and Wright have written of dimensions of mathematisation (Ellemor-Collins & Wright, 2011b; Wright & Ellemor-Collins, 2018) and themes of mathematisation (Wright et al., 2012). In our instructional designs, the identification of dimensions has arisen from analysis of teaching experiments, tracking the task adjustments and progressions in instruction developed by teachers when teaching in a domain. When investigating moment-to-moment adjustments in tasks, the notion of dimensions of progression is related to notions of *affordances* in tasks (Chick, 2007), and of *dimensions-of-possible-variation* in tasks (Marton & Booth, 1997; Mason & Johnston-Wilder, 2006). However, when investigating how moment-to-moment adjustments develop into longer progressions in
instruction, the dimensions span further than individual tasks. They are, perhaps, the key affordances within a whole topic.

The notion of mathematical dimensions of progression has also drawn on studies of particular forms of mathematisation prominent in instruction. Research has established the importance of several particular forms of mathematising for learning arithmetic, including: unitising (e.g. Lamon, 1996; Mulligan & Mitchelmore, 2009; Norton et al., 2015; Steffe, 1992; Wheatley & Reynolds, 1999); decimalising to develop base-ten thinking (e.g. Beishuizen, 1993; P. Cobb & Wheatley, 1988; Freudenthal, 1991); symbolising (e.g. Gravemeijer et al., 2000; Gray & Tall, 1994; Sfard, 2000; Tillema & Hackenberg, 2011); and refining computation strategies (e.g. Beishuizen & Anghileri, 1998; Buys, 2001; Klein, 1998; Threlfall, 2002).

Ellemor-Collins and Wright developed an experimental framework of ten dimensions of mathematisation for arithmetic instruction (Ellemor-Collins & Wright, 2011b). The framework is “intended to indicate productive dimensions for developing tasks to elicit mathematisation in interactive teaching, within all domains of arithmetic. More broadly, we intend the framework to characterise the key dimensions of progressive mathematisation involved in learning whole number arithmetic” (p. 314). The ten dimensions in the framework are listed in Table 2.2.

To contribute to an instructional framework in multiplication and division, a central aim of the present study is to identify and describe key mathematical dimensions for instruction in multiplication and division. The framework of ten dimensions was an early source to consult in the study.

Table 2.2 Ten dimensions of mathematisation

<table>
<thead>
<tr>
<th>Dimension</th>
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</thead>
<tbody>
<tr>
<td>Complexifying arithmetic—more parts or more directions of operation in tasks</td>
</tr>
<tr>
<td>Distancing the setting</td>
</tr>
<tr>
<td>Extending the range of numbers</td>
</tr>
<tr>
<td>Formalising arithmetic—especially in notations and terminology</td>
</tr>
<tr>
<td>Organising and generalizing</td>
</tr>
<tr>
<td>Notating</td>
</tr>
<tr>
<td>Refining computation strategies</td>
</tr>
<tr>
<td>Structuring numbers</td>
</tr>
<tr>
<td>Decimalizing numbers</td>
</tr>
<tr>
<td>Unitizing numbers</td>
</tr>
</tbody>
</table>

Table from Ellemor-Collins and Wright (2011b).
Note that, in the present study, I have modified the terminology to refer to *mathematical dimensions of progression* or *mathematical dimensions of instructional progression*, rather than dimensions of mathematisation. I aim for and discuss mathematisation in students’ activity. Meanwhile, I am investigating task adjustments and progressions enacted by the teacher. I do not want to refer to the teacher’s progressions as mathematisation because I do not want to confuse the adjustments made by the teacher with those of the student. Instead, I use the more neutral term, dimensions of progression.

### 2.6 Research Aims

This chapter has reviewed the several bases of the present research: the need for instructional design in intervention for low-attaining students; principles of highly responsive instruction; aims of instruction in arithmetic; issues of instruction in multiplication and division; and potential forms of instructional design. Gathering these bases, I can articulate the motivation for the present research project.

Within a research program developing arithmetic intervention for low-attaining students, the main goal of the present research is to design instruction to support the critical transition from counting-based strategies to efficient multiplicative strategies. I understand that this transition involves the development of different strategies with different sets of factors, over time. In keeping with the aims of arithmetic instruction in general, I expect the instruction to involve structuring numbers, formalising numbers, and refining strategies, before moving towards automatising basic facts. I want the instruction to be a highly responsive instruction which can adjust sensitively to a student’s cutting edge, and can guide the progressive mathematisation of the student’s multiplicative reasoning. And I want the instructional design to be in the form of a fine-grained instructional framework, organised around a few key mathematical dimensions of progression.

At the crux of this design task, I need to understand how the highly responsive instruction progresses towards multiplicative strategies: What are the dimensions of that instructional progression, and how can I describe the progression as it makes its individualised, micro-adjusted, labyrinthine way?
devised the present study to illuminate this instructional progression. I framed the study with three research aims, to:

1. Identify key mathematical dimensions of instructional progression.

2. Describe the progressions in each dimension.

3. Characterise the instructional progression, using the dimensions.

By identifying and describing the dimensions involved, the study can inform the design of the instructional framework for the domain of multiplication and division. By characterising the instructional progression using the dimensions, the study contributes to the development of local instructional theory, elaborating how the dimensions can progress, how they can interact, and how they are woven together into coherent responsive instruction.

The study was framed within the analysis of a teaching experiment in arithmetic intervention. In the next chapter, I describe the methodology and procedures of the teaching experiment, and the approach to the data analysis.
Chapter 3 – Methodology

Just wait, let me do it this way. (Blair)

In this chapter, I address the methods and methodology of the study, in six sections. I give an account of the methodology as a form of design research. I describe the procedures of the teaching experiment, and the instructional design used in the teaching experiment. I explain the case selected for the case study. I describe the process and methods involved in analysing the data. Finally, I address questions of the methodological quality and rigour of the study.

3.1 Design Research Methodology

The study is a form of design research. Design research seeks to develop both instructional design and instructional theory, by experimenting in an authentic learning context over cycles of design and analysis (P. Cobb et al., 2003; P. Cobb & Gravemeijer, 2008; Design-Based Research Collective, 2003). The present study has grown out of a larger program of design research, and aims to develop both design and theory for use by practitioners, so design research is an appropriate methodology. In this section, I acknowledge the methodological traditions I draw on, and describe the main elements of the methodology.

3.1.1 An evolving teaching experiment methodology

The methodology of the present study draws from a tradition of constructivist teaching experiments, initiated by Steffe and his collaborators, which involves teaching students individually over extended periods, recording sessions on video, and seeking to develop models of the students’ mathematics (Steffe, 1991; Steffe & Cobb, 1988; Steffe & Thompson, 2000; Steffe, von Glasersfeld, Richards, & Cobb, 1983). I also draw on the Dutch tradition of developmental research, inspired by Freudenthal and articulated by Gravemeijer, which seeks to design innovations in instruction in a particular topic, drawing on instructional principles and design heuristics, and then to refine the instructional design over a series of trials (Freudenthal, 1991; Gravemeijer, 1994a, 2001). The collaborations of Cobb and
Gravemeijer have combined the close analysis of student activity with the cyclic instructional design in a powerful methodology for research in classroom instruction (P. Cobb, 2003; P. Cobb & Gravemeijer, 2008; Gravemeijer & Cobb, 2006). Wright’s research methodology has drawn on Steffe, Cobb, and Gravemeijer, using individual teaching experiments to develop pedagogical tools for intervention instruction, including assessment tasks, instructional procedures, and learning frameworks (Wright, 1994a, 1994b; Wright et al., 2007; Wright et al., 1996). Wright’s major research projects have been multi-tiered (Lesh & Kelly, 2000), addressing theory, instructional design, professional learning, and intervention for students (Wright et al., 2011).

The research method used in the present study draws on this evolving teaching experiment methodology, involving close analysis of cycles of individual teaching experiments to develop an instructional framework for intervention in a particular topic. Unlike Steffe’s work, the aim in this study is to understand and develop instruction, not just to understand children’s mathematics. Unlike Cobb and Gravemeijer’s work, the setting is one-to-one rather than the classroom; and the instructional design is a framework for responsive individual instruction, rather than a classroom learning trajectory. Unlike Wright, the study itself is not multi-tiered; however it draws on and contributes to the larger multi-tiered MISP program. The study is distinctive in developing an instructional design in terms of multiple dimensions of progression, and the analysis of those dimensions has been an innovation in the methodology.

To describe the methodology, I draw on the broader design research literature that has developed over the last 25 years (Barab & Squire, 2004; Brown, 1992; P. Cobb et al., 2003; Confrey, 2006; Design-Based Research Collective, 2003; Kelly, Lesh, & Baek, 2008; Lesh, 2002; van den Akker, Gravemeijer, McKenney, & Nieveen, 2006). As well, I take heart from earlier educators who also developed design and theory in experimental school settings, such as Dewey (1900/1956, 1902/1956), Montessori (1912/1964), Vygotsky (1978), and Bruner (1966).
3.1.2 Twin aims of instructional design and theory

The design research methodology has twin aims:

- the development of an instructional design; and
- the development of instructional theory.

These two aims contribute to each other. On the one hand, teaching with a design furnishes a context for research into the learning and instruction. On the other hand, analysis of the learning and instruction informs the revision of the instructional design. This intertwining of research into both theory and design, in an authentic learning context, is a defining feature of design research (P. Cobb et al., 2003; P. Cobb & Gravemeijer, 2008; Design-Based Research Collective, 2003).

The instructional design can be on a large scale, such as a major framework, or on a small scale, such as a particular set of tasks, but it needs to be a significant and coherent set of innovations (P. Cobb & Gravemeijer, 2008; Middleton, Gorard, Taylor, & Bannan-Ritland, 2008). In this study, I began with an instructional framework for the whole domain of multiplication and division. The study then aimed to refine a coherent aspect of this framework: the key mathematical dimensions of progression for the development of multiplicative strategies. The research seeks to draw on the experience of the teaching experiment and the analysis of instruction to develop a coherent, grounded design that can guide future instruction. The aim is not to assess whether a planned design worked, but rather to understand and improve the design (P. Cobb et al., 2003; Gravemeijer, 1994b; Lesh & Sriraman, 2010).

As Walker (2006) argues, designing instruction is “not a work of imagination or a straightforward deduction from theory”; rather, it is “an R&D endeavour” (p. 9).

The instructional theory of interest in the present study is local instructional theory (Gravemeijer, 2004; Gravemeijer & Cobb, 2006) about how learning and teaching can work in a specific mathematical domain. In this case, the study aims to characterise how instruction in multiplicative strategies progresses responsively, in terms of mathematical dimensions of progression. Thus, as well as designing a framework of dimensions, I can develop theory explaining how the framework works: how each dimension progresses, how
the dimensions interact, how the adjustments on the dimensions support learning, why each dimension is significant. The local instructional theory develops in the context of the more generic instructional principles pursued in the study, such as inquiry-based and responsive intervention instruction (P. Cobb & Yackel, 1996; Gravemeijer & Cobb, 2006). The local theory is intended to support other researchers and teachers when they adapt the design for use in other contexts with other students, so developing local theory is critical for the goal of developing designs that are useful for others (P. Cobb & Gravemeijer, 2008; Design-Based Research Collective, 2003; Lesh & Sriraman, 2010). As Confrey (2006) asserts, “one cannot prescribe practices, but one can guide practice by means of explanatory frameworks accompanied by data, evidence, and argument” (p. 139).

3.1.3 The design research cycle
Design research is cyclic, developing designs and theory over multiple iterations (Design-Based Research Collective, 2003; Gravemeijer & Cobb, 2006; McClain, 2002; Middleton et al., 2008; Walker, 2006). The design research cycle of the present study can be described in three phases:

- Phase 1. Planning an initial instructional design;
- Phase 2. Conducting the teaching experiment working with the initial design;
- Phase 3. Retrospectively analysing the instruction and refining the instructional design.

The thesis reports a case within a single cycle of research. The initial design drew on earlier design research work in the MISP program, and I anticipate the results of this study contributing to future cycles of research in multiplication and division instruction.

3.1.4 Phase 1: Initial instructional design
Here, I describe the methodology for the initial design phase. I describe the initial instructional framework in Section 3.3.

Design research begins by developing an initial instructional design with which to experiment (Gravemeijer et al., 2000; Walker, 2006). In some design studies, the initial design is developed from scratch, constituting a
significant phase of research. As the current study arose out of the ongoing design and development within MISP, the latest version of the MISP instructional framework was used as the basis for the initial design.

The MISP instructional framework took account of students’ potential starting points in multiplication and division (Gravemeijer et al., 2000; McClain, 2002; Simon, 1995), drawing on research literature on the development of students’ multiplicative reasoning and strategies, as surveyed in Section 2.4, as well as knowledge from the MISP program of low-attaining students’ profiles in additive and multiplicative domains. The instructional framework also had specific instructional goals in terms of the sophisticated multiplicative knowledge identified in research on the multiplicative domain, and aligned with broader principles in arithmetic instruction (P. Cobb & Gravemeijer, 2008; Gravemeijer, 2001; Thompson, 2008). The instructional framework is an imagined plan progressing from the starting points to the instructional goals, in the spirit of a hypothetical learning trajectory (Gravemeijer, Bowers, & Stephan, 2003; Gravemeijer & Cobb, 2006; Simon, 1995).

The initial instructional framework organised the instruction in multiplication and division with domains, phases, and a scheme of ranges, and proposed instructional settings and task types for each phase. In the mode of Gravemeijer’s *bricoleur* designer (1994b, 2001), I proposed modifications to this instructional framework by drawing on a range of sources, including published teaching materials, materials being used in the school site, and tasks that have proved successful in related domains. Aspects of the initial design were speculative, and I anticipated that some would be refuted or revised in the course of the teaching experiments (Confrey, 2006; Gravemeijer, 1994b; Steffe & Thompson, 2000).

Phase 1 of the research also involved preparing the theoretical intent of the study (P. Cobb & Gravemeijer, 2008). As described earlier, I framed the study principally to contribute to local instructional theory. I anticipated that, while the instruction of each student would be individualised, cases in the teaching experiment would illuminate how intervention instruction in multiplication and division can be organised for other students.
Gravemeijer and Cobb (2006) describe how design research can also contribute to broader theoretical issues, and in this light, I became interested in how this research might contribute to our developing notions of learning trajectory and instructional framework. The form of instructional framework being developed here, using dimensions of progression to guide intensive one-to-one intervention instruction, seems a potentially distinctive contribution. While I have not made this broader theoretical issue an explicit research aim of the case study reported here, this intention remained in the teaching experiment overall. I will offer reflections on this issue in the discussion chapter, and address it in subsequent reports.

3.1.5 Phase 2: Teaching experiment

Phase 2 involves conducting the teaching experiments, drawing on and continuing to revise the instructional design. The methodology of the teaching experiment is described below. The procedures for conducting the teaching experiment in the present study are described in Section 3.2.

A fundamental interest of design research is to experiment with a design in the intended authentic setting (Amit, 2010; Brown, 1992; Design-Based Research Collective, 2003; Middleton et al., 2008). Classrooms, and even individual tutorials, are recognised as messy and complex, with the actions of teachers and students highly interdependent. Design research seeks to experiment in the context of that complexity, and find approaches to analysis and design that respect that complexity (Lesh & Sriraman, 2010; Steffe & Thompson, 2000). The present study conducted the teaching experiment as an experimental intervention program within a regular public school, involving two terms of intervention instruction.

Participants in a teaching experiment are selected to match the intended student profile of the instructional design. In this case, I selected low-attaining students who appeared ready to make progress with establishing multiplicative thinking. The student participants were assessed at the beginning and the end of the teaching cycle. As well as informing the teaching (Wright, Martland, Stafford, et al., 2006), the assessments became important data in charting the progression of students’ learning over the course of the teaching cycle (P. Cobb & Gravemeijer, 2008).
In a teaching experiment, the teacher is a central instrument of the research (Steffe & Thompson, 2000), and plays a “central mediating role” (P. Cobb & Gravemeijer, 2008, pp. 70–71) between the instructional design and the students. The teacher’s sensitivity to the students, and to the instructional design, is the basis of the sensitivity of the research when analysing the students’ activity and the instruction. So the pedagogical principles of allowing extended time for the students’ problem-solving, attending to the students’ reasoning, and directing attention towards mathematical sophistication, become important methodological principles as well. This points to the fundamental interrelationship between the research methodology, the instructional design, and the pedagogy (P. Cobb & Steffe; Gravemeijer, 1994b).

The principal data collected are video recordings of the teaching sessions. Assessments, written work, and logbooks are also kept. The intention is to collect a thorough corpus of longitudinal data with which to reconstruct and analyse the instruction (P. Cobb & Gravemeijer, 2008; Shavelson, Phillips, Towne, & Feuer, 2003). The research seeks to investigate beyond students’ behaviour and answers, to conjecture their reasoning and conceptual development, and to understand the interaction between the student and the teaching. For such close analysis, video recording, combined with the assessments and written material, is an appropriate and rewarding form of data (Clement, 2000; P. Cobb & Whitenack, 1996; Lesh & Lehrer, 2000; Powell, Francisco, & Maher, 2003).

In the teaching experiment phase, each lesson can be conceived as a micro-cycle of design, experimental teaching, and analysis (Design-Based Research Collective, 2003; Gravemeijer, 2001; Gravemeijer & Cobb, 2006). In the present study, after each lesson, the teacher-researcher reflected on episodes in the lesson, and made new plans for the next lesson, perhaps to continue with some instruction that seemed to be working, perhaps to try a new idea to attend to a persistent challenge (Simon, 1995). Over the course of three or four lessons, aspects of the initial instructional design could be affirmed, clarified, or modified. These day-to-day and week-to-week cycles are nested within the single large cycle of design, teaching and analysis that comprises the whole study. Gravemeijer and Cobb (2006) highlighted the
interdependencies between the instruction, the instructional design, and underlying instructional theory throughout these micro-cycles. The instruction is informed by the instructional design; however, in turn, the instructional design is clarified and modified based on reflections on the instruction. Similarly, the interpretations of the teaching and learning are informed by the local instructional theory and general instructional principles; these too may in turn be clarified and revised based on reflections on the teaching. With attention to documenting these reflections and refinements, and awareness of the different aspects involved, a great deal can be learned about the instruction.

3.1.6 Phase 3: Retrospective analysis

Phase 3 of the design research cycle, after the teaching experiment is completed, involves an extensive retrospective analysis of the longitudinal data. The analysis of the data needs to be systematic (P. Cobb & Gravemeijer, 2008). The approach to data analysis is informed by the method described by P. Cobb and Whitenack (1996) for analysing longitudinal video data on instruction, which draws on the constant comparative method of Glaser and Strauss (1967). The methodology involves a form of case study, selecting from the extensive data interesting episodes and sequences for closer analysis (P. Cobb & Gravemeijer, 2008; Steffe & Thompson, 2000). The analysis comprises at least two layers: firstly, to investigate the development of students’ reasoning; and secondly, to investigate how the instruction supported students’ learning. Students’ activity influences the teachers’ teaching as much as the reverse, so understanding the instruction involves recognising these reflexive relationships (P. Cobb & Gravemeijer, 2008; Steffe & Thompson, 2000). The analysis is an emergent and iterative process, cycling between conjectures, codes, and transcripts, developing increasingly coherent insights into how the instruction worked and how it might be improved, grounded in exemplar episodes. The analysis culminates in responses to the twin aims of the design study: revisions of the instructional design, and contributions to local instructional theory (McClain, 2002). The details of the analysis process used in the present study are described in Section 3.5 below.
3.1.7 A summary of the methodology

The methodology of the study accords with the five principles of design research developed by van den Akker and colleagues (2006): (1) The method is interventionist, in actively teaching the students, and changing the instructional design, during the study; (2) The method is iterative, through the micro-cycles, the larger cycle of three phases, and the cycles within the retrospective analysis; (3) The method is process-oriented, in using those cycles to understand and improve the instructional design, rather than to merely test or evaluate it; (4) The method is utility-oriented, in seeking to develop an instructional design that works in an authentic school setting; and (5) The method is theory-oriented, by drawing on instructional theory to inform the design, and in turn seeking to contribute to instructional theory.

The rationale for using a design research methodology for the study can be summarised thus. The aims of the study are to refine an instructional design and to develop associated local instructional theory. The instructional design is a substantial framework, not a single aspect or tool. I wish to engage in authentic school settings, and study the interdependence of instruction and learning, in order to inform and change education practice. I also wish to do research that can contribute to broader theoretical issues, such as developing our notions of a learning trajectory or instructional framework. Design research methodology has developed for just such aims and convictions as these (P. Cobb et al., 2003; Gravemeijer & van Eerde, 2009).

3.1.8 Distinguishing from other methodologies

In many respects, the methodology resonates with other qualitative social research methods: the rich data, the use of case study, the interpretive analysis, the interest in understanding and in theory (Flick, 1998; Patton, 2002). The interest in utility is not shared by all qualitative research, but is certainly common: a major concern of Glaser and Strauss (1967), for example, was to develop grounded theory that would work for practitioners. The interventionist approach of the methodology is different from much ethnographic or phenomenological research. The study does not seek to understand instruction and design by observation and introspection alone; rather, it seeks to understand them by changing them (Design-Based Research Collective, 2003; Gravemeijer & Cobb, 2006; Steffe & Thompson, 2000).
The methodology has parallels with action research in education (Kemmis & McTaggart, 1992; Kincheloe, 2003), which is also interventionist, and uses a similar cycle of planning, teaching, and reflecting. That I have been both teacher and researcher in this study also resembles action research. However, action research is centred in a local context, and aims for change for the participants and collaborators in the research. This design research, by contrast, is centred in instructional design, and aims to inform a specific design and related theory. Nevertheless, I have kept an extensive journal throughout the teaching and analysis phases of this study, and the project has been a source of significant reflection and change in my own teaching practice.

3.2 **Teaching Experiment—Organisation**

The data for this study were drawn from a teaching experiment, conducted as a small experimental intervention program for low-attaining students in a primary school. In this section I describe the procedures of the teaching experiment, including the setting, the selection of students, the program schedule, and the collection of data. I leave a description of the instructional design to the next section.

3.2.1 **Roles in the teaching experiment**

I was the teacher-researcher for the teaching experiment, involved in all aspects of the study. I managed the arrangements with the school, administered the assessments, taught the intervention lessons, and collected the digital video recordings and other data. In turn, I made the ongoing analyses of the instruction and revisions to the instructional design.

I was supported at the school by the principal and the Year 5/6 classroom teachers. The teachers helped make the initial selection of participants, and acted as liaisons with the students and parents. In turn I shared with the teachers what I learned about the participants’ mathematics development, through informal discussion and a final formal report.

I was supported in the teaching and analysis by my supervisor. Through each teaching term, we had fortnightly meetings. In the meetings, we reviewed the latest lessons, and made initial analyses of students’ current activity. In turn,
we discussed further instruction, and potential revisions of the instructional design.

3.2.2 School setting
The experimental intervention program was conducted in a public primary school in a regional town. The school had over 300 students from Kindergarten (5-year-olds) to Year 6 (11- and 12-year-olds). The school was chosen as a school that was accessible to the researcher, and was sufficiently large and diverse to have a range of student attainment, including more than six students in Year 5 who could be considered low-attaining. The intervention program ran for two 10-week terms: Term A was term 3 of 2013, and Term B was term 1 of 2014.

3.2.3 Criteria for student selection
The criteria for selection in the program were as follows. I sought Year 5 students who were low-attaining in mathematics, achieving below curriculum expectations. In the experience of MISP intervention in other schools, most Year 3 and 4 students identified as low-attaining still used weak counting-by-ones strategies for simple addition and subtraction, and were unsuccessful with two-digit addition and subtraction tasks. These students typically needed instruction focused on addition and subtraction for at least one term of intervention. For the research on instruction in multiplication and division, I wanted students who had made some progress with non-counting strategies for simple addition and subtraction, so that I could focus instruction on multiplication and division. Also, I did not want to work with students with exceptional difficulties with behaviour or language. Such students would require more time and support than I could accommodate in the study, and at this point in the design research, I was content to develop instruction suited to students with fewer difficulties. Finally, I did not include students with poor school attendance, as I wanted sufficient teaching sessions with each student to genuinely experiment with the instruction. These restrictions on participants were in accord with Siegler’s (2005) recommendation to select students expected to benefit within the time frame of the experiment. While I decided not to include students with exceptional difficulties or disruptions, the instruction and theory I develop may still be of use for teaching such students.
3.2.4 Initial assessments

Six low-attaining Year 5 students were selected for initial assessment, based on their general attainment in number and mathematics in the school, and consultation with the classroom teachers. The students and parents were informed about the program, and gave consent to participate. I then assessed these six students individually.

The main assessments were individual task-based interviews (Ginsburg, 1997), using the assessment method of our prior research and intervention programs, as described in Wright, Martland, and Stafford (2006). The assessments lasted approximately 45 minutes, and were recorded on video. I posed mathematical tasks, and observed student responses. Sometimes, I asked students how they arrived at an answer. The assessment method involved a discipline of neutrality, not indicating whether a student’s answer was correct or otherwise, but maintaining my sole interest in observing and understanding the student’s unassisted mathematical activity. Assessment tasks were drawn from schedules of tasks already developed for MISP, addressing students’ knowledge of five domains: structuring numbers 1 to 20, conceptual place value, addition and subtraction to 100, multiplicative strategies, and multiplication basic facts. Topics addressed are summarised in Table 3.1. The five assessment schedules are included in the appendices.

### Table 3.1 Topics addressed in each of the five assessment schedules

<table>
<thead>
<tr>
<th>Schedule</th>
<th>Topics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Addition and subtraction 1 to 20</td>
<td>Key combinations: doubles, partitions of 10, additions to 10.</td>
</tr>
<tr>
<td></td>
<td>Written addition and subtraction tasks in the range 1 to 20.</td>
</tr>
<tr>
<td>Conceptual place value</td>
<td>Incrementing and decrementing by 10, and by 100,</td>
</tr>
<tr>
<td></td>
<td>in the context of base-ten materials, and in bare numbers.</td>
</tr>
<tr>
<td>Addition and subtraction to 100</td>
<td>Inc-/decrementing two-digit numbers by one-digit numbers.</td>
</tr>
<tr>
<td></td>
<td>Mental computation of two-digit addition and subtraction.</td>
</tr>
<tr>
<td>Multiplicative strategies</td>
<td>Multiplicative tasks in the context of equal groups of dots.</td>
</tr>
<tr>
<td></td>
<td>Multiplicative tasks in the context of arrays of dots.</td>
</tr>
<tr>
<td></td>
<td>Number word sequences by 2s, 3s, and 5s.</td>
</tr>
<tr>
<td></td>
<td>Mental computation of written multiplication and division tasks</td>
</tr>
<tr>
<td></td>
<td>beyond the range of basic facts.</td>
</tr>
<tr>
<td>Multiplication basic facts</td>
<td>Mental computation of written basic facts tasks, with both factors</td>
</tr>
<tr>
<td></td>
<td>in the range 0 to 10.</td>
</tr>
</tbody>
</table>
Most assessment time was devoted to the last two domains addressing knowledge of multiplication and division, as this was the knowledge under study. Other domains were included to make a thorough formative assessment of each student’s number knowledge, to inform the subsequent instruction. In accordance with the assessment method, I did not pose every task on the schedules. My purpose was to assess the range of each student’s current knowledge: I did not pose tasks that I judged to be too easy or too hard to be informative.

As well as the interview assessment, I also assessed each student using the One Minute Tests of Basic Facts (Westwood, 2003). These have a separate test for each of addition, subtraction, multiplication, and division, each presenting a single page of 33 basic facts tasks, with the student to answer as many as possible in one minute. These tests have been used with large cohorts of students, providing statistics of mean score and spread for each age in primary school. A student’s scores indicate the student’s relative computational fluency.

After reviewing the six initial assessments, I needed to select three students for the intervention program. The program was limited to three students by scheduling and resources, and the teaching experiment required a close study of extended teaching with only one or two students to investigate the aims of the design research, so selecting three intervention students was sufficient. One assessed student succeeded on all multiplication basic facts so, while she revealed some confusions in multiplicative tasks, I did not select her for the program. Another student’s knowledge of simple addition and subtraction was still particularly weak, so I did not select him. I made a final selection of three students, based on the criteria explained above, and use anonyms for these intervention students: Blair, Lee, and Hailey.

3.2.5 Intervention schedule

The three intervention students were taught individually in Term A (term 3 2013). They were then in Year 5, all aged 10. Only two of the students could continue in Term B (term 1 2014), when they were in Year 6, and had turned 11. For each student, the program scheduled three 30-minute lessons per week, for eight weeks of each term. Actual lesson lengths and lesson numbers
varied, depending on varying circumstances for the students and the school. The numbers of lessons for each student are shown in Table 3.2. An example of one student’s record of lessons in one term is given in Table 3.3.

Table 3.2  
Number of lessons in each term for each student

<table>
<thead>
<tr>
<th>Student</th>
<th>Term A lessons</th>
<th>Term B lessons</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blair</td>
<td>20</td>
<td>24</td>
</tr>
<tr>
<td>Lee</td>
<td>20</td>
<td>23</td>
</tr>
<tr>
<td>Hailey</td>
<td>17</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 3.3  
Record of Blair’s lessons in Term B

<table>
<thead>
<tr>
<th>Week</th>
<th>Week 2</th>
<th>Week 3</th>
<th>Week 4</th>
<th>Week 5</th>
<th>Week 6</th>
<th>Week 7</th>
<th>Week 8</th>
<th>Week 9</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10min</td>
<td>30min</td>
<td>60min</td>
<td>90min</td>
<td>120min</td>
<td>150min</td>
<td>180min</td>
<td>210min</td>
</tr>
<tr>
<td>2</td>
<td>40min</td>
<td>45min</td>
<td>50min</td>
<td>55min</td>
<td>60min</td>
<td>65min</td>
<td>70min</td>
<td>75min</td>
</tr>
<tr>
<td>3</td>
<td>45min</td>
<td>50min</td>
<td>55min</td>
<td>60min</td>
<td>65min</td>
<td>70min</td>
<td>75min</td>
<td>80min</td>
</tr>
<tr>
<td>4</td>
<td>30min</td>
<td>40min</td>
<td>50min</td>
<td>60min</td>
<td>70min</td>
<td>80min</td>
<td>90min</td>
<td>100min</td>
</tr>
</tbody>
</table>

Lessons were conducted in a spare classroom adjacent to other classrooms. Students left their regular maths class for the period of their intervention lesson; otherwise they continued with regular class routine.

The intervention students were assessed again three times: at the end of Term A, the beginning of Term B, and three weeks after the end of Term B. These assessments were largely parallel to the initial assessments: individual task-based interviews, drawing from the same schedules of assessment tasks, recorded on video. The tasks posed in the four assessments were mostly the same, but a few tasks were different, to respond to the students’ changing knowledge. The One Minute Tests of Basic Facts were also administered again in the final assessment.

3.2.6  
Data collection

All the lessons and assessment interviews were recorded on video, with the frame capturing the teacher, student, and desk (see Figure 3.1 showing arrangement of desk and video camera). The recordings were stored as a database of digital video files, totalling approximately 75 hours. The video recordings constituted the principal data of the study.
I maintained a file of lesson plans for each student, and an attendance roll of all assessments and lessons. All writing during the lessons, by the teacher and by the students, was kept in student workbooks, with each entry dated. Finally, I maintained an extensive logbook of notes written outside lesson times, totalling over 30,000 words, writing routinely on:

- lesson planning;
- immediate observations and reflections after lessons;
- ongoing analysis of students’ learning; and
- ongoing revisions to the instructional design.

### 3.2.7 Intervention instruction

The intervention teaching followed a basic approach in keeping with MR and MISP practice. Lessons were planned, offering inquiry-based tasks in number, often involving simple settings like ten-frames and n-tiles. The instruction was highly responsive to the student’s activity, adjusting tasks and sequences of tasks to challenge the cutting edge of each student’s developing knowledge. The principles of the instructional approach were discussed in the review of literature, Section 2.2.

Lessons were planned to address three to five topics in number instruction. I tended to persist with the same topics over several lessons, but over time, some topics tapered out and others were introduced, so that over the whole two terms, there was an overlapping progression of several topics. The three intervention students were not facile with addition and subtraction, still using
counting-based strategies for two-digit and for some one-digit tasks. As suggested in the review of literature, students’ multiplicative strategies depend on knowledge of two-digit addition and subtraction, so I thought it important to include instruction to improve this knowledge. In the first half of Term A, instruction focused on additive topics. In the second half of Term A, I continued with additive topics, while introducing some early topics in multiplication and division. Term B was devoted entirely to multiplicative topics, particularly to the development of multiplicative strategies for basic facts tasks. Figure 3.2 indicates this progression of topics over Terms A and B.

**Figure 3.2 Progression of domains of instruction over Terms A and B**

<table>
<thead>
<tr>
<th></th>
<th>Structuring numbers 1–20</th>
<th>Higher decade add &amp; subtract</th>
<th>Conceptual place value</th>
<th>Skip-counting</th>
<th>Low-range basic facts (BFs)</th>
<th>Strategies for mid-range BFs</th>
<th>Strategies for high-range BFs</th>
<th>Further topics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Term A early</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Term A late</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Term B early</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Term B late</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The instruction in these topics drew on the instructional designs already developed in MISP intervention programs, as described in Wright, Ellemor-Collins, and Tabor (2012). The design for the multiplication and division topics is described in the next section. During the course of the teaching terms, I continued to adapt and revise the instruction. Each individual lesson was planned, taught, and then reviewed. These daily mini-cycles were central to the design research approach. These cycles were informed by ongoing attention to the student’s thinking; by the initial instructional design; by overarching instructional aims and principles; and by instructional theory, especially local instructional theory about multiplication and division.

### 3.3 Teaching Experiment—Instructional Design

Within MISP, we had already experimented with teaching multiplication and division, and developed an instructional framework and materials for this domain. The initial instructional design for the present teaching experiment
was based on that existing instructional framework, in keeping with the design research methodology. The purpose of the teaching experiment overall was essentially to experiment with this design, in order to better understand it, and to refine it.

3.3.1 Initial instructional design

The existing design was described in two documents: Chapter 7 of the Wright, Ellemor-Collins, and Tabor text (2012); and a pair of MISP teaching charts, labelled 3E and 3F, (included in the appendices). I summarise the design below.

The starting points for the instruction were students’ early counting-based strategies in the context of counters and related settings. The instructional aims were to develop sophisticated multiplicative reasoning, efficient multiplicative strategies extending beyond the range of the basic facts, and fluent automatised knowledge of the basic facts.

Instruction in the domain overall was organised into six phases, shown in Table 3.4 (Wright et al., 2012, p. 149). To organise instruction within the phases, a scheme of four ranges was also used, as shown in Table 3.5.

<table>
<thead>
<tr>
<th>Table 3.4</th>
<th>Initial instructional framework: six phases</th>
</tr>
</thead>
<tbody>
<tr>
<td>Phase 1</td>
<td>Building on students’ emergent strategies for multiplying and dividing.</td>
</tr>
<tr>
<td>Phase 2</td>
<td>Instruction on sequences of multiples (skip-counting).</td>
</tr>
<tr>
<td>Phase 3</td>
<td>Structuring numbers multiplicatively.</td>
</tr>
<tr>
<td>Phase 4</td>
<td>Developing multiplicative strategies for one-digit factors.</td>
</tr>
<tr>
<td>Phase 5</td>
<td>Habituation of basic facts for multiplication and division.</td>
</tr>
<tr>
<td>Phase 6</td>
<td>Extending multiplication and division to multi-digit factors.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 3.5</th>
<th>Initial instructional framework: scheme of four ranges</th>
</tr>
</thead>
<tbody>
<tr>
<td>Range 1</td>
<td>Low × low: Both factors in the range 2–5</td>
</tr>
<tr>
<td>Range 2</td>
<td>1s, 10s, high 2s, and high 5s</td>
</tr>
<tr>
<td>Range 3</td>
<td>High 3s and high 4s</td>
</tr>
<tr>
<td>Range 4</td>
<td>High × high: both factors in the range 6–9</td>
</tr>
</tbody>
</table>
The scheme of ranges divides the numbers 1–10 into the low factors, 1–5, and the high factors, 6–10. Sets of multiples can likewise be divided into low multiples and high multiples. For example:

the low 5s are: \(5 \times 1, 5 \times 2, 5 \times 3, 5 \times 4, \) and \(5 \times 5\);

the high 5s are: \(5 \times 6, 5 \times 7, 5 \times 8, 5 \times 9, \) and \(5 \times 10\).

The focus of intervention instruction was expected to be Phases 3, 4, and 5: the development of multiplicative structuring, multiplicative strategies, and habituation, for the basic facts. Phase 3, structuring numbers multiplicatively, was intended to develop knowledge of multiplicative relations within specific sets of multiples, such as the multiples of 5. An exemplar task type in Phase 3 was called *incrementing and decrementing with 5-tiles* (a 5-tile is a card with a row of 5 dots). The teacher places, say, four 5-tiles, totalling 20 dots. After screening the tiles, he adds two tiles, and the student is to say both the total number of dots (30) and the number of 5-tiles (6). The teacher can continue to increment and decrement numbers of tiles, with the student tracking the total and the number of tiles. These tasks are intended to develop awareness of relations in the set of multiples of 5.

Instruction in Phase 4, developing multiplicative strategies, also focuses on specific sets of multiples, working through Ranges 1 to 4. An exemplar task type involves posing multiplicative tasks with arrays of dots, and as the student answers, inquiring about the student’s strategies. Over a sequence of tasks, the teacher seeks to develop informal notations for the strategies, creating opportunities to develop student awareness of strategies, and to refine strategies for whole sets of related multiples.

Once effective strategies have been developed, instruction enters Phase 5, habituation of basic facts. Phase 5 instruction works through Ranges 1 to 4 again, using common techniques to habituate sets of facts, such as daily rehearsals in answering sets of cards showing tasks in standard expressions. Division basic facts are included as well as multiplication basic facts.

The instructional design included suggestions of the mathematical dimensions of instructional progression in these phases. The dimensions were indicated in the two teaching charts: Chart 3E included Phases 3 and 4, while Chart 3F
included Phase 5. The dimensions were different within each phase, and included:

- progressions through ranges;
- distancing the setting through screening;
- formalising the presentation of tasks;
- making tasks more arithmetically complex;
- developing notations;
- refining strategies.

3.3.2 Design revisions during the teaching experiment

I modified and refined the instructional design during the teaching experiment, with modifications on a range of scales, from the broad organisation of topics down to fine details in the handling of particular notations. These modifications were of interest for the broader project of design research, of course, but a detailed account of them is unnecessary for the present study. Here, I will describe the three significant modifications that formed the basis of the sequence which became the focus case for the present study.

Firstly, I found the instruction more clearly organised by range, rather than by phase. Early lessons were planned with segments on different phases, but I found instead a need to organise segments on different ranges, for example: five minutes rehearsing knowledge of Range 1 and 2 facts, then ten minutes refining strategies for a set of Range 3 facts, and finally five minutes on initial strategies for some Range 4 facts.

Secondly, within each range, I would work on the Phase 3, 4, and 5 aims as intended—multiplicative structuring, development of strategies, and habituating basic facts—but I found the distinctions between these phases less compelling in actual instruction. Structuring and strategy development, in particular, seemed to be largely intertwined.

Thirdly, I reorganised the scheme of ranges. The new scheme is shown in Table 3.6. One change was to make the multiples of 2 and of 10 the first range. These multiples are largely translated from known additive facts. A second change was to move high 5s to later, in Range 3, along with high 3s
and 4s. Further, I recognised that each of the high 3s, high 4s, and high 5s might warrant their own attention in the development of structuring, strategies, and habituation. This is the scheme of ranges to which I will refer in the remainder of this dissertation.

<table>
<thead>
<tr>
<th>Range 1</th>
<th>Multiples of 2 and multiples of 10.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Range 2</td>
<td>Low × low: Both factors in the range 2–5</td>
</tr>
<tr>
<td>Range 3</td>
<td>High 3s, high 4s, and high 5s.</td>
</tr>
<tr>
<td>Range 4</td>
<td>High × high: both factors in the range 6–9</td>
</tr>
</tbody>
</table>

With these revisions in place from early in Term B, I ended up pursuing a sequence of lesson segments on each of the new ranges. Instruction within those ranges had a strong sense of progression, moving from development of structuring and strategies toward habituation. But trying to describe this progression in terms of three phases was less convincing. I became interested in how to describe the progression of instruction within these range-focused sequences, in terms of dimensions of progression. The study presented here is a close study of one such sequence with one student.

### 3.4 Case Study

The extensive teaching experiment over two terms provided an opportunity to investigate several issues in instructional design for intervention in multiplication and division, beyond the focus of the present study. In the concluding chapter, I will touch on some of these issues in suggesting directions for further research. Meanwhile, the particular aims of the present study were to characterise an instructional progression toward multiplicative strategies, in terms of a few key mathematical dimensions of progression. To this end, I sought a case study within the teaching experiment, of successful progression toward multiplicative strategies.

#### 3.4.1 The case of Blair’s High 5s Sequence

I selected as a case the instruction with one student, Blair, to develop multiplicative strategies for multiples of 5, over segments of eight lessons in Term B. The instruction in the case focused in particular on developing Blair’s knowledge of what I called the high 5s: the set of products 6×5, 7×5,
8×5, 9×5, 10×5, and their commutations and corresponding quotients. So I refer to the collection of lesson segments addressing this topic as Blair’s *High 5s Sequence*. Furthermore, within the High 5s case, I selected three segments of one lesson with Blair, Lesson 10, for a closer analysis of finer-scaled adjustments in the instruction. I labelled this the *Lesson 10 Sequence*. Thus, the *Lesson 10 Sequence* was nested within the *High 5s Sequence*. Figure 3.3 shows a summary of Blair’s instruction in Term B, with the segments of the High 5s Sequence boxed in red, and the Lesson 10 segments circled.

This thesis is principally a close analysis of the Blair’s High 5s case. In Chapter 5 I will give an overview of the High 5s Sequence and of Blair’s progress with high 5s. Subsequent chapters present the rich detail of the analysis. In the remainder of this section, I explain the rationale for the selection of the case, as part of explaining my research method.

**Figure 3.3 Lesson segment topics over Blair’s 24 lessons in Term B, highlighting the High 5s Sequence and Lesson 10 Sequence**

![Lesson segment topics over Blair’s 24 lessons in Term B, highlighting the High 5s Sequence and Lesson 10 Sequence](image)
3.4.2 Selecting the case study

The case was selected as the clearest and most complete example of instruction with a student progressing from counting-based strategies to multiplicative strategies. Over the course of Term B, Blair made significant progress across many aspects of multiplication and division. However, from the beginning of the intervention, he already had some knowledge of small basic facts, and multiplicative strategies for some multiples of 3 and 4. With high 5s tasks, on the other hand, he appeared initially to be limited to skip-counting-based strategies. Over the course of the instruction in Term B, Blair developed multiplicative strategies for high 5s tasks, and achieved considerable fluency by the end of the term. His knowledge of basic facts with higher factors, such as 6×7, 6×9, 7×8, also improved from initial counting-based strategies; however he had not reached the same fluency with these tasks by the end of the intervention. Thus, Blair’s progression with high 5s provided a unique opportunity to study the complete progression from a reliance on counting-based strategies to successful multiplicative strategies.

The high 5s also emerged in the instruction as a coherent topic of instruction. I can clearly identify lesson segments focused on the high 5s, and trace Blair’s progress with insights and strategies specific to the high 5s. So Blair’s High 5s makes a coherent case to extract for closer analysis, from amidst the whole teaching experiment.

The analysis of teaching experiments commonly involves identifying several distinct key episodes, which illuminate the issues under investigation. This study aims to characterise instructional progressions: how instruction can adjust and respond over sequences of tasks and lessons. I found it was not helpful to select separate episodes: instead, I needed an extended sequence of instruction, in order to document the twists and turns that are so characteristic of instructional progression. I settled on making a detailed case study of this one, relatively complete, sequence, the High 5s Sequence. I expect the instructional progression in the high 5s to inform the progression in developing strategies for other families of basic facts. While particular number relations and computation strategies may differ, the dimensions of instructional progression revealed for the high 5s can inform the dimensions of progression across multiplication and division more generally.
The selection of a case that can address the research aims accords with a methodology in qualitative research of purposeful sampling: selecting cases because they are rich and illuminating (Patton, 2002; Strauss & Corbin, 1998). In particular, teaching experiment methodology makes a practice of finding clear and successful cases to study, which show a viable approach to instruction (Steffe & Thompson, 2000). The analysis of such cases does not seek to compare or evaluate instructional designs. Rather, I seek to better understand how instruction can support students’ learning, and to develop a coherent, viable instructional design (P. Cobb & Gravemeijer, 2008). The analysis of the case of Blair’s High 5s serves this purpose well.

The future investigation of my research aims does not need to be limited to this case. My first recommendation for further research is to investigate my proposed characterisation of the instructional progression in other cases, even drawing from this teaching experiment. What I contend, as the basis of the present study, is that this case study alone is already a coherent and useful contribution to the investigation of these aims. It has proven a substantial analysis, even on an apparently small slice of the teaching experiment. I have found it a neat, significant study to which to devote my thesis.

3.4.3 Depth and breadth
I find it instructive to consider how this choice of case study pursues both depth and breadth of analysis. The case study pursues depth of analysis by focusing in minute detail on a small selection of instructional episodes. I am trying to understand responsive instructional progression by looking at one progression very closely. On the other hand, I am trying to attend to both learning and teaching, and to several strands in both: computational strategies, structuring of numbers, knowledge of basic facts, use of settings and tools, use of notations, and so on. Each of these strands could be analysed separately. By trying to address all these strands together, I am pursuing a breadth of analysis as well as a depth of analysis. The breadth involves treating all of the instructional progression as a coherent whole phenomenon, weaving together each of its dimensions and strands. The depth involves focusing on only a relatively small example of instruction, so I can capture sufficient detail and insight across that breadth.
Adding it Up (National Research Council, 2001), a major US report on the state of mathematics education research at the turn of the millennium, offered a significant conclusion:

Although much is known about characteristics of effective instruction, research on teaching has often been restricted to describing isolated fragments of teaching and learning rather than examining continued interactions among the teacher, the students, and the mathematical content. (p. 9)

The combination of depth and breadth in the present case study is a response to this research need: I examine a sequence of continued interactions among a teacher, a student, and the mathematics of multiplicative strategies.

3.5 Data Analysis
The methods of data analysis were based in the teaching experiment methodology, and drew on coding and constant comparison methods common to many qualitative methodologies. The data analysis was an emergent process (Patton, 2002; Strauss & Corbin, 1998), with key analytic questions and analytic tools developed as part of the analysis process. Below I outline how the data analysis developed over four phases. Then I describe the scheme used for segmenting the data. Finally, I describe the data analysis methods in terms of ten aspects.

3.5.1 Four phases of analysis
Following the teaching experiment, the data analysis proceeded over an extended period. In retrospect, I can outline the process in the following four phases.

3.5.1.1 Phase 1: Preliminary transcriptions, identifying issues and cases
The initial phase of analysis involved developing lesson summaries, and overviews of the instruction of each student. At the same time, I sought episodes of interest, and completed preliminary transcriptions of many episodes. I generated memos on many issues that emerged from initial partial analyses, and proposed early responses to some of my research questions.

Goodness, I’m beginning to swim with ideas I want to write about.
This is a good long term sign. (Research journal, 16/6/2014)

I moved towards a focus on the particular research aims of this study, and identified a case study: Blair’s High 5s.
3.5.1.2 Phase 2: High 5s Sequence case study
In the second phase, I completed transcriptions and summaries of all the lesson segments in the High 5s Sequence. I developed a detailed account of the progression of Blair’s thinking over the sequence. I also developed a partial account of the instructional progression in high 5s, and an improved characterisation of the dimensions of progression involved.

Now, bringing this research question back to my 5s case study. I have refreshed myself with the puzzles I had in creating a map of the progressions, back in August. I certainly think I can get clearer with how that map should go for the 5s. So that’s what I should press ahead to do. (Research Journal, 15/12/2014)

During this phase, I decided I wanted a closer study of one lesson to clarify my understanding of the instruction, and identified the Lesson 10 segments as a case for that study.

3.5.1.3 Phase 3: Lesson 10 Sequence case study
In the third phase of analysis, I became intimately acquainted with the three high 5s segments of Lesson 10. I developed my practice of layered, multidimensional analysis, and completed a thorough, detailed account of progressions in Blair’s activity and progressions in my instruction over the Lesson 10 Sequence. During this phase, I settled on the research aims of this study, and on my main conclusions.

So far for Lesson 10 I have done Level B in segment chunks; then segment sections on thinking with Levels C and D woven together somewhat; and Level E as final paragraph of that section. (Research Journal, 20/8/2015)

3.5.1.4 Phase 4: Integration and writing
When I completed the case study of the Lesson 10 Sequence, I returned to analysing the whole High 5s Sequence. I brought my clearer research aims and proposed conclusions to the high 5s material; I also brought my clarified practice of layered analysis to organise the study.

To write up Range, and then Setting, I’ve found myself going back through my older notes, and the big chart…it feels like primary analysis again. Discussion ideas turn up: new ways of organizing, creating coherence across the analysis between Lesson 10 and the whole High 5s Sequence. (Research Journal, 18/5/2016)

This final phase became a period of refining and integrating my findings, and of organising the writing of analysis and discussion chapters.
3.5.2 Scheme for segmenting records of instruction

Analysis was organised using a simple scheme for segmenting records of instruction into manageable units: lessons, lesson segments, and tasks. The scheme is described below, with reference to the compound Figure 3.4, which illustrates the nesting of the units.

3.5.2.1 Lessons

The lesson is a natural unit: the period spent with one student in one day, usually 15–45 minutes long. Lessons were numbered in sequence through the term. The data analysed in this study is drawn from the 24 lessons with Blair in Term B, over nine weeks. Figure 3.4A shows a record of the first 11 lessons.

3.5.2.2 Lesson segments

Lessons were divided into lesson segments. A lesson segment is a passage of a lesson consisting of a series of tasks of similar type, addressing the same basic topic. For example, see Figure 3.4B showing the segmenting of Blair’s Lesson 10. Lesson 10 was organised into seven segments: Segment 10.1 was about two minutes, 10.2 was about three minutes, and so on. Lesson segments range from about 2 to 15 minutes in length; lessons typically comprise three to six segments (Lesson 10 was an unusually long lesson).

Each lesson segment is devoted to a single topic. For example, in Lesson 10, the last three segments all address the same basic topic of high 5s products. They are divided into different segments, because they each pursued a different task type. A task type is a specific procedure for posing arithmetic tasks. For example, Segment 10.6 involved a type of division task, where the teacher places some number of 5-tiles behind a screen—say, eight 5-tiles—says the total number of dots—40 dots—then asks the student to calculate the number of 5s—eight. A task type can generate different particular tasks, by varying the numbers involved. I labelled 10.7 as a new segment, because the task type shifted at this point, to involve posing multiplications as well as divisions, and also to involve recording results.
Figure 3.4  Segmenting instruction into lessons, segments, and tasks
Lesson segments and task types were already basic constructs in the instructional approach, and my lesson plans involved essentially planning lesson segments with topics and task types. Nevertheless, given that lessons varied from plans, and some tasks evolved spontaneously during lessons, I needed to clarify the segments and task types for each lesson during the data analysis.

### 3.5.2.3 Tasks
Lesson segments can be divided into tasks. Figure 3.4C shows Segment 10.6 divided into five tasks: 10.6.1–10.6.5. Figure 3.4D presents the transcript for two of those tasks, 10.6.2 and 10.6.3. Each task is essentially a passage in which the teacher poses a single task of arithmetic, and the student responds. Tasks can vary considerably in duration. Some tasks consist simply of one question by the teacher, one answer by the student, and one moment of checking the answer, as in the example transcript for Task 10.6.2. However, more commonly there are further exchanges, as the teacher enquires about the strategy, or the student checks his answer, and so forth, as in the example transcript in Task 10.6.3. These exchanges can involve the teacher adjusting the task, by for example rephrasing the question, or by lifting a screen to make a material setting visible to the student. These adjustments are considered still part of the one task. If the teacher changed the numbers involved in the arithmetic, then this was generally considered a new task. Occasionally, the instruction pursued a lengthy sequence of follow-up questions, or a sequence of confused answers and adjustments. Pragmatic decisions were made about subdividing tasks, to be amenable for referencing passages and summarising the lesson segments. I have not numbered lines of transcript. I have found that the task numbering is sufficient reference for moments and quotations from the transcript.

Lesson segments and tasks became the basic organisers for all the data analysis. The High 5s Sequence is a study of a selection of lesson segments all addressing the same basic topic, involving a handful of task types that evolved through that sequence of segments. The Lesson 10 Sequence is a study of the sequence of tasks from Segments 10.5, 10.6, and 10.7, including close analysis of instructional adjustments within as well as between those tasks.
3.5.3 Aspects of analysis

I will describe the analysis process in ten aspects. These include:

- aspects creating records of what happened in the lessons:
  - developing lesson summaries,
  - transcribing episodes,
  - developing summary charts;
- aspects for identifying episodes to study:
  - defining the unit of analysis,
  - selecting cases for study;
- core aspects of analysing the cases:
  - coding and refining concepts,
  - charting progressions,
  - layering the analysis;
- and aspects of refining the research questions and answers:
  - refining the study aims,
  - refining theory and design.

These ten aspects share similarities with the four-phase model for analysis of video data described by P. Cobb and Whitenack (1996), and the seven-phase model described by Powell, Francisco, and Maher (2003).

These aspects of analysis did not proceed stepwise; rather, the analysis work involved moving back and forth between the aspects, and progressing all the aspects together over time. Each of the four phases in the analysis involved most of the aspects, but the approach to each aspect became clearer and more focused as the phases progressed. While the analysis was emergent and recursive in this way, it was rigorous, as it followed the principles of engaging the emerging design and theory with the data (Flick, 1998; Glaser & Strauss, 1967).

3.5.3.1 Developing lesson summaries

A basic step in analysis was to generate a lesson summary of each lesson. Each lesson summary was presented in a single page, and divided the lesson into lesson segments. For each segment the summary indicated what main topics were addressed, what task type was used, an abbreviated list of the
tasks posed, and what kinds of responses the student gave. Typically there was room to mark significant moments of interest.

Lesson summaries were developed using the lesson plans and post-lesson notes written during the teaching terms, and reviews of the video recording of each lesson. Most lesson summaries were created in the first phase of analysis. Summaries of key lessons were refined and extended in later phases.

In the process of creating lesson summaries, I identified the recurring task types used. Once task types were identified, summaries and charts and accounts of the instruction were simplified by referring to task types, rather than needing to describe the task procedures each time. A description of the task types relevant to the present study is included in Section 4.4.

3.5.3.2 Transcribing episodes
A second basic step in analysis was to transcribe the activity in the video recordings. Transcribing all the recordings in full would have been neither practical nor helpful. Instead, I pursued transcriptions at three different levels of detail (Flick, 1998; Pirie, 2001; Strauss, 1987).

A basic level was task transcription. At this level, I would list each of the arithmetic tasks posed in the episode, and typically indicate the student’s success on each task with ticks, crosses, and other simple codes familiar from our video-based assessments. Task transcriptions provided a useful overview of what instruction was pursued in each lesson. These were only a level more detailed than the lesson summaries.

A richer level of transcription I called preliminary transcription. At this level, I would try to note what was said and done by both teacher and student, without requiring every word spoken. I used preliminary transcriptions of many episodes in the initial phase of analysis, which helped me identify episodes and cases of interest. Likewise, I completed preliminary transcriptions of all relevant segments in the high 5s case, as a first entry into the study of that case.

The third level of transcription was a full transcription, aiming to record words and actions thoroughly. All lesson segments used in the final written analysis in the thesis were fully transcribed.
The transcriptions were written in two columns, one for the teacher, one for the student, so simultaneous activity could be recorded, and the progression of activity in each actor can be readily followed. The transcriptions also had regular time-codes for easy reference to the video, and room for extensive memos about any moment.

### 3.5.3.3 Developing summary charts

As I was interested in tracking progressions in instruction over several lessons, I needed more condensed views of several lessons. In the initial phase of analysis, I created weekly summaries. These condensed the lesson summaries by about a third, so I could fit three lessons from one week onto a single page, and thus summarise a whole term of teaching in ten pages. The weekly summaries were important as a kind of index for locating which episodes happened in which lessons; they also helped trace longer sequences, including identifying the sequence of high 5s segments that became the case study.

In the High 5s Sequence analysis phase, I developed an instruction summary chart, which gave a brief list of tasks and responses in each segment of the sequence. In turn, in the Lesson 10 Sequence analysis phase, I developed an instruction summary chart for the Lesson 10 Sequence, which is more fine-grained, summarising task-by-task, rather than segment-by-segment. These instruction summary charts became central references for the subsequent analysis, and the basis of the instructional analysis charts described further below.

### 3.5.3.4 Defining the unit of analysis

The simplest unit of analysis was the task. For each task, I was tracking exactly what task was posed by the teacher, including uses of settings, notations, choices of numbers and so on; and in turn exactly how the student responded to the task. However, I sought to analyse instructional progressions: not to describe what happened in any single task per se, but to describe how the instruction adjusted to pose the next task, and adjusted again to pose the task after that. Thus, the true unit of analysis became sequences of tasks. At the same time, the study became characterised by analysis at different scales: at times, I tracked the micro scale of moment-to-moment adjustments within single tasks, and at other times I tracked the broader scale...
of segment-to-segment adjustments over several lessons. Keeping the analysis coherent across different scales and different sequences was a challenge, managed through the selection of case studies at two different scales, and the scheme of analysis layers, which are described further below.

3.5.3.5 Selecting cases for study

A basic technique in the teaching experiment methodology is to seek interesting cases to study (Steffe & Thompson, 2000). Without claiming generality, we find that the study of successful, viable cases of instruction contributes to instructional design and theory. So, from the beginning of the analysis of data, I was interested in identifying successful periods of instruction. I compared initial and final assessments to identify ways each student had progressed. I also reviewed episodes I had marked as breakthroughs during the teaching experiment. For a time I sought to assemble a series of brief key episodes. I also made initial studies of aspects of one other student, and of two other instructional topics. As I clarified my research focus on progressions, I recognised the need to study an extended sequence, where many aspects of progression could be tracked within the single sequence. So I sought an explicit case study, progressing from non-multiplicative strategies to successful multiplicative strategies, and decided Blair’s instruction in High 5s was the clearest example.

Likewise, in the course of the high 5s study in the second analysis phase, I decided I needed a chance to look closely at task-by-task progressions, and sought single lesson segments for this case study within a case study.

I remain very tempted coming back to the idea of doing a case study of one lesson segment, in depth. That could be nested within this [High 5s Sequence]. That’s a way to then draw on a much more closely argued episode as well. (Research Journal, 11/12/2014)

I settled on using the three high 5s segments of Lesson 10, which included moments of both insight and confusion for Blair, and a range of instructional adjustments, within a continuous sequence of instruction.

During the third and fourth phases I continued to clarify, within these selections of cases, the clearest beginning and end for each case. For example, the end of Blair’s High 5s Sequence was not clearly determined, as high 5s
material continued to arise in lessons until the end of Term B. During analysis in the fourth phase, I decided to end the case study sequence at Lesson 17, because later lessons gave less focus, emphasis, and time to high 5s instruction. So this aspect of selecting case studies, like the other aspects of analysis, continued in some form to be part of the analysis process to the end.

3.5.3.6 Coding and refining concepts

In the first phase of analysis, and into the second phase, I pursued an open coding (Flick, 1998) of the transcripts and summaries, that is, marking and annotating a range of points of interest in the student’s activity and in the teacher’s activity. Examples of student codes are: “uses skip-counting”, “notices a relationship”, and “initiative with new task”. Examples of teaching codes are: “uses screening”, “asks about strategy”, and “retreats challenge”.

As research focus clarified around characterising the instructional progression in the Blair High 5s case, the coding began to narrow toward key threads in Blair’s learning of high 5s, and key aspects of adjustment and progression in the teaching. In particular, I pursued a more selective coding for the dimensions of progression involved in instruction. I tried coding dimensions in tasks, and also coding at the broader scale of dimensions in each segment. Initially I had more than five dimensions proposed. For a while I also distinguished major and minor dimensions: for example, changes in the number size was a major dimension, whereas changes from even to odd multiples was a minor dimension. Extensive memos were kept on each proposed dimension, with links to the many coded episodes.

Before I put the chart aside, I wanted to think further about the dimensions in it. Ended up checking through all 10 dimensions of the dimensions framework, and getting straight which ones I might want to indicate in the progressions chart. (Research Journal, 22/8/2014)

A central task of the whole analysis process became the refining of the dimension concepts. Refinement was driven by a form of the constant comparative method (P. Cobb & Whitenack, 1996; Glaser & Strauss, 1967). Ideas for dimensions would emerge from some episodes, and when compared with other episodes, those ideas would develop and clarify. Concept refinement involved winnowing—for example, letting go of distinguishing
major and minor dimensions. Refinement also involved combining—for example, aspects of instruction I had been coding as “formalising” were absorbed into the dimension “notating”. The development of the progressions chart (the next analysis aspect described) became central in refining the dimensions. By the middle of the third phase, the Lesson 10 case study, I had settled on the final five dimensions. Through the Lesson 10 analysis, and especially through the fourth phase analysis linking from Lesson 10 back to the whole High 5s Sequence, I identified distinctions between the dimensions, and interactions between the dimensions, and deepened my use of the dimension concept in general. The process overall of coding and refining concepts was in accord with Flick’s (1998, p. 542) account of qualitative data analysis, that repeated coding of data leads to denser concept-based relationships and hence to theory.

While the concept-refining method used was in accord with the constant comparative method first described by Glaser and Strauss for developing grounded theory, it is worth clarifying that I was not developing grounded theory. In particular, I did not arrive at the data without prior theory and concepts in hand, as Glaser and Strauss require. Instead, I was developing instructional theory in accord with the methodology of design research. I began with an initial design, and with instructional theory about potential key dimensions. These informed both my teaching and my analyses of the data. The development of the dimensions and local theory reported here is best understood as a cycle within multiple design cycles.

3.5.3.7 Charting progressions

In an earlier study analysing dimensions of instruction in early addition and subtraction (Ellemor-Collins et al., 2013), I developed a chart which summarised adjustments in three dimensions over a series of lessons. The chart proved illuminating of patterns in the use of the dimensions, and became a central tool of analysis and presentation in that case study. In the first analysis phase of the present study, I sought again to create a chart to illuminate adjustments and progressions in dimensions of instruction. I was not immediately successful. Coming to understand why this study was harder to chart than the previous study became part of the research work. In retrospect, I can list the following initial difficulties: the case was not as well
defined; the progression of learning was more convoluted and involved a more significant shift in mathematical sophistication; the progression of instruction was, similarly, more convoluted; there were more dimensions involved; and I knew less about those dimensions.

As my research aims focused toward describing dimensions of progression in the second phase of analysis, coding of dimensions and memos on dimensions became insufficient to reveal the progressions I sought. The need for a chart to track several dimensions simultaneously became pressing:

I try picturing again some chart that tracks movements on the dimensions. There will have to be such a chart, surely. And a rough version of that chart could be what I want for my research searching. (Research Journal, 4/2/2015)

Over time, I developed the instructional analysis charts. The analysis charts were built off the instruction summary charts. A final version of the instructional analysis chart for the Lesson 10 Sequence is included in Section 5.1, and the chart for the High 5s Sequence is in Section 7.1. A simplified portion of the Lesson 10 chart is shown in Figure 3.5, as an example. Each row of the chart presents a task and the analysis of that task. Left-hand columns summarise each task and response. Five coloured columns in the middle track the five dimensions of instruction, each column indicating for each task whether the dimension is involved in the instruction, and whether the dimension has advanced or retreated from the previous task. The right-hand column tracks key strands in the student’s learning. The instructional analysis charts enabled me to track the progression in each dimension separately running down each column; to examine interactions between the dimensions across each row; and to relate instructional adjustments in the middle columns to the student’s activity in the right-hand column.

Figure 3.5  Example portion of an instructional analysis chart
I sought ways to use visual codes, such as graph lines, symbols, and colours, to help reveal what was happening in the dimensions. Deciding what to track and creating effective codes required clearly defined dimensions and dimension attributes. So, I needed to define the dimensions in order to set up the analysis charts. On the other hand, I wanted to use the analysis charts to help the research define the dimensions. Given this familiar circularity, the data analysis process in the third and fourth phases can be understood as a series of cycles which involved revising the analysis charts then revising the dimensions in turn. The analysis charts became the central references for the data analysis.

3.5.3.8 Layering the analysis
The study aimed to characterise the instruction, and by this I increasingly meant the progression of the instruction in the High 5s case. In accord with design research methodology, there were two basic layers to this analysis: first, to develop an account of the student’s learning, and second, to develop an account of the instruction (P. Cobb & Gravemeijer, 2008). However, as I worked on these accounts for the High 5s case, I realised there were more layers in my approach. I sought a multidimensional account of the instructional progression, but I needed to analyse the dimensions of the instruction separately, to better understand each of them, before I could characterise their interaction. By the third phase of analysis, I had developed an explicit practice of analysis involving six layers. Figure 3.6 represents these six layers of analysis.

Layer A: Observation. The disciplined practice of transcribing and summarising what can be observed in the teacher and student activity.

Layer B: Local interpretation. Interpreting what is observed in an episode as a meaningful instructional exchange, in terms of basic categories of interest: what dimensions of progression appeared to be involved in the instruction, whether the teacher appeared to be trying to increase or decrease the challenge, what computation strategy the student appeared to use.

Layer C: Progressions in student activity. Developing accounts of the longitudinal development of the student’s knowledge by tracing strands through the Layer B interpretations of each episode in the case. For example,
tracing the progression in Blair’s use of a new computation strategy over the course of the High 5s Sequence.

**Layer D: Progressions in each dimension of instruction.** Developing accounts of how each single dimension was adjusted and progressed over a case. Layer D involves tracing each dimension through the Layer B interpretations, and relating the instruction to the Layer C accounts of learning.

**Figure 3.6 Six layers of analysis**
Layer E: Interactions between dimensions. Identifying and describing characteristic interactions between particular dimensions, over a case. For example, describing how advances in the setting dimension tended to be coordinated with advances in the notation dimension.

Layer F: Multidimensional progressions of instruction. Developing an account of how the instruction overall adjusted and progressed over the case, in terms of multiple dimensions. Layer F involves bringing together the Layer C accounts of learning, the Layer D accounts of each dimension, and the Layer E accounts of interactions between dimensions, to see how a view of the interaction across all the multiple dimensions and strands can illuminate the progression of instruction.

This layering of analysis became the central organiser of the study. In Section 4.7, I describe how I use these six layers to organise the analysis chapters of the thesis.

3.5.3.9 Refining the study aims
The aims of the study were refined and delimited over the course of the data analysis (Flick, 1998, p. 48). The teaching experiment was designed to respond to a broader research task of refining an instructional framework for multiplication and division. I brought a range of questions to the research, and throughout the teaching experiment and the first phase of analysis, I kept a list of further questions and issues arising, including questions about task types, instructional settings, multiplicative structuring, and phases of instruction, as well as about dimensions of instruction. In the first phase of analysis, I identified a focus for this study: to understand the mathematical dimensions of instructional progression supporting the transition from counting-based to multiplicative strategies. Over the second and third phases, as I analysed the High 5s and Lesson 10 cases, I clarified what story I wanted to tell (Flick, 1998; Strauss & Corbin, 1998). In conjunction with refining my conclusions, I refined my aims to the set of three linked research aims of the present study:

1. Identify key mathematical dimensions of instructional progression.
2. Describe the progressions in each dimension.
3. Characterise the instructional progression, using the dimensions.
Also, as I recognised the complexity of characterising a multidimensional instructional progression, I decided to restrict this study to the case of Blair’s High 5s, and leave other interesting episodes in the teaching experiment to subsequent studies.

3.5.3.10 Refining theory and design

Iteratively through the four phases of analysis, I returned to the task of proposing and revising my conclusions. The entire process of coding and re-coding, charting progressions, layering analyses, and condensing and refining memos, worked to generate instructional theory for this design research. From the beginning of the data analysis, I was outlining key shifts in teaching Blair, and sketching possible frameworks of dimensions. Toward the end of the second phase, I wrote the following in my research journal:

I have been writing out the story of the instruction, in terms of the dimensions. I find this basically satisfying, interesting, and heartening. This feels like it can be the main story I want to tell. I can make some headway tracing the progression of individual dimensions, especially setting, and notation. And I get the sense that the clearest way to understand the instruction is as a deliberate interaction among several dimensions: holding some steady while advancing/retreating on others. This could end up being my main ‘point’. (Research Journal, 4/2/2015)

This has indeed emerged as my main point: that the responsive instruction can be characterised as the interwoven calibration of several dimensions. The considerable subsequent work with the data, first in the Lesson 10 Sequence, and then in the High 5s Sequence overall, formulated the five dimensions involved, and established accounts of how those dimensions progressed and interacted in the nested cases. The final phase of analysis worked through the many higher-level memos about instructional progression, selecting the most pertinent, categorising them and integrating them, toward conclusions that were grounded in the data, and could offer insight for the instructional design. The work of presenting the analysis and discussion in a written dissertation further crystallised the characterisation of the multidimensional instructional progression.
3.6 Methodological Quality and Rigour
The aims of design research differ from those of conventional experimental research. The aims are also different from those of ethnographic or other less interventionist methodologies. Consequently, the criteria for quality and rigour are distinctive. The scholarly community of design researchers has debated and refined the criteria for quality of design research over the last 20 years (P. Cobb et al., 2003; Goodchild & English, 2002; Kelly, 2006). Below, I draw on the literature to establish the quality of my methodology. I consider in particular the broad criteria of trustworthiness, generalisability, and reflexivity.

3.6.1 Trustworthiness
A critical discipline in establishing credibility and trustworthiness is making the analysis of data systematic (P. Cobb & Gravemeijer, 2008). As described above, the approach to data analysis sought to be sufficiently flexible and creative to respect the non-linear ways that qualitative research works, and to support the generation of theoretical innovation and insight. At the same time, the approach needed to be sufficiently principled and systematic to challenge my assumptions and maintain sensitivity to the data (Confrey, 2006). The achievement of a combination of creative and systematic analyses is a hallmark of qualitative research methodologies, as articulated in Glaser and Strauss’s (1967) seminal account of grounded theory. To be systematic, I attended to each of the phases and aspects of data analysis described above. The analysis followed the discipline of constant comparisons, seeking to check and refute conjectures against the extensive corpus of data. Each phase of analysis needed to be well documented, including the development and refutation of conjectures. Gravemeijer and Cobb (2006) ask that documentation be sufficient that “Final claims and assertions can … be justified by backtracking through the documentation, back to original videotapes if necessary” (p. 38). The final presentation and discussion of the data involves extensive descriptions of episodes in the teaching. Systematic attention to comparisons and documentation can establish the credibility of these episodes as representative or significant, rather than as convenient or tangential moments selected through sloppy analysis.
The methodology need not pretend that another researcher given the same data would have arrived at the same analysis. However, I do ask my analysis to be in a form that other researchers can monitor and critique. A disciplined documentation of analysis supports external monitoring. I have documented the discussions with other researchers, particularly those I had with my supervisor. Being disciplined about making explicit the criteria and arguments for my claims also enables external critique (P. Cobb & Gravemeijer, 2008; Shavelson et al., 2003).

Teaching experiments in authentic settings cannot be replicated in the sense of traditional experimental methods. Nevertheless, there are aspects of replication in the teaching experiment methodology which serve to improve the rigour of the analysis. Firstly, in the extended time of a teaching cycle, of over 20 lessons, each lesson can be in part an attempt to experiment further with what was attempted in the previous lesson. As Cobb and Gravemeijer (2008) suggest, the micro-cycles of enactment and reflection within one teaching cycle can have a role analogous to the variations of a controlled experiment. In turn, completing a second full teaching cycle strengthens the methodology. As well, Steffe argues studying three individual students operating at a similar level is a form of replication (Steffe, 1991; Steffe & Thompson, 2000).

A related aspect of trustworthiness is making the research virtually replicable, in the sense that other researchers could retrace and re-imagine it. Freudenthal (1991) has a similar aspiration in educational research, of “experiencing the cyclic process of development and research so consciously, and reporting on it so candidly that it justifies itself, and this experience can be transmitted to others to become like their own experience” (p. 161). The methodology aspires to offer thick descriptions of instructional episodes and students’ mathematical activity. Thick descriptions are a feature of many approaches to qualitative research (Geertz, 1973; Patton, 2002). Through the thick description of instructional episodes, the research can establish ecological validity (Gravemeijer & Cobb, 2006): a coherent and credible account of how instruction could occur in an authentic setting.
The trustworthiness of my accounts is reinforced by the prolonged engagement with the students in the study, a strength of the teaching experiment methodology (Cohen, Manion, & Morrison, 2011; Gravemeijer & Cobb, 2006; Shavelson et al., 2003). Steffe and Thompson (2000) write of making “records of the living models of students’ mathematics that illustrate aspects of the claim available to an interested public” (p. 303). I aspire to presenting accounts of living instruction in mathematics.

3.6.2 Generalisability
Teaching experiments cannot be simply repeated. A basic assumption of design research is that authentic educational settings are sufficiently complex that no two moments of instruction are ever exactly alike. Nevertheless, one of the goals of the research is to identify aspects of the instruction that can be repeated in, or can inform, future practice. The thick description provides a basis for adapting the instruction to other settings. Furthermore, by trying to identify which aspects of instruction were necessary for the learning and improvement, and which were contingent, I can support this form of generalisability (P. Cobb et al., 2003; P. Cobb & Gravemeijer, 2008). As Steffe and Thompson (2000) point out, teaching experiments do not try to ascertain facts that hold for a sample, and that might generalise to apply to a larger population: “It is not a matter of generalising the results in a hypothetical way, but of the results being useful in organising and guiding our experience of students doing mathematics” (p. 304). I am seeking to develop ways of explaining students’ activity, which might generalise by being useful in explaining other students’ activity. Likewise, the instructional tools I design might generalise by being useful in instruction with other students. Gravemeijer and Cobb (2006) ask that aspects of the study be framed as exemplars of broader issues. Then, in turn, the ways of interpreting or explaining those aspects can be generalised; the approach to understanding the cases presented can inform approaches to other cases. In this study, I frame one sequence of instruction as a case, such that the instructional framework devised from that case could generalise to instruction with other students.
3.6.3 Reflexivity

A feature of qualitative research is reflexivity (Flick, 1998; Guba & Lincoln, 1985; Patton, 2002). As researchers, we are aware that our particular knowledge and interests influence the shape of our research. We are interested in reflecting on our influence, and our reflections on our influences can in turn become part of the research. Steffe and Thompson (2000) emphasise the reflexivity in the practice of teaching experiments.

At every point when interacting with students in a teaching experiment, the students’ and teachers’ actions are co-dependent. The realization that the researchers are participants in the students’ constructions and the students are active participants in the researcher’s constructions is precisely what recommends the teaching experiment methodology. (p. 305)

A basic discipline of the methodology is to maintain a distinction between my understanding, and students’ understandings, of the mathematics. A further, reflexive discipline is to seek to make explicit my mathematical understanding, and in the same way I make explicit an analysis of a student’s understanding. Further still, I can recognise that my analysis of a student’s understanding must still be the product of my mathematical thinking. As I consider a student’s particular responses and persistent errors, and try to construct an account of the mathematical conceptions that could lead to such responses and errors, I will be using my own mathematical thinking to construct that account. This recognition need not collapse the distinction between my mathematics and the students’ mathematics. However, such reflexivity does point to the centrality of mathematical thinking in the research: to understand and teach a student in mathematics is itself a task of mathematical thinking (Steffe & Thompson, 2000, p. 270). I take this centrality of mathematical thinking to be consistent with the perspective on mathematics education invoked as my instructional approach in Chapter 2, in accord with the insights of Freudenthal, Steffe, Mason, Sfard, and others.

Steffe and Thompson (2000) link reflexivity to our broader aims as researchers, to change the culture of mathematics teaching and educational research. “Rather than being regarded as a weakness of the methodology, [reflexivity] is one of its greatest strengths because it provides researchers the possibility of influencing the education community’s images of mathematics
teaching, learning, and curricula” (p. 305). Similarly, researchers in other forms of qualitative research make recommendations that the reflexivity of researchers contributes to the potential for the research to genuinely engage with social change (Erickson, 2011a; Reason, 1998). As my aims of educational change are important to me, I take seriously the concern for reflexivity.
Chapter 4 – Cases, Layers, and Key Terms for Analysis

But to use the thing that I saw, I need to have like the answers first.
(Blair)

This chapter prepares the groundwork for the substantial presentation of data, analysis and discussion in the subsequent chapters. The chapter includes an orientation to the case study, explanations of key terms and codes used in the analysis, and an overview of the organisation of the analysis. The chapter is in seven sections:

- Analysis of Blair’s Progress Across Term B;
- Orientation to the Case of Blair’s High 5s;
- Instructional Aims in the High 5s Sequence;
- Descriptions of the Task Types Used in the High 5s Sequence;
- Descriptions of the Main Computation Strategies for High 5s Tasks;
- Introduction to the Five Dimensions of Progression;
- Organisation of the Analysis.

4.1 Analysis of Blair's Progress Across Term B

As explained in Section 3.2, I administered four interview-based assessments of Blair’s number knowledge: Assessments 1 and 2 at the beginning and end of Term A, and Assessments 3 and 4 at the beginning and end of Term B. Assessment 4 was actually three weeks after the end of Term B, following two weeks of school holidays. A comparison between his responses in Assessment 3 and Assessment 4 indicates his progress over Term B.

As an illustration of his progress, consider his responses to a pair of tasks involving multiples of 5: a 5×3 array was flashed, the student was told there are five rows and three columns, and asked how many dots altogether, then a supplementary task posed the same question for a 5×6 array. In Assessment 3, Blair called the 5×3 array the “15-dot card”, so answered immediately. For the 5×6 array, he said:

After some delay, I asked “Is there any relationship between these?”, placing out the 5×3 and 5×6 arrays side by side. He said

Blair: Yes actually. Turn that that way, that that way … ah, you could have just kept them that way … You could just join them on and make that, um … cos it’s the same, um … it’s the same, um, width, five dots. Five dots by three, five dots by six. Not that different. But if you do that [placing arrays edge-to-edge] it makes a bigger-[array].

Interpreting these responses, the 5×6 task was found by adding five 6s, apparently with a short cut of finding four 6s as double two 6s. When asked directly about a relationship between 5×3 and 5×6, he was focused on the physical arrangement of the array cards, and did not recognise a multiplicative relationship.

In Assessment 4, Blair answered the 5×3 task immediately “15”, and answered the 5×6 task “30”, explaining

Blair: It’s basically the same as the fifteen [indicating the previous array card] but times 15 by 2. Or, 5 times 5 equals 25, plus one 5.

He now recognised a multiplicative relationship between the 5×6 and 5×3 tasks, which he used to solve 5×6, and he also offered an alternative multiplicative strategy for calculating six 5s, with his reasoning more fluent, and independent of the array setting.

A similar shift was evident on other tasks. For a task to share 12 apples between four children, in Assessment 3 he tested adding up four 2s, four 4s, and four 3s, and concluded 3s works, whereas in Assessment 4, he answered 3 as a known fact. For a task to find how many rows of 2 in a 12-dot array, in Assessment 3 he counted by 2s up to 6, making six strokes over the screened array, whereas in Assessment 4 he answered 6, explaining that five 2s is 10 and six 2s is 12. In general, his reasoning progressed from skip-counting-based strategies, with a dependence on the setting, to multiplicative strategies more independent of the setting.

The assessments also posed many multiplication basic facts as written bare number tasks, and in each range of basic facts there was improvement from
Assessment 3 to Assessment 4 (refer to Table 3.6 for the scheme of ranges). For Range 1 and Range 2 facts—that is, multiples of 2, multiples of 10, and facts with both factors in the range 3 to 5—in Assessment 3, Blair was already successful, while in Assessment 4 he was noticeably more fluent. For Range 3 high 3s and 4s tasks, in Assessment 3 he generally tried relevant multiplicative strategies but laboured in managing the mental computation, whereas by Assessment 4, he was generally more fluent, and some tasks appeared to have become known facts. For Range 4 tasks—that is, with both factors in the range 6 to 9—in Assessment 3, he sought to use multiplicative reasoning, but his strategies were uncertain, his computation laborious, and his success only partial. By Assessment 4, a few Range 4 tasks were known facts, and for the remaining tasks, while not fluent, he used effective multiplicative strategies successfully.

For high 5s basic facts, the focus of this case study, there was a clear shift in strategies and fluency. In Assessment 3, he solved $8 \times 5$ via “5 times 5 is 25, just add three more 5s: 30, 35, 40”. He solved $5 \times 9$ via “9, 18, 27, thirty—[then starting again, this time in coordination with raising successive fingers on his right hand] 9, 18, 27, 36, 45. 45!” That is, he used skip-counting-based strategies, and in the second task he needed to keep track of the count using his fingers. In his final lesson in Term B, when posed a series of mixed basic facts multiplications and divisions, he answered high 5s tasks fluently, giving multiplicative explanations, such as for $8 \times 5 = 40$: “Half [pointing to the numeral 8], four, add the zero”, a reference to the $\frac{1}{2}F \rightarrow n$ strategy he developed during the term. In Assessment 4, administered after two weeks of holidays, his answers were not as quick as in the final lesson, but his multiplicative strategies remained: he solved $8 \times 5$ via 50 less 10; and explained his solution for $5 \times 9$ as “If that is up to 90, then that is 45”, presumably recognising a multiplicative relation to halve $10 \times 9 = 90$. Over Term B across the multiplication topics in general, and in the high 5s in particular, Blair showed significant progress in the critical transition from counting-based strategies dependent on context, to multiplicative strategies independent of context.
4.2 Orientation to the Case of Blair's High 5s

This section provides an orientation to the case of Blair’s High 5s, the case to which all subsequent analysis and discussion are devoted. I begin with an overview of how the high 5s instruction arose in the context of Blair’s Term B. I specify what lesson segments are included in the High 5s Sequence, and in the Lesson 10 Sequence which forms a focus case nested within the High 5s Sequence. I also introduce a chart of the High 5s Sequence which serves as a first map for navigating the story and analysis.

4.2.1 Overview of Blair's high 5s instruction in Term B

As described in Section 4.1 above, at the beginning of Term B, Blair demonstrated knowledge of lower basic facts (Ranges 1 and 2), and he also had some multiplicative strategies for high 3s and 4s tasks in Range 3. By contrast, Blair did not demonstrate any high 5s as known facts, and his main strategy for solving high 5s product tasks was counting-on by 5s from 25. He did not demonstrate any awareness of connections between 5s products and 10s products.

In planning instruction at the beginning of the term, I chose to work first on establishing his multiplicative knowledge of high 3s and 4s, as well as on consolidating his knowledge of the smaller Range 1 and 2 products. In Lesson 8, week 4, I began to address high 5s. Lesson segments on high 5s became a focus of lessons through weeks 4, 5, and 6, alongside continued work on Range 1 and 2 and high 3s and 4s, and some other multiplicative investigations. By Lesson 17, at the beginning of week 7, Blair’s knowledge of high 5s had progressed considerably: he had structured relationships between products of 5s and 10s, he had strong strategies for calculating high 5s products and some quotients, and he had begun to habituate some products. From week 7, I shifted the focus of instructional attention to Range 4, the basic facts with higher factors. Meanwhile, I continued instruction on high 5s in rehearsal mode, working to habituate some relations as known facts, and consolidate his knowledge of these relations when mixed among other tasks. By the end of term, Blair had good strategies for, if not habituated knowledge of, any high 5s product or quotient in bare number form. Thus, as explained in Section 3.4, Blair’s High 5s was a valuable case to observe a student successfully progressing from counting-based strategies to
multiplicative strategies. Table 4.1 presents a simplified outline of this story of teaching and learning of high 5s:

<table>
<thead>
<tr>
<th>Week</th>
<th>Lessons</th>
<th>Instruction in high 5s</th>
</tr>
</thead>
<tbody>
<tr>
<td>1–3</td>
<td>1–7</td>
<td>Occasional mention.</td>
</tr>
<tr>
<td>4</td>
<td>8–10</td>
<td>First segments devoted to high 5s. Use of 5-tiles setting. Initial confusions and insights relating 5s and 10s.</td>
</tr>
<tr>
<td>5</td>
<td>11–13</td>
<td>Informal notations. Establishing relations of 5s and 10s.</td>
</tr>
<tr>
<td>6</td>
<td>14–16</td>
<td>Transition to bare numbers and to formal notation. Refining strategies for products and quotients.</td>
</tr>
<tr>
<td>7</td>
<td>17–18</td>
<td>Rehearsal of bare number tasks, mixed with other ranges. Mostly division.</td>
</tr>
<tr>
<td>8</td>
<td>19–21</td>
<td>Rehearsal of bare number tasks, mixed with other ranges. Most becoming fluent.</td>
</tr>
<tr>
<td>9</td>
<td>22–24</td>
<td>Brief rehearsal.</td>
</tr>
</tbody>
</table>

4.2.2 The High 5s Sequence

The *High 5s Sequence* is the sequence of main lesson segments addressing high 5s, from Lessons 8 through 17. Figure 4.1 summarises the topics addressed in all Blair’s lessons in Term B, and the segments constituting the High 5s Sequence are highlighted and numbered. There are 18 segments in total, from eight lessons: Lessons 8, 10, 11, 12, 14, 15, 16, and 17. There were no segments on high 5s in Lessons 9 and 13, which were shorter lessons and only addressed other topics. From Lesson 18 onward, the segments involving high 5s were in rehearsal mode, and I decided not to include these in the focus of the analysis. The instruction in the later lessons remains of interest to understanding Blair’s development overall, and included some interesting moments in reflecting on his strategies. Nevertheless, the main progressions in the instruction in multiplicative strategies in high 5s reached an end by Lesson 17, so ending the case sequence here offers sufficient material for the analysis of the instructional progression.
4.2.3 The Lesson 10 Sequence

I have selected the three segments on high 5s in Lesson 10—10.5, 10.6, and 10.7—as a case for more detailed analysis, nested within the High 5s Sequence. I refer to these as the Lesson 10 Sequence, circled in Figure 4.1. The Lesson 10 Sequence was an extended passage of instruction, during which Blair had both significant insights and significant confusions, and overall made progress. The instruction involved some significant adjustments during the segments: extensive adjustments between screened and unscreened materials; adjustments in the orientation of tasks; adjustments in the range of numbers involved in tasks; a progression from use of the 5-tile setting to bare number tasks; an introduction of informal notation; and two main changes of task types. These are just the sorts of adjustments I want to study, and they prefigured many of the adjustments in instruction that arose in subsequent segments in the High 5s Sequence. So, the Lesson 10 Sequence is a rich
source for analysis, and serves well to gain insight into the instructional progression over the whole High 5s Sequence.

4.2.4 The Basic Chart of the High 5s Sequence

Figure 4.2 is a basic chart of the High 5s Sequence. The chart shows each of the 18 lesson segments. The Lesson 10 segments are highlighted in red. The instruction in each segment is summarised by indicating task type, settings, notations, and range of numbers involved. The task types are described in Section 4.4 in this chapter.

This basic chart offers an orientation to the sequence of instruction I describe and analyse in detail in the remainder of this dissertation. I will also generate the more elaborated instructional analysis charts based on this chart, which will include listing individual tasks, notes on Blair’s responses, and columns tracking adjustments in the five dimensions of progression. These later charts become central to organising and tracking the analysis.

Figure 4.2 Basic chart of the High 5s Sequence

<table>
<thead>
<tr>
<th>Seg</th>
<th>Task type</th>
<th>Setting, notation, range</th>
<th>Orientation</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.8</td>
<td>Multiplication with 5-tiles/ bare numbers</td>
<td>Number sentences</td>
<td>Given (even) F, find T.</td>
</tr>
<tr>
<td><strong>Lesson 10 Sequence</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10.5</td>
<td>Inc- &amp; decrementing with 5-tiles</td>
<td>Visible --&gt; screened</td>
<td>Say N, say F.</td>
</tr>
<tr>
<td>10.6</td>
<td>Division with 5-tiles</td>
<td>Screened</td>
<td>Given N, find F.</td>
</tr>
<tr>
<td>10.7</td>
<td>Varied orientations with 5-tiles</td>
<td>Screened; arrow sentences</td>
<td>Varying given: N, F, T.</td>
</tr>
<tr>
<td>11.3</td>
<td>Multiplication with 5-tiles</td>
<td>Screened</td>
<td>Given F × 5, find N.</td>
</tr>
<tr>
<td>11.4</td>
<td>Varied orientations with 5-tiles</td>
<td>Screened; parallel expressions</td>
<td>Varying given: N, F, T.</td>
</tr>
<tr>
<td>12.4</td>
<td>Division with bare numbers</td>
<td>Parallel expressions</td>
<td>Given N, find F, T.</td>
</tr>
<tr>
<td>12.5</td>
<td>Multiplication with bare numbers</td>
<td>Number sentences</td>
<td>Given F × 5, find N.</td>
</tr>
<tr>
<td>12.6</td>
<td>Multiplication with bare numbers</td>
<td>Parallel expressions</td>
<td>Given F × 5, find T, N.</td>
</tr>
<tr>
<td>14.3</td>
<td>Varied orientations with 5-tiles/ bare numbers</td>
<td>Visible --&gt; screened --&gt; bare number</td>
<td>Varying given: N, F, T.</td>
</tr>
<tr>
<td>15.3</td>
<td>Varied orientations with bare numbers</td>
<td>Screened --&gt; bare; parallel expressions</td>
<td>Varying given: N, F, T.</td>
</tr>
<tr>
<td>15.4</td>
<td>Varied orientations with bare numbers</td>
<td>Number sentences, including + sign</td>
<td>Varying given: N, F, T.</td>
</tr>
<tr>
<td>15.5</td>
<td>Discussion of strategy</td>
<td>5-tiles used; various notations.</td>
<td></td>
</tr>
<tr>
<td>15.6</td>
<td>Multiplication with bare numbers</td>
<td>Various notations; notating strategies.</td>
<td>Given F × 5, find N.</td>
</tr>
<tr>
<td>16.3</td>
<td>Multiplication with 5-tiles.</td>
<td>Screened</td>
<td>Given F × 5, find N.</td>
</tr>
<tr>
<td>16.4</td>
<td>Varied orientations with a fixed table</td>
<td>Number sentences; High 3s, 4s, 5s</td>
<td></td>
</tr>
<tr>
<td>17.1</td>
<td>Varied orientations with a fixed table</td>
<td>Number sentences; High 3s, 4s, 5s</td>
<td></td>
</tr>
<tr>
<td>17.2</td>
<td>Varied orientations with varied n-tiles</td>
<td>High 3s, 4s, 5s</td>
<td></td>
</tr>
</tbody>
</table>

The chart uses the codes N (total number), F (number of 5s), and T (number of 10s). These codes are explained further in Section 4.3.2 below.
4.3 **Instructional Aims in the High 5s Sequence**

The aim of the segments of the High 5s Sequence was to develop Blair’s knowledge of what I called the *high 5s*. I intended the high 5s range to include the set of relations: six 5s, seven 5s, eight 5s, nine 5s, ten 5s, and their turn-arounds and inversions. That is, Blair’s knowledge of high 5s was to include: that six 5s are 30, that five 6s are 30, that 30 divided into six parts has 5 in each part, that the number of 6s in 30 is five, and so on.

During the teaching experiment in Term B, I had three instructional aims for each range of basic facts, as explained in Section 3.3 in describing the initial instructional design. For the high 5s, these aims were:

a) Multiplicative structuring: developing number relationships around 5s products;

b) Multiplicative strategies: developing sophisticated strategies for calculating high 5s products and divisions;

c) Habituation: habituating knowledge of the high 5s basic facts.

In line with the basic instructional principles, these aims were interrelated. I anticipated that Blair’s development of multiplicative strategies for high 5s would draw on his structuring of multiplicative relations with high 5s, and vice versa. Habituation of some facts could arise spontaneously over the course of instruction, and could support refinement of some strategies. I intended attention to habituating high 5s facts to come later in instruction, to build on the development of efficient strategies. Instruction in the High 5s Sequence focused on the first two aims: structuring multiplicative relations and developing multiplicative strategies. Attention shifted towards the third aim, habituation, in later lesson segments after the sequence analysed here.

4.3.1 **Anticipated multiplicative relations and strategies for high 5s**

Multiplicative computation strategies with high 5s tasks commonly relate multiples of 5 to multiples of 10. This could be by thinking, for example, that:

- eight 5s will be the same as four 10s, which is 40.

Or by a related route, by thinking:

- eight 5s will be half of eight 10s, that is half of 80, which is 40.

Or, the turn-around of this, that:

- five 8s will be half of ten 8s, that is half of 80, which is 40.
The reasoning for such strategies depends on some proportional understanding of doubling and halving: for example, that if two 5s makes one 10, then \( F \) 5s will make \( \frac{1}{2}F \) 10s.

Odd multiples of 5 can be solved via some of these same strategies:

seven 5s will be half of seven 10s, that is half of 70, which is 35.

Another strategy for odd multiples is to derive them from neighbouring even multiples of 5:

seven 5s might be solved thinking of six 5s and one more 5;
nine 5s as ten 5s less one 5.

Before the High 5s Sequence began, Blair had not shown any indication of realising such relationships between multiples of 5 and multiples of 10. I sought to develop his insight into the rich network of these relations, and to enable him to develop his computation strategies based in these relations.

The aim to cultivate his understanding of the network of relationships is in accord with the pedagogical principle of reaching for deep conceptual learning (Section 2.3). I did not need the rich insight into relations to be remembered in some explicit form in the long term. Rather, I believed that, following deeper insights, the nearer ground would become more familiar (Watson & Geest, 2005). So, for example, Blair could come to associate 40 with 8 5s via doubling the 4, though the immediate logic of this doubling may not remain explicit. Likewise, he could come to recognise that the even multiples of 5 are decuples—six 5s are 30, eight 5s are 40—though he may not immediately be able to explain this pattern. This is indeed what appeared to happen for Blair over the term.

4.3.2 Codes for N-F-T terms and relations

Most of the instruction in the High 5s Sequence is centred on multiplicative tasks involving some number of 5s, an equivalent number of 10s, and a total number. I use a system of codes to manage writing about these different quantities. These codes are used frequently throughout the analysis and discussion. The tables below explain these codes, using the example relations \( 8 \times 5 = 40 \) and \( 40 \div 5 = 8 \).
I use these basic codes in abbreviations for different task orientations, as in the following examples. The task orientation codes use a dotted arrow.

<table>
<thead>
<tr>
<th>Code</th>
<th>Description</th>
</tr>
</thead>
</table>
| $F \rightarrow N$ | A task giving $F$, and seeking the value of $N$.  
E.g. *What is eight 5s?* |
| $N \rightarrow F$ | A task giving $N$, and seeking the value of $F$.  
E.g. *How many 5s in 40?* |
| $T \rightarrow F \& N$ | A task giving $T$, and seeking the values of $F$ and $N$.  
E.g. *I have four 10s. How many 5s? How many altogether?* |

I also use codes for relationships and computation strategies involving these terms. The relation codes use a solid arrow, and are bold.

<table>
<thead>
<tr>
<th>Code</th>
<th>Description</th>
</tr>
</thead>
</table>
| $2n \rightarrow F$ | Doubling $n$ gives $F$.  
E.g. Given 40, doubling the 4 gives 8, which is the number of 5s. |
| $2T \rightarrow F$ | Doubling $T$ gives $F$.  
E.g. Given four 10s, doubling 4 gives 8, which is the number of 5s. |
| $\frac{1}{2}F \rightarrow T$ | Halving $F$ gives $T$.  
E.g. Given eight 5s, halving 8 gives 4, which is the number of 10s. |
| $n \rightarrow T$ | The digit $n$ is equivalent to $T$.  
E.g. Given 40, the left digit 4 links to the number of 10s being four.  
We cannot assume that the student is always aware of this relation. |

I refer to this set of relations and strategies collectively as the $N$-$F$-$T$ relations. The instruction through the High 5s Sequence was focused on developing these $N$-$F$-$T$ relations and strategies. Mathematically, these relations can be understood as a single coherent system of relations. For example, $2T \rightarrow F$ and $\frac{1}{2}F \rightarrow T$ can be understood as two inversions of the same bijective relationship.

Note that, when I use one of these codes to refer to a strategy Blair used, we must be clear that Blair himself did not reason in terms of such a code. While Blair did achieve some recognition of his strategies, and of the generalised units and relations involved in high 5s tasks, he never articulated anything about the tasks with such brevity and formality. These codes are abbreviations for our purposes in analysis and discussion.
Chapter 4 – Cases, Layers, and Key Terms for Analysis

4.4 Seven Task Types in the High 5s Sequence
Below I describe seven task types identified in the High 5s Sequence of lesson segments. Charts and discussions will refer to these task types. The descriptions here serve as a reference.

The first three task types involve a setting of 5-tiles. The 5-tiles are typically arranged in pairs, to support efficient calculation, and to support relating 5s to 10s.

4.4.1 Incrementing and decrementing with 5-tiles
See for example Lesson Segment 10.5. This task type was in the initial instructional design, described as *structuring numbers multiplicatively* (Wright et al., 2012, p. 150). An initial task is to lay down 5-tiles one by one, with the student to keep track of both the cumulative number of 5s, $F$, and the cumulative total number of dots, $N$. Tiles can be decremented as well—that is, taken away. The instruction can then adjust three dimensions: the extending of the range, the distancing of the setting, and the complexity of the increments.

- To extend the range, the teacher might progress from a range of five tiles, to ten tiles.
- To distance the student from the setting, the teacher screens the tiles, with judicious use of flashing and unscreening.
- To make the tasks more complex, the teacher increments and decrements more than one tile at a time.

4.4.2 Multiplication or division with 5-tiles (fixed orientation)
See for examples Lesson Segments 8.8, 10.6, and 11.3. The teacher places a number of 5-tiles and poses a multiplicative question about the tiles. The sequence of tasks in a lesson segment keeps a constant orientation. Two main orientations are possible:

- Multiplication $F \rightarrow N$. E.g. Six 5-tiles, what is the total?;
- Quotitive division $N \rightarrow F$. E.g. 30 made with 5-tiles, how many tiles?

Tasks can also consider the number of 10s $T$, as in:

- $F \rightarrow T$. E.g. Six 5-tiles, how many 10s?
Screening may be used, and the setting may become distanced to the point of ignoring the 5-tiles. Answers may be merely verbal, or answers may be recorded.

4.4.3 **Varied orientations with 5-tiles**

See for examples Lesson Segments 10.7, 11.4, and 14.3. A variation of the previous task type, where the teacher presents a sequence of tasks with varying orientations, for example, an $N \rightarrow F$ task, then a $T \rightarrow F$ task, then an $F \rightarrow N$ task.

4.4.4 **Multiplication or division of 5s with bare numbers (fixed orientation)**

See for examples Lesson Segments 12.4, 12.5, and 15.6. The teacher poses a multiplicative task involving multiples of 5, without reference to a setting. The sequence of tasks in a lesson segment keeps a constant orientation. Tasks are typically recorded with number sentence notation or an informal notation. For example, the following tasks are posed, and written as horizontal expressions: $6 \times 5$, $9 \times 5$, $12 \times 5$.

4.4.5 **Varied orientations of 5s with bare numbers**

See for examples Lesson Segments 15.3 and 15.4. This is a variation of the previous task type, where the teacher presents a sequence of bare number 5s tasks with varying orientations.

4.4.6 **Varied orientations with a fixed table of multiplications**

See for example Lesson Segments 16.4 and 17.1. A set of candidate multiplications is listed in a table, such as the high 3s, 4s, and 5s as shown in Figure 4.3.

**Figure 4.3** Example of a fixed table of multiplications: the high 3s, 4s, and 5s

<table>
<thead>
<tr>
<th>3×6</th>
<th>3×7</th>
<th>3×8</th>
<th>3×9</th>
</tr>
</thead>
<tbody>
<tr>
<td>4×6</td>
<td>4×7</td>
<td>4×8</td>
<td>4×9</td>
</tr>
<tr>
<td>5×6</td>
<td>5×7</td>
<td>5×8</td>
<td>5×9</td>
</tr>
</tbody>
</table>

Then multiplication and division tasks are posed verbally, using the candidates in various orientations, for example:

- What is 3 times 6?;
• 5 times what equals 35?;
• 32 divided by 8?

The student can refer to the table of multiplications to help solve the tasks. Typically, the student is to write answers as number sentences.

4.4.7 Varied orientations with varied n-tiles
See for example Lesson Segment 17.2. A range of sizes of n-tile are selected—for example, 3-tiles, 4-tiles, and 5-tiles. The teacher sets up a task behind a screen, by arranging a number of tiles of one size. A task is then posed verbally, with three possible orientations:
• Multiplication. E.g. Six 5-tiles, what is the total?;
• Quotitive division. E.g. 30 made with 5-tiles, how many tiles?;
• Partitive division. E.g. 30 made with six tiles, how many dots on each tile?

Thus, compared with the task types above with 5-tiles only, the varied size of n-tile here introduces the possibility of partitive division.

4.5 Five Computation Strategies for High 5s Tasks
One aim of the high 5s instruction was to develop multiplicative strategies for calculating high 5s products. There are a range of possible strategies, some involving additive reasoning, some multiplicative. I describe five below, in order of mathematical sophistication, which were observed in Blair’s case. Describing them here allows me to recognise them and refer to them in later analysis.

4.5.1 Counting by 5s from 5
For example, calculate 8×5 by thinking “5, 10, 15, 20, 25, 30, 35, 40!”; keeping track of the number of counts on fingers, and stopping the count when the fingers reach a pattern for 8.

This strategy requires knowing how to skip-count by 5s, and being able to keep track of eight counts. The 5s are treated as abstract composite units, which can be incremented, keeping track of both the total number (up to 40), and the number of units (up to eight). Thus, this strategy involves operating at
Level 4 in Wright et al.’s model of conceptual development in multiplication and division. It is an inefficient skip-counting-based strategy.

4.5.2 Counting-on by 5s from a lower multiple
For example, calculate 8×5 by thinking “Five 5s is 25, then 30, 35, 40”, keeping track of making three more counts, perhaps as a known rhythmic pattern of three.

This was a common strategy for Blair when the high 5s instruction began. As with the previous strategy, this strategy requires knowing how to skip-count by 5s, and being able to keep track of counts. The strategy also requires having a multiple of 5 as a known fact—in the example, knowing that five 5s is 25—and being able to count-on when keeping track of the multiplier—counting-on from five 5s to eight 5s. By taking as given that five 5s is 25, nested within the eight 5s, the reasoning requires an explicit number sequence in Steffe’s (1992) terms, showing a valuable conceptual advance on the previous strategy. However, it is still an inefficient strategy involving skip-counting.

4.5.3 Using a neighbouring multiple
For example, calculate 9×5 by thinking “Eight 5s is 40, so nine 5s is 45”, or “Ten 5s is 50, so nine 5s is 45”.

This strategy seems conceptually similar to the previous one, in this case counting-on or back a single count. Nevertheless, it is a more efficient strategy, and shows more number sense in recognising the neighbouring multiple to begin with. In the case of Blair, I made an aim of Blair improving his strategies from always counting-on from 25, to finding closer known multiples, which he did achieve.

4.5.4 Summing partial products
For example, calculate 8×5 by thinking “Five 5s is 25, three more 5s is another 15, 25 and 15 makes 40”.

The total product is constructed as a sum of partial products. This strategy begins like the counting-on by 5s strategy, from a lower known multiple. However, instead of counting-on the remaining 5s one unit at a time, the
product of the remaining 5s is calculated as well, and added on as a single addend. In real time, the difference between the two strategies is subtle, and a student may not be able to recall or articulate his own thinking clearly enough to discern the difference. Nevertheless, this distinction indicates an advance in multiplicative sophistication. By partitioning eight 5s into five 5s and three 5s, the eight is treated as an abstract composite unit; in fact, a unit (the eight as a whole) of units (the 5s as wholes) of units (the elements); and thus the reasoning requires a *general number sequence* in Steffe’s (1992) terms. This coordination of both factors as composite units is the distinguishing characteristic of conceptual Level 5, regarded as the mark of true multiplicative thinking (Wright, Martland, & Stafford, 2006, p. 125). Blair used this strategy for some tasks later in the High 5s Sequence.

4.5.5 **Calculating as a proportion of a multiple of 10**

For example, calculate 8×5 by thinking “Eight 5s is four 10s, which is 40”.

Such a strategy requires proportional reasoning, recognising a proportional relationship between two quantities—for example, that the number of 10s, \( T \), will be half the number of 5s, \( F \). The strategy also requires having the multiple of 10 as a known fact. The proportional reasoning requires a general number sequence, and is certainly multiplicative and efficient.

4.6 **Introducing the Five Dimensions of Instructional Progression**

The aims of this study are to identify and describe key mathematical dimensions of progression in the instruction. As described in Section 3.5, the data analysis responding to these aims has been an emergent and recursive process. I began by tracking dimensions proposed in the initial instructional design, and over the course of the analysis, I identified other possible dimensions, adjusted my understanding of each dimension, combined some dimensions and split others, and continued to return to the data to establish coherent accounts of the instructional progression. I have settled on five dimensions that characterise the instructional progression in the case of Blair’s High 5s, which I have called:

- range;
- orientation;
Below I will introduce each of these five dimensions. These are my first answers to the research aims: in sympathy with Blair’s concern in the epigraph beginning this chapter, I am giving my answers first. The analysis and discussion in the subsequent chapters will track each of these dimensions in the data to reveal their significance in the instruction, how they each progress, and how they interweave to characterise the instructional progression.

Each dimension has a three-letter code. I use these codes throughout the analysis and discussion chapters to facilitate the analytical attention to the dimensions. Also, when I need to work through all five dimensions systematically, I address them in the order given here. I find this consistency supports the clarity of the presentation.

4.6.1 Range (RNG)

The range dimension refers to the range of numbers involved in arithmetic tasks. In the instruction on multiples of 5, I distinguish three basic ranges, drawing from the scheme of ranges in the initial instructional framework (see Section 3.3.1):

- low 5s: 1×5, 2×5, 3×5, 4×5, 5×5;
- high 5s: 6×5, 7×5, 8×5, 9×5, 10×5;
- beyond 50: 11×5, 12×5, 13×5 … 20×5.

Another adjustment in range that emerged as significant was between:

- even multiples: 2×5, 4×5, 6×5, 8×5, 10×5 … ; and
- odd multiples: 1×5, 3×5, 5×5, 7×5, 9×5 ….

When instruction shifts between different sets of numbers, I label this an adjustment in RNG, the dimension of range.
4.6.2 Orientation (ORN)

The orientation dimension refers to the choice of which values are given, and which are to be determined, in an arithmetic task. For example, tasks about the relation $8 \times 5 = 40$ could be presented in several different orientations, including:

- $F\rightarrow N$: What is eight 5s?
- $N\rightarrow F$: How many 5s in 40?
- $T\rightarrow F, N$: I have four 10s. How many 5s? How many altogether?

When instruction changes orientation for a new task, I label this an adjustment in ORN, the dimension of orientation.

I have used the term orientation, rather than inversion, for two reasons. Firstly, inversion is an important formal mathematical term typically applied to an operation or a function, not simply to a situation. A mathematician is likely to interpret the situation “given $T$, what is $N?$” as requiring the inverse operation from the situation “given $N$, what is $T?$”. But we cannot assume that a student makes this interpretation. So, I prefer the more open term of changing the orientation. A second reason is that often—as in an example above—the task involves three numbers: one known and two unknowns. There are more than two possible orientations for these tasks. The term “inversion” does not capture this clearly.

4.6.3 Setting (SET)

The setting dimension refers to the materials in which the task is set, such as 5-tiles. If no materials are used, I speak of a setting of bare numbers. The main progression in this dimension is to distance the setting, shifting from visible materials, to screened materials, to bare numbers. The use of the setting in the case study revealed considerable scope for adjustments: how many of the 5-tiles were screened; flashing the 5-tiles briefly then screening again; invoking the tiles without presenting them; and so on. I have labelled these kinds of subtleties as adjustments in the SET, the dimension of setting.
4.6.4 Notation (NTN)

The notation dimension refers to a variety of uses of written notation in a task. The main uses are:

- when posing a task, such as the teacher writing $8 \times 5$, or $8 \times \_ = 40$, and asking the student to calculate an answer;
- when recording a task, such as the student writing “$=40$”; or the teacher recording an answer in a table of answers; and
- when notating a strategy or pattern being discussed, such as the teacher writing “$8 \times 5 \rightarrow 25 + 15 \rightarrow 40$”.

Notation can be written by the teacher or the student. Notation can be informal, such as the arrow sentence, or more formal, such as a conventional horizontal number sentence. Since the instruction aims for the student to be able to work with formal notation, progressions in the dimension of notation NTN can include:

- shifting from not using notation to using notation;
- shifting from informal notation toward formal notation; and
- shifting from teacher writing to student writing.

4.6.5 Attention to structure and strategies (STR)

The structure and strategies dimension refers to structuring multiplicative relations and refining multiplicative strategies. As well as structuring relations in bare numbers, as it were, structuring can involve recognising a pattern in a written table of results, or organising the setting, by putting 5-tiles in pairs for example.

The teacher cannot directly structure the student’s knowledge of relations, nor directly refine the student’s strategies. Instead, the teacher can draw attention to structuring or strategies, through questions, comments, and gestures like:

- “How did you do that?”;
- “Why is that?”;
- “What arrangement of tiles are you going to see under the screen?”;
- “Try using a closer multiple”;
- Pointing to a previous answer, to raise awareness of a pattern.
When analysing the instruction, it is the teacher’s efforts to draw attention to structuring and strategies that I label as adjustments in STR.

### 4.7 Organising the Analysis

I am about to embark on the core of the study over the remaining chapters of the thesis: the analysis of the dimensions of progression in the case of Blair’s High 5s. Below I explain how the analysis in these chapters is organised.

#### 4.7.1 Three research aims

As introduced in Section 2.6, this study has three research aims. In studying the progression of responsive instruction supporting the development of multiplicative strategies, I aim to:

1. Identify key mathematical dimensions of instructional progression.
2. Describe the progressions in each dimension.
3. Characterise the instructional progression, using the dimensions.

The basic approach to the analysis is to track, throughout a sequence of instruction, progressions in Blair’s learning, alongside tracking adjustments in the five dimensions of instruction. By tracking Blair’s activity, I can indicate how the instructional adjustments were responding to and supporting his learning over the course of the instructional sequence. I can then argue how each of the five dimensions was significant in the interactions with Blair, and so establish them as key dimensions of instructional progression, achieving the first research aim. By tracking each of the dimensions, I can develop an account of the kinds of adjustments made in each dimension, and how each dimension can progress overall, achieving the second research aim. Finally, having tracked the dimensions separately, I can develop an account of the whole instructional progression in terms of interwoven adjustments across the five dimensions, and so achieve the third research aim.

#### 4.7.2 Six layers of analysis

Developing this analysis involves a challenge in managing the different layers involved: tracking multiple threads in Blair’s learning, and five dimensions in instruction, across a whole sequence of lesson segments. To manage this challenge in my data analysis process, I developed an explicit labelling of six layers of analysis, which I could work through systematically, as described in
threads through a labyrinth

the methodology chapter, section 3.5.3.8. i present the analysis in the following chapters using the same layering. figure 3.6 represents these six layers of analysis.

the six layers of analysis are:

**layer a: observation.** transcriptions and summaries of teacher and student activity.

**layer b: local interpretation.** interpretations of what happened within single tasks or segments: what dimensions the teacher used, and what relations and strategies the student appeared to use.

**layer c: progressions in student activity.** tracing the longitudinal development of strands of the student’s knowledge, over the whole sequence.

**layer d: progressions in each dimension of instruction.** tracing the longitudinal progression of each of the five dimensions of instruction separately, over the whole sequence.

**layer e: interactions between dimensions.** describing characteristic interactions between particular dimensions.

**layer f: multidimensional progressions in instruction.** an account of how the instruction overall progressed, through interacting adjustments across all five dimensions.

each layer builds on the previous layers. layer b is based on the layer a observations, interpreting what the teacher and student are doing in each task or segment. layer c is based on layer b, tracing through the layer b interpretations of blair’s activity in each episode, to identify the significant progressions in blair’s learning over the sequence. layer d is also based on layer b, tracing through the layer b interpretations of the dimensions of instruction in each episode, to develop accounts of the progression in each of the five dimensions. layer d also draws on layer c, using the layer c insights into blair’s activity, to illuminate how the instructional progressions in layer d interacted with the student. layer e elaborates on layer d, revealing ways that progressions in two or more dimensions could be coordinated or complementary. finally, layer f weaves together layers c,
Threads Through a Labyrinth

Chapter 4 – Cases, Layers, and Key Terms for Analysis

D, and E, to develop an account of the coordination of the five dimensions as a responsive instructional progression. Layers D, E, and F are the layers which directly address the research aims of investigating the dimensions of instructional progression. Layer D aligns with Research Aim 2, and Layers E and F align with Research Aim 3.

4.7.3 Presentation of data through the layers of analysis

A second challenge in presenting this analysis is that I am studying longitudinal progressions, so I need to present data on whole sequences of instruction. As explained in Section 3.4.2, it is not sufficient to select separate episodes to illuminate the progressions. To track several strands in Blair’s learning, and five dimensions of progression in instruction, I need to draw on every lesson segment in the High 5s Sequence. I have decided to present long sections devoted to Layer A and Layer B accounts of sequences of instruction before I present Layers C, D, E, and F. The drawback of this is the lengthy presentation of transcripts and local interpretations, without an ongoing rich analysis to motivate it. Nevertheless, I think these basic accounts of the instruction are already interesting stories to read, and they do at least make sense when told chronologically. The advantage is that, once the lengthy data are presented, the episodes are easily located in sequence, for all the later references back to them during the Layer C, D, E, and F analyses.

As I present the layers of analysis, each lesson segment will appear in every layer of analysis. A given segment will appear in Layer A with transcripts from the segment, and in the Layer B interpretations of the segment. Then it will appear in Layer C as I track several threads of Blair’s learning through the segment; in Layer D as I track how each dimension progressed through the segment; and in Layer E as I track how some dimensions interacted through the segment. Finally, it will appear in Layer F as I track how all five dimensions interwove through the segment. Later, in the discussion chapter, I may refer to the segment several times again to illustrate my points. This is the basic texture of these analysis chapters: a layering up of understanding about each segment, and how each segment is a way station in several simultaneous progressions through the whole High 5s Sequence. By layering up this understanding of each segment, I come to my main conclusion of how
the responsive instruction in any segment can be characterised as an interweaving of dimensions of progression.

Throughout this layering of analysis, the main reference for identifying segments is a numbering scheme, which numbers the segments within each lesson, and the tasks within each segment. I often also refer to the task or task type of an episode, to help recall what it was about. For some episodes, I have singled out a distinctive quote of the teacher or student, to stand for the episode in later discussion.

### 4.7.4 Five core chapters of analysis and discussion

The analysis is presented over the next four chapters. Chapters 5 and 6 address the close-up case of the Lesson 10 Sequence. Chapters 7 and 8 address the longer case of the High 5s Sequence.

The chapters work through all of Layers A to F for both cases, but the presentation is organised differently according to the structure of each case. To address the Lesson 10 case, Chapter 5 works through Layers A, B, C, and D for each of the three lesson segments separately. Chapter 6 then analyses the three segments as a continuous sequence, developing Layers D, E, and F. To address the High 5s Sequence, Chapter 7 works through Layers A, B, and C for the whole sequence. Chapter 8 then presents Layers D, E, and F for the whole sequence. Following these chapters, the Discussion Chapter 9 discusses each of the three research aims in turn, drawing on the analyses to understand the instructional progressions in terms of the five dimensions.

Figure 4.4 is a diagram showing an example of how this layering of analysis over Chapters 5 to 8 develops towards findings in the discussion in Chapter 9. The box at the bottom of Figure 4.4 represents a paragraph on a single finding in Chapter 9, within the section addressing Research Aim 2, and the subsection describing the progressions in the dimension of orientation ORN. The paragraph begins with the claim that

*instruction can switch back and forth between orientations to promote learning about the whole multiplicative relationship* [in a given range] (Section 9.2.2.1).
Figure 4.4  The development of analysis through Chapters 5 to 8 toward one finding in the Discussion, Chapter 9
As indicated in Figure 4.4, the paragraph continues by supporting that claim with references to a point of analysis of Segment 10.7 and a point of analysis of Lessons 10 to 15. Figure 4.4 shows how these two references link back to a paragraph in Chapter 6 and a paragraph in Chapter 8, respectively. In turn, Figure 4.4 shows how the paragraph in Chapter 6 is reached, through the Layers A, B, C, and D on Segment 10.7 in Chapters 5 and 6. Likewise, the paragraph in Chapter 8 is reached through the Layers A, B, and C in Chapter 7, and the Layer D section in Chapter 8 on the progression in ORN through the High 5s Sequence. This kind of back story of layered analysis of progressions, linking back to transcribed sequences of instruction, backs up each of the findings in Chapter 9.

For both the Lesson 10 Sequence, and the High 5s Sequence, I will present a large detailed instructional analysis chart. These charts become the central references for following the sequence of tasks and segments being analysed, and for tracking the adjustments in the dimensions. You may want to bookmark these charts in the text for ready reference.

Such a multi-layered, multidimensional analysis is not easy to organise in writing. I think my approach is reasonable, but not ideal. By the time I am discussing the final layer of the multidimensional progressions, the earlier layer analysing students’ learning, which is a prerequisite for understanding the final layer, is many pages back. There seems to be no way to avoid this stretch. I have tried to furnish each layer with a summary that can be carried forward to serve in later layers. Also, I hope the careful organisation at least keeps clear what layer I am up to.

4.8 Chapter Summary: Bon Voyage
In this chapter I have prepared the groundwork for the analysis I am about to present. I have given an overview of the case: the Lesson 10 Sequence, nested in the whole High 5s Sequence. I have described the instructional aims that guided the instructional progression through the case. I have described the main task types and computation strategies referred to throughout the analysis, and the $N$-$F$-$T$ codes used for these. I have introduced the five dimensions, which are the centre of the analysis. Finally, I have explained how the presentation of analysis is organised over the next four chapters,
addressing six layers of analysis over each of two nested cases, to then discuss three research aims. Table 4.2 below summarises all these key terms and codes.

We are now ready to embark. The presentation of analysis will demand some patience. I hope it is rewarding, with its insight into the inner workings of some highly responsive mathematics instruction. The real jewels, for me, are the multidimensional accounts, Layer F. If you are prepared to believe my interpretations of the students’ activity and the teacher’s instructional responses, you could skip to these accounts, to read the main story I want to tell.
Table 4.2  A summary of the terms and codes used throughout the analysis

<table>
<thead>
<tr>
<th>Ranges of multiples of 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low 5s</td>
</tr>
<tr>
<td>High 5s</td>
</tr>
<tr>
<td>Beyond 50</td>
</tr>
</tbody>
</table>

Terms in multiplicative relationships involving multiples of 5

Using the example relations 8×5=40 and 40÷5=8.

<table>
<thead>
<tr>
<th>Term</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td>The total, that is, the number of ones. In the examples, N is 40.</td>
</tr>
<tr>
<td>n</td>
<td>The tens digit of the numeral for N. In the examples, n is 4.</td>
</tr>
<tr>
<td>F</td>
<td>The number of 5s. In the examples, F is 8.</td>
</tr>
<tr>
<td>T</td>
<td>The number of 10s. In the examples, T is 4.</td>
</tr>
</tbody>
</table>

N-F-T relations

<table>
<thead>
<tr>
<th>Relation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>N-F-T</td>
<td>The network of multiplicative relationships between N, F, and T for even multiples of 5, such as that double n gives F, double T gives F, half of F gives T, and T gives n (see list of relations next).</td>
</tr>
</tbody>
</table>

Multiplicative N-F-T relations and strategies

<table>
<thead>
<tr>
<th>Relation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>2n→F</td>
<td>Doubling n gives F. E.g. Given 40, double the 4 gives 8, which is the number of 5s.</td>
</tr>
<tr>
<td>2T→F</td>
<td>Doubling T gives F. E.g. Given four 10s, double 4 gives 8, which is the number of 5s.</td>
</tr>
<tr>
<td>½F→T</td>
<td>Halving F gives T. E.g. Given eight 5s, half 8 gives 4, which is the number of 10s.</td>
</tr>
<tr>
<td>n→T</td>
<td>The digit n is equivalent to T. E.g. Given 40, the left digit 4 links to the number of 10s being four. This link is not always taken for granted.</td>
</tr>
</tbody>
</table>

Orientations for multiplicative tasks

<table>
<thead>
<tr>
<th>Task</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>F→N</td>
<td>Giving F, and seeking the value of N. E.g. What is eight 5s?</td>
</tr>
<tr>
<td>N→F</td>
<td>Giving N, and seeking the value of F. E.g. How many 5s in 40?</td>
</tr>
<tr>
<td>T→F&amp;N</td>
<td>Giving T, and seeking the values of F and N. E.g. I have four 10s. How many 5s? How many altogether?</td>
</tr>
</tbody>
</table>

Strategies for multiplicative tasks involving 5s

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Counting by 5s from 5</td>
<td>E.g. To solve eight 5s, counting “5, 10, 15, 20 … 40!” keeping track of eight counts.</td>
</tr>
<tr>
<td>Counting-on by 5s from a lower multiple</td>
<td>E.g. To solve eight 5s: “Five 5s is 25, then 30, 35, 40!”, keeping track of making three more counts.</td>
</tr>
<tr>
<td>Using a neighbouring multiple</td>
<td>E.g. To solve nine 5s: “Eight 5s is 40, so nine 5s is 45”, or “Ten 5s is 50, so nine 5s is 45”.</td>
</tr>
<tr>
<td>Combining two known multiples</td>
<td>E.g. To solve eight 5s: “Five 5s is 25, three more 5s is another 15, 25 and 15 makes 40”.</td>
</tr>
<tr>
<td>Calculating as a proportion of a multiple of 10</td>
<td>E.g. To solve eight 5s: “Eight 5s is four 10s, which is 40”. i.e. using the relation ½F→T</td>
</tr>
</tbody>
</table>
### Dimensions of instructional progression

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>RNG</td>
<td>Range: The range of numbers involved in tasks, as in low 5s, high 5s, even multiples.</td>
</tr>
<tr>
<td>ORN</td>
<td>Orientation: For a task about an arithmetic relationship, which values are given, and which are to be determined.</td>
</tr>
<tr>
<td>SET</td>
<td>Setting: The materials in which a task is set, such as screened 5-tiles; or if no materials are used, a setting of bare numbers.</td>
</tr>
<tr>
<td>NTN</td>
<td>Notation: Notations written to pose or record or otherwise accompany a task.</td>
</tr>
<tr>
<td>STR</td>
<td>Attention to structure &amp; strategies: Words or actions by the teacher to draw attention to the structuring of number relationships, or to computation strategies.</td>
</tr>
</tbody>
</table>
Chapter 5 – Lesson 10 Sequence, Layers A, B, C, and D: Analysis of Each Segment

With this chapter, I begin the main work of my thesis: the analysis of the nested cases of the Lesson 10 Sequence and the High 5s Sequence. This chapter is the first of the two chapters analysing the Lesson 10 Sequence. The purpose of these two chapters is to illuminate the role of the five dimensions in the instructional progression through the sequence.

The Lesson 10 Sequence is an extended episode of instruction from the second lesson of the High 5s Sequence, parsed into the three segments 10.5, 10.6, and 10.7. As explained in Section 4.2, Blair arrived at significant challenges and insights over the course of these three segments, and the issues that arose in the instruction here prefigured much of what followed in subsequent segments on high 5s, so the segments make a rich case for analysis.

As described in Section 4.7, the analysis of the case is organised in six layers:

- Layer A: Observation;
- Layer B: Local interpretation;
- Layer C: Progressions in student activity;
- Layer D: Progressions in each dimension of instruction;
- Layer E: Interactions between dimensions;
- Layer F: Multidimensional progressions in instruction.

This chapter, Chapter 5, addresses Layers A to D for each of the three Lesson 10 segments in turn. These close analyses of each segment lay the groundwork for the next chapter, Chapter 6, which addresses Layers C to F for the sequence of segments as a whole, characterising the instructional progression running through the Lesson 10 Sequence.
The chapter is in four main sections. A brief first section gives an outline of the Lesson 10 Sequence and introduces the major Lesson 10 Analysis Chart which summarises the sequence. The first section also explains how the layers of analysis are presented in the remainder of the chapter. Following this first section, the three main chapter sections analyse the three segments in turn: Segments 10.5, 10.6, and 10.7.

5.1 Lesson 10 Sequence: Outline, Chart, and Organisation of Analysis

5.1.1 Outline of the Lesson 10 Sequence

Before launching into the analysis, I offer the following outline of the story of the Lesson 10 Sequence. In Section 4.2 I gave an overview of Blair’s High 5s instruction overall. There I explained that, before the High 5s Sequence began, Blair did not have an efficient multiplicative strategy for multiplication or division of high 5s. The first lesson segments in the sequence were Segment 8.8, followed by the Lesson 10 segments 10.5, 10.6, and 10.7. Segment 8.8 began with a setting of 5-tiles, but quickly progressed to a sequence of bare number tasks: given even number of 5s \( F \), calculate the number of 10s \( T \). Blair had difficulty with the tasks, and did not appear to recognise a relationship between \( F \) and \( T \). The plan for the high 5s topic in Lesson 10 was to return to the 5-tile setting, and experiment with the task type of incrementing and decrementing with 5-tiles. The lesson turned out to have extra time, and we made considerable progress with the high 5s.

Lesson 10 was long—52 minutes—and after four shorter segments on other topics, the majority of the lesson was devoted to high 5s—some 33 minutes. I have divided the high 5s instruction into three segments: 10.5 (10 minutes), 10.6 (11 minutes), and 10.7 (12 minutes). Segment 10.5 involved incrementing and decrementing with 5-tiles, both visible and screened. Segment 10.6 switched the task type to division with screened 5-tiles. Segment 10.7 continued this task type, including multiplication and division with 5-tiles, and incorporated recording tasks in an impromptu table.

At the end of 10.5, Blair announced an insight concerning the relation of multiples of 5 and multiples of 10. He used this insight successfully to solve the five tasks of 10.6, but then in the last task had a significant block in
answering a supplementary question: how many 10s in 90. Having finally resolved this block, he solved the tasks of 10.7 with increasing insight and fluency, and ended the segment articulating a deeper explanation of the relation of 5s and 10s. Many passages in the instruction will serve as rich examples for analysis of the instructional progression.

5.1.2 Lesson 10 Analysis Chart

The large Lesson 10 Analysis Chart (Figure 5.1) summarises the task-by-task Layer A and Layer B analyses for all the tasks in the Lesson 10 Sequence. This chart becomes an important reference when following the later Layer C, D, E, and F analyses of the progressions in Blair’s activity and the progressions in instruction across the tasks.

Overall, the rows of the chart are grouped into the three horizontal segments: 10.5, 10.6, and 10.7. The columns of the chart are grouped into Layer A and Layer B sections. The Layer A section has the grey column Tasks on the left and the white column Blair’s responses on the right; and the Layer B panel echoes this arrangement, with the rainbow and grey task dimension columns on the left, and the white response column on the right.

Each row of the chart addresses a single task or sub-task, with the task number shown at the beginning of the row. Across each row, the Layer A and B analyses for the task are summarised, with key quotes included to help recall key episodes. The first column summarises the task posed. The second column summarises Blair’s response to the task. Response codes used are listed at the foot of the chart: ✓ for correct, ✓✓ for correct with confidence, and so on. The coloured columns in the middle indicate how the task instruction was calibrated on each of the five dimensions: RNG, ORN, SET, NTN, and STR. Distinct calibration codes are used in each dimension, and are explained in the chart footer. The next grey column records further key notes on the dimensions which are not captured in the calibration columns, such as what kind of notation was used, or how attention to strategies was given. The last column summarises key aspects of my interpretation of Blair’s responses.
### Lesson 10 Analysis Chart: Tasks, responses, and dimensions

<table>
<thead>
<tr>
<th>Layer A Analysis: Observations</th>
<th>Blair’s responses</th>
<th>Layer B Analysis: Local Interpretations</th>
<th>Notes on dimensions</th>
<th>Interpretations of Blair’s responses</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Tasks</strong></td>
<td><strong>RING</strong></td>
<td><strong>ORN</strong></td>
<td><strong>SET</strong></td>
<td><strong>NTN</strong></td>
</tr>
<tr>
<td><strong>19.5 Incrementing &amp; decrementing with 6-tiles</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.1 Visible: One 6, two 2s...six 6s.</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>How can you see 30 there?</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td><strong>5.2 Soundbox:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6s</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>6s</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td><strong>5.3 Two 4s (40, 8 6s)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>5.4 Two 4s (40, 8 6s)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>5.5 Six 4s (60, 6 6s)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>5.6 Two 6s (40, 8 6s).</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Let’s write this down, this is important.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>&amp;“Take the zero off the 60, then just double the 3.”</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>19.5 Division with 6-tiles: N=6</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6d</td>
<td>8 cards</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>6g</td>
<td>10 cards &amp;“Six, it’s working!”</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>6h</td>
<td>100, no...14.</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Counted tiles in pairs “10, 15, 20...6 2 6...”</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>19.6 Labeled boxes: N=6</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6e</td>
<td>12 cards</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>What are you going to see there?</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Two cards here, 2 6 that’s 30, then double that.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>How many cards for 30? for 60?</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Two, four, six...</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Two, four, six...twelve.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>How could you get that 12 just from 60?</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>“Take the zero off the end of the 60...double the 6.”</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6f</td>
<td>90 dots</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>How many 10s don’t count.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>&amp;“Stack...that is 18 X 9 =”</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5g</td>
<td>Labeled box: How many dots?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>How many 5s?</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5, 10, 15...8...</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>How many 10s?</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8h</td>
<td>Worked: 60 x 18 = 9 = 6 x 10 = 90.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>90 is nine 10s. Is that surprising?</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>19.7 Varied orientations with 6-tiles, recording in a table</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7.1</td>
<td>Find T, F.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Labeled box: Check Y, F.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7.2</td>
<td>Find T, F.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7.3</td>
<td>Find T, F.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7.4</td>
<td>Find T, F.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7.5</td>
<td>Find T, F.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Labeled box: Check N, F, F.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Check N, F, F.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>“Everything is correct.”</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Notes on dimensions</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Interpretations of Blair’s responses</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**CODES**
- Codes as per Table 4.2.
- Proportional N-F-T strategies: ✓ correct ✔ correct with confidence ✗ incorrect...thinking time
- N= total n= tens digit of number N F= number of 5s T= number of 10s
- N6: Blair noticed 6 x 5 = 30.
- STR: Inspired about visual structuring. Counted pairs of 5-tiles as 10s.

---

**Lesson 10 Analysis Chart**

This chart is reproduced in full A3 size in Appendix 5.
The chart can be read horizontally, to recall key episodes, such as: in which task Blair had his $2n \to F$ insight; in which task I introduced the arrow sentences; and what STR question I posed then. This is how the chart was constructed: task by task.

The chart can also be read vertically down the dimension calibration columns, to track the progression in each dimension. This is how the chart supports the Layer D analysis of progressions in dimensions, and leads toward the Layer E and F analyses of the multidimensional progression. The dimension calibration codes have been presented to assist reading these vertical progressions. For four of the dimensions, the calibrations have been summarised on three-point scales:

- **RNG:** $L_0 =$ low 5s, $H_1 =$ high 5s, $50+ =$ beyond 50
- **SET:** $\Box =$ 5-tiles visible, $\Box =$ screened, $\_ =$ bare numbers
- **NTN:** $\_ =$ no notation, $\Rightarrow =$ some notation, $\Rightarrow =$ task based in notation
- **STR:** $\_ =$ no STR attention, $\star =$ some STR, $\star =$ task focused on STR

The three point codes are offset horizontally within each column, so that the progression of adjustments in calibrations is visible down each column. The other dimension, ORN, uses the regular orientation codes rather than a three-point scale, but these codes have also been offset within the ORN column to give a visual cue to adjustments in ORN.

### 5.1.3 Organisation of the analysis: Layers A, B, C, and D

The multi-layered, multidimensional analysis in this chapter requires clear organisation. The basic organisation is to analyse separately each of the three segments 10.5, 10.6, and 10.7. For each segment, the analysis is made in three sections.

The first section works through the segment task by task, giving a Layer A account and a Layer B local interpretation for each task in turn. The Layer A account of what the teacher and student did and said in a task includes considerable detail, since the study is interested in several aspects of the reasoning and teaching. Layer B interprets how the task instruction was pitched on each of the five dimensions, and interprets Blair’s activity in the task in terms of usefulness to the study, such as his use of settings or notation,
his apparent understanding of multiplicative units, or his fluency with a multiplicative strategy. These Layer A and B analyses are summarised in the Lesson 10 Analysis Chart, Figure 5.1.

Following this first task-by-task section, the second section for each segment addresses Layer C, analysing Blair’s activity across the whole segment. The Layer C analysis draws together the Layer B interpretations of his activity on each task, to suggest evidence of his engagement with the target mathematics, evidence of his working at his cutting edge, evidence of his current levels of knowledge, and evidence of progress in his knowledge.

The third section for each segment addresses Layer D, analysing the instructional progressions through the segment in each of the five dimensions. The analysis draws on Layers A and B to develop a rich account of each dimension, and to better understand how each dimension supported the reasoning and learning identified in Layer C.

5.2 Segment 10.5. Incrementing and Decrementing With 5-tiles

5.2.1 Overview of Segment 10.5

Segment 10.5 involved the task type incrementing and decrementing with 5-tiles. While this task type had been used with 4-tiles in the previous term, this was the first time, and in the end the only time, this task type was pursued in this term. There were six tasks. After an introductory task with visible tiles, the remaining five tasks involved screened tiles, sometimes unscreening to check answers, with Blair keeping track of both the number of 5-tiles, \( F \), and the total number of dots, \( N \). I developed a practice of arranging the 5-tiles in pairs, which became standard in the three Lesson 10 segments, and in later segments in the High 5s Sequence. After solving the sixth task, Blair announced an insight into the multiplicative relation between \( N \) and \( F \).

Blair’s reasoning in the segment appeared embedded in the setting of 5-tiles. His interest in structuring the setting seems important, especially his final insight into the relation of \( N \) and \( F \), which was a breakthrough in his learning about multiplicative relations. Of interest in the instructional progression is how the instruction used screening of the setting, how it challenged Blair to track both \( F \) and \( T \), and how it attended to Blair’s structuring.
Segment 10.5: Task-by-task account (Layer A) and interpretations (Layer B)

Task 10.5.1 (Layer A): Incrementing by 5s up to six 5s.
5-tiles were placed one by one up to six 5s, in a pair-wise arrangement, with Blair saying at each increment both the current number of 5s, and the total number of dots. After six 5s, I asked:

T\(^1\):  How can you see 30 there?
Blair:  10, 20, 30 [in coordination with touching the three pairs of 5-tiles in succession].

Blair then remarked that you could also calculate the number of dots as five 6s, which led to a short discussion of arrays, and Blair drawing a 6×5 grid with the annotation “= 6 × 5 = 30”.

Task 10.5.1 (Layer B): Local interpretations

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Calibrations in this task</th>
</tr>
</thead>
<tbody>
<tr>
<td>RNG</td>
<td>Low 5s</td>
</tr>
<tr>
<td>ORN</td>
<td>Given increments in (F), find (N)</td>
</tr>
<tr>
<td>SET</td>
<td>Visible</td>
</tr>
<tr>
<td>NTN</td>
<td>Incidental: Letting student record (6 \times 5 = 30)</td>
</tr>
<tr>
<td>STR</td>
<td>Attention with question <em>How can you see 30?</em></td>
</tr>
</tbody>
</table>

**Blair’s activity.** In this visible tile setting, in the range up to six 5-tiles, Blair could fluently name both the number of 5s \((F)\), and the total number \((N)\). Furthermore, he could count \(N\) by counting the pairs of 5-tiles as tens. He also used a formal number sentence to express the multiplication of six 5s.

5.2.2.2  Tasks 10.5.2, 10.5.3 (Layer A): Six 5s and another 5, and another 5.
Returning to the incrementing tasks, the six 5-tiles were screened (Fig. 5.2).

Figure 5.2  Task 10.5.2. Screening the six 5-tiles.

\(^1\) ‘T’ stands for teacher throughout the transcripts.
A 5-tile was added, and he calculated that there were 35 dots. Another 5-tile was added, and he calculated that there were now 40 dots. I then asked

T: How many 5s?
Blair: At 30 you had six 5s. Then you added another two 5s, so that’s eight 5s.

When the tiles were unscreened, showing four pairs of 5-tiles, he confirmed his answers (see Figure 5.3).

**Figure 5.3** Task 10.5.3. Unscreening the eight 5-tiles, with Blair confirming his answers.

---

### 10.5.2, 10.5.3 (Layer B): Local interpretations

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Calibrations in this task</th>
</tr>
</thead>
<tbody>
<tr>
<td>RNG</td>
<td>Advanced to high 5s</td>
</tr>
<tr>
<td>ORN</td>
<td>Continued: Given increments in $F$, find $N$</td>
</tr>
<tr>
<td></td>
<td>Introduced finding $F$</td>
</tr>
<tr>
<td>SET</td>
<td>Advanced by screening, while unscreening to check answers.</td>
</tr>
<tr>
<td>NTN</td>
<td>-</td>
</tr>
<tr>
<td>STR</td>
<td>-</td>
</tr>
</tbody>
</table>

**Blair’s activity.** With the tiles now screened, Blair successfully kept track of increments in both $F$ and $N$. At 40 dots, he did not find $F$ directly from $N$; rather, he summed the increments in $F$.

**5.2.2.3 Task 10.5.4 (Layer A): Eight 5s and another 5**

With eight 5-tiles screened again, another 5-tile was added.

T: Another 5. How many 5s now?
Blair: If you added another 5, it wouldn’t be even, there would be just another one there. So if you had eight 5s, you would have nine 5s. So that would be about … ah, what number? … 50.

T: [Unscreens tiles.]
Blair: [Touching the four pairs of tiles] 10, 20, 30, 40, [then the lone tile] 45. Oh!

T: [Smiling.] You even described that we didn’t have a whole 10 at the end there. So what is it?

Blair: Ah, 45.

T: How many 5s?

Blair: How many 5s are there? There’s nine 5s.

### 10.5.4 (Layer B): Local interpretations

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Calibrations in this task</th>
</tr>
</thead>
<tbody>
<tr>
<td>RNG</td>
<td>Continued: high 5s</td>
</tr>
<tr>
<td>ORN</td>
<td>Continued: Given increments in $F$, find $N$ and $F$</td>
</tr>
<tr>
<td>SET</td>
<td>Continued: Screening, with unscreening to check.</td>
</tr>
<tr>
<td>NTN</td>
<td>-</td>
</tr>
<tr>
<td>STR</td>
<td>Attention with comment: <em>You even described that we didn’t have a whole 10 at the end there.</em></td>
</tr>
</tbody>
</table>

**Blair’s activity.** His first comment “If you added another 5, it wouldn’t be even, there would be just another one there”, suggests he was visualising the tiles, and had developed a distinction between an even arrangement—presumably when all the 5-tiles are paired—and an uneven arrangement—when there is a lone 5-tile. Nevertheless, while he recognised that nine tiles “wouldn’t be even”, he incorrectly calculated $N$ to be an even 50. We know that adding 5 to 40 was normally straightforward for Blair, so I presume he was trying to do more than add 5 to 40. It is possible that his reasoning involved adding one unit to 40, but instead of incrementing by the unit of 5, he incremented by the unit of 10, to arrive at 50. If so, this was a task where he could not keep clear the distinction between the units of 5 and 10.

**5.2.2.4 Task 10.5.5 (Layer A): Nine 5s take three 5s**

The nine 5-tiles were screened again.

T: I’m gonna take three 5s off. [Removes three 5-tiles.]

Blair: Nine 5s … Now you have six 5s.

T: And how many dots?

Blair: [Pointing to recent notation in book “$6 \times 5 = 30$”] Six 5s: 30.

T: Very nice.
### 10.5.5 (Layer B): Local interpretations

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Calibrations in this task</th>
</tr>
</thead>
<tbody>
<tr>
<td>RNG</td>
<td>Continued: high 5s</td>
</tr>
<tr>
<td>ORN</td>
<td>Continued: Given increments in $F$, find $N$ and $F$</td>
</tr>
<tr>
<td>SET</td>
<td>Screening.</td>
</tr>
<tr>
<td>NTN</td>
<td>-</td>
</tr>
<tr>
<td>STR</td>
<td>-</td>
</tr>
</tbody>
</table>

**Blair’s activity.** Blair appeared to calculate $F$ as six by decrementing three from nine. However, he calculated $N$, not by decrementing, but rather as a known fact that six 5s is 30, pointing with a gesture of confirmation to the number sentence $6 \times 5 = 30$.

### 5.2.2.5 Task 10.5.6 (Layer A): Six 5s add two 5s

Six 5-tiles were still screened.

- **T:** Now I’m gonna add two 5s on. [Places two more 5-tiles behind screen.]
- **Blair:** Two 5s … 30, fif… ah, 30, 35, 40. 40.
- **T:** How many 5s?
- **Blair:** … Eight. Eight 5s.

The tiles were unscreened again, and looking at them he confirmed his calculations. He then exclaimed about a new insight.

- **Blair:** Say, if you said there are 30, there are 30 dots, on 5-, on 5-dotted cards, how many cards were there? You said 30 dots [showing three fingers]? Six cards. You double the—you take the zero off the 30 [making a wipe down gesture in the air], then you just double the three.

He gave another example, calculating how many 5-dot cards for 40 dots as “80—8 … you have to take the zero off the end”. The teacher said *Let’s write this down, this is important* and wrote an informal notation for the strategy Blair had explained, shown in Figure 5.4. Blair then copied this notation into his own notebook, saying “I feel really good taking notes. Feel like I’m in high school”.

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10.5.6 (Layer B): Local interpretations

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Calibrations in this task</th>
</tr>
</thead>
<tbody>
<tr>
<td>RNG</td>
<td>Continued: high 5s</td>
</tr>
<tr>
<td>ORN</td>
<td>Continued: Given increments in $F$, find $N$ and $F$</td>
</tr>
<tr>
<td>SET</td>
<td>Screening, with unscreening to check.</td>
</tr>
<tr>
<td>NTN</td>
<td>Incidental: Teacher notated Blair’s new strategy, and Blair copied this notation</td>
</tr>
<tr>
<td>STR</td>
<td>Attention was drawn to Blair’s structuring insight, affirming <em>This is important.</em></td>
</tr>
</tbody>
</table>

**Blair’s activity.** Blair was successful in the task, keeping track of the increment of both $N$ and $F$. He then had an insight about a general relationship when given $N$: how to find $F$. I have coded the tens digit of $N$ as $n$, and thus coded the relation he describes here as $2n \rightarrow F$; that is, doubling the digit $n$ generates $F$. The paired tile setting was important for Blair’s new insight. The insight arose in the context of the pair-wise 5-tiles, and was expressed in these terms—the cards and dots as he called them. The insight came immediately after tasks having six 5s/30 dots, and eight 5s/40 dots, and these were also the two examples he gave to explain his insight. I conjecture that he recognised a doubling pattern in how 30 linked with three pairs and with 6, while 40 linked with four pairs and with 8, and that having recognised the doubling pattern, he anticipated the pattern would generalise. He
expressed his procedure in terms of the numeral for \(N\): “You have to take the zero off the end”. This appeared to be his way of articulating the three-ness of 30. I suspect he saw this three-ness in his visualisation of three pairs of 5-tiles—note that while proposing the example of 30 dots, he held up three fingers. However, he did not yet articulate 30 explicitly as three tens, and I do not presume he yet recognised a relation directly with the number of tens \((T)\), as such.

### 5.2.3 Segment 10.5 (Layer C): Progressions in Blair’s activity

The examples discussed in this Layer C analysis are generally summarised in the Lesson 10 Analysis Chart (Figure 5.1), so the chart will be a helpful reference throughout this section.

A feature of Segment 10.5 was how Blair’s reasoning appeared embedded in the 5-tile setting. He interpreted his task as keeping track of how many tiles, and how many dots. His language referred to the tiles, and he kept close attention on the tiles when they were unscreened for checking.

Over the course of Segment 10.5, Blair made initial progress in his knowledge of the \(N\)-\(F\)-\(T\) relations. In Task 10.5.1, the unscreened task, his use of counting the pairs of tiles as 10s to calculate \(N\) indicated an initial implicit knowledge of relations between 5s, 10s, and \(N\). In Task 10.5.4, his distinction between even and uneven arrangements showed an important early structuring of the 5s–10s relationship. He was generally able to coordinate keeping track of both \(F\) and \(N\), but this was not straightforward for him, as indicated by his difficulties at Task 10.5.4, eight 5s and another 5.

The tasks were posed in terms of increments and decrements by 5s, but there were opportunities for using multiplicative reasoning. Blair’s calculations of \(F\) and \(N\) did not use a multiplicative strategy. He found \(F\) by incrementing from the previous \(F\), as in Task 10.5.3, when he said “At 30 you had six 5s. Then you added another two 5s, so that’s eight 5s”, and in Task 10.5.4, when he calculated eight 5s and one more 5 made nine 5s. Similarly, he calculated \(N\) by incrementing by 5s from the previous \(N\), as in Task 10.5.6, six 5s add two 5s, when he said “30, 35, 40”. That is, he was calculating \(F\) and \(N\) by increments, which are additive; he was not using a multiplicative relation from \(N\) directly to \(F\), or vice versa. One notable exception to this additive...
strategy was in Task 10.5.5, nine 5s take three 5s, when having calculated $F$ to be six 5s, he appeared to recognise that six 5s is 30, without calculation.

His $2n \rightarrow F$ insight at the end of the segment was the first time he suggested a multiplicative relationship directly from $N$ to $F$, a relationship he expected to work across other examples. This seems a significant breakthrough. Note that his insight addressed the task in the orientation $N \cdot F$, which is the orientation adult mathematics would label as division.

In summary, in Segment 10.5, Blair’s reasoning appeared embedded in the setting of paired 5-tiles. There were interesting moments of early structuring involving $N$-$F$-$T$ relations, such as: calculating $N$ by counting pairs of 5s as 10s, visualising 5-tile arrangements as even or uneven, and keeping track of $F$ and $N$ simultaneously. Finally, he made an important breakthrough to multiplicative reasoning with the $2n \rightarrow F$ insight.

5.2.4 Segment 10.5 (Layer D): Dimensions of instruction

The examples discussed in this Layer D analysis are generally summarised in the Lesson 10 Analysis Chart (Figure 5.1), so the chart will be a helpful reference throughout this section.

5.2.4.1 RNG. Range

These segments in Lesson 10 were planned as instruction in high 5s, so the RNG dimension would be expected to remain mostly in the high 5s. The introductory Task 10.5.1 was posed in the low 5s, and Blair responded with fluency. The subsequent tasks progressed to the high 5s range, and became more challenging for Blair. He did not use knowledge of the high 5s basic facts to solve the tasks, with the exception of recognising six 5s as 30. Thus, the initial shift of RNG from low 5s to high 5s confirmed that the high 5s were an appropriate topic at the cutting edge of Blair’s knowledge, and RNG then remaining steady helped keep the tasks at his cutting edge.

5.2.4.2 ORN. Orientation

In Segment 10.5, the ORN was changed by regularly switching the sought value between the number of dots, $N$, and the number of 5s, $F$. Seeking both these units was important in revealing and supporting Blair’s reasoning. In Task 10.5.1, incrementing visible 5-tiles up to six 5s, Blair reported both $F$
and $N$ at each increment. Task 10.5.2, with screened tiles, only asked for $N$. At Task 10.5.3, after finding $N$, the ORN shifted to ask also for $F$. Blair’s answer revealed that he did not use a relation from $N$ directly to $F$, but tracked the increments in $F$ independently. At Task 10.5.4, eight 5s and another 5, with the ORN to find both $F$ and $N$, Blair correctly calculated $F$ as 9, but miscalculated $N$ as 50, revealing a potential confusion of units of 5 and 10. In both Tasks 10.5.5 and 10.5.6, the ORN continued to involve finding both $F$ and $N$. Subsequently, Blair had his $2n \to F$ insight, recognising a relationship directly between the values of $N$ and $F$. I conjecture that the ORN of keeping track of both values throughout the segment contributed to Blair’s reaching this insight.

Blair’s insight was essentially in the inverse orientation to the prevailing orientation in the segment. Each task in the segment gave an increment in $F$, so the prevailing orientation was from partly known $F$, to find $N$. On the other hand, when Blair had his $2n \to F$ insight, he expressed it in the inverse orientation: given $N$, how to find $F$. Thus, he here made a shift from tasks in a multiplication orientation to an insight about the division orientation. I think it is significant to observe that students can have insights about one orientation that arise from tasks in another orientation.

5.2.4.3 SET. Setting

The instructional setting of 5-tiles was established as the context for the tasks in Segment 10.5. Using this setting meant the tasks could be expressed in terms of numbers of dots and tiles, expressions which could be curtailed toward the more abstract constructs of total number, and number of 5s. Blair could treat the tiles as the reference for checking his answers. His expression of his final insight in terms of the tile setting indicates the importance of the setting to his reasoning at this point.

The 5-tiles were arranged in pairs, and this pair-wise arrangement appeared significant for Blair’s reasoning. For example, after first reaching six 5s, he confirmed there were 30 dots by counting the three pairs of tiles as “10, 20, 30”. Also, his comment on the task of eight 5s and another 5, that “it wouldn’t be even, there would be just another one there” suggests he was
visualising pairs of 5s, and trying to use this structure to help his calculations. His final insight arose closely linked to the pair-wise arrangement.

The distancing of the setting along the SET dimension was a feature of the instruction. By presenting the first tasks with the 5-tiles visible, the tiles were established as the context for the tasks, and Blair established his pair-wise structure for counting the tiles. Note that with the tiles visible, the task did not actually require calculating an increment from the previous total: the answers could be calculated directly from counting the visible tiles. It was only by screening the tiles that the task required the calculation of an increment. Screening the tiles—advancing along the SET dimension—appeared to challenge Blair’s reasoning: he had to think hard to visualise the pair-wise tiles, and to keep track of both $F$ and $N$. Blair succeeded at seven 5s, and eight 5s, but at nine 5s made the interesting error of 50 for $N$.

The temporary unscreening—stepping back along SET—appeared important for his reasoning. Unscreening after eight 5s, he confirmed his answer and clarified his pair-wise visualisation, which then featured in his reasoning about the task of eight 5s and another 5. After this task, unscreening again allowed Blair to confirm that it was an uneven arrangement of tiles, as he had visualised, but the total did not amount to 50, rather to 45. I expect such immediate feedback on the aspects he had been thinking about was important for learning. After tasks 10.5.5 and 10.5.6, another unscreening to check the eight pair-wise tiles appeared to prompt Blair to his insight about $2n \rightarrow F$.

Thus, the strategic ratcheting back and forth along SET by screening and unscreening the setting had Blair at his cutting edge, thinking hard about the topic of instructional interest: the structuring of 5s in relations to 10s, and $F$ in relation to $N$.

5.2.4.4 NTN. Notation

There were two instances of notating, each supporting subsequent problem-solving. The first occurred in Task 10.5.1, after the discussion of alternative ways to count six 5s. The teacher gave Blair time to make notes: he drew a $6 \times 5$ grid, and wrote the number sentence “$6 \times 5 = 30$”. These notes made the first explicit link between the task in 5-tiles, and the formal written expression
of multiplication. Also, in Task 10.5.5 these notes supported Blair recalling that he knew what six 5s was without calculation.

The second instance of notating occurred at the end of the segment, when the teacher notated Blair’s $2n \rightarrow F$ insight. The value of making this notation is evident in the subsequent segment, so this instance is analysed in the discussion of that segment.

5.2.4.5 STR. Attention to structuring and strategies

There were three instances of STR in Segment 10.5, when the instruction attended to structuring. The first STR comment was in Task 10.5.1 when, after incrementing to six 5s, I asked How can you see 30 there? The question served as an assessment, revealing that Blair noticed that the tiles were paired, and that he could count the pairs of 5-tiles as 10s. Furthermore, Blair suggested a second way of seeing 30 without further prompting—as five 6s—taking for granted that such an alternative was of interest. Thus, the teacher’s attention to structuring the setting created an opportunity for Blair to investigate further structuring.

A second STR comment came at the end of Task 10.5.4, eight 5s and another 5. After the tiles were unscreened and Blair realised that $N$ was 45 not 50, I said You even described that we didn’t have a whole 10 at the end there. This remark affirmed that Blair’s reasoning about the last 5-tile being uneven had been correct, and significant. Further, the remark introduced referring to the paired 5s as “a whole 10”.

A third STR comment came after Blair explained his insight structuring the $2n \rightarrow F$ relation, when the teacher affirmed Let’s write this down. This is important. I conjecture that this recurring instructional attention to STR supported Blair’s evident attention to structuring the $N$-$F$-$T$ relations over the course of this segment, and the subsequent segments.

5.3 Segment 10.6. Division With 5-tiles: $N \rightarrow F$.

5.3.1 Overview of Segment 10.6

The previous task, 10.5.6, had ended with me informally notating Blair’s $2n \rightarrow F$ insight. I continued the lesson, wanting to attend to the new insight. Saying Let’s try it a few times, I posed tasks in the same form that he had
used to describe his insight. This amounts to the task type *division with 5-tiles*, where given the total number of dots $N$ on screened 5-tiles, he is to say the number of 5s $F$. I posed five such tasks: 40 dots, 50 dots, 70 dots, 60 dots, and 90 dots, and he succeeded on each. However, in the 90-dot task, I introduced a supplementary question of finding how many tens $T$, and Blair got stuck. Instruction addressing this same 90-dot task continued for some seven minutes. I have divided these seven minutes into three passages labelled 10.6.5a, 10.6.5b, and 10.6.5c. The problem-solving exchanges in these passages are revealing, and led to the introduction of some arrow sentence notation in 10.6.5c, which became significant in Segment 10.7.

Through Segment 10.6, Blair showed excitement at the success of his new multiplicative strategy, which contrasted with the protracted difficulties of the final task. Of particular interest in the instructional progression are: the development of a sequence of tasks on which to try Blair’s new strategy, the introduction of the supplementary question of finding $T$ in Task 10.6.5, and the responses to Blair’s subsequent difficulties.

5.3.2 **Segment 10.6: Task-by-task account (Layer A) and interpretations (Layer B)**

5.3.2.1 **Task 10.6.1 (Layer A): Given 40 dots, find $F$**

- **T:** [Screened: Four pairs of 5-tiles.] I’ve got 40 dots.
  - Blair: Eight. Eight cards.
- **T:** [Unscreens tiles.] You right?
  - Blair: Yes. Two, four, six, eight [in coordination with two-fingered pointing to each of the four pairs of tiles].

**10.6.1 (Layer B): Local interpretations**

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Calibrations in this task</th>
</tr>
</thead>
<tbody>
<tr>
<td>RNG</td>
<td>Continued: High 5s</td>
</tr>
<tr>
<td>ORN</td>
<td>$N \rightarrow F$</td>
</tr>
<tr>
<td>SET</td>
<td>Continued: Screening, with unscreening to check.</td>
</tr>
<tr>
<td>NTN</td>
<td>-</td>
</tr>
<tr>
<td>STR</td>
<td>Attention was drawn to Blair’s new $2n \rightarrow F$ strategy, by the explicit framing of the task as trialling the strategy</td>
</tr>
</tbody>
</table>

**Blair’s activity.** Blair answered fluently, presumably using his new strategy of doubling the 4 in 40 to find the answer of 8.
5.3.2.2 Task 10.6.2 (Layer A): Given 50 dots, find $F$

T: [Screened: Five pairs of 5-tiles.] I’ve got 50 dots.
Blair: 50 dots, that’s ten cards.
T: [Unscreens tiles.]
Blair: Two, four, six, eight, ten [in coordination with pointing to the five pairs of tiles]. See, it’s working! [Clapping.]

10.6.2 (Layer B): Local interpretations

The five dimensions remained the same as they were in Task 10.6.1.

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Calibrations in this task</th>
</tr>
</thead>
<tbody>
<tr>
<td>RNG</td>
<td>Continued: High 5s</td>
</tr>
<tr>
<td>ORN</td>
<td>Continued: $N \rightarrow F$</td>
</tr>
<tr>
<td>SET</td>
<td>Continued: Screening, with unscreening to check.</td>
</tr>
<tr>
<td>NTN</td>
<td>-</td>
</tr>
<tr>
<td>STR</td>
<td>Continued: Trialling the new $2n \rightarrow F$ strategy.</td>
</tr>
</tbody>
</table>

Blair’s activity. Again, Blair presumably used his new strategy, doubling the 5 in 50 to find the answer of 10. With the comment “See, it’s working”, he showed he viewed these tasks as a trial of his strategy, and was pleased that his strategy was producing correct answers.

5.3.2.3 Task 10.6.3 (Layer A): Given 70 dots, find $F$

T: [Screened: A row of five pairs and a row of two pairs of 5-tiles.] I have 70 dots.
Blair: 70 dots, 140. No no no no—Fourteen.
T: Fourteen?
Blair: Yes, fourteen.
T: [Unscreens tiles.] Fourteen cards?
Blair: [In coordination with pointing to the pairs of tiles in sequence] 10, 15, 20, 25, 30, 35, fort—Oh, no, no, no. Sorry, sorry. Er … [Beginning the pointing sequence again] two, four, six, eight, ten, twelve, fourteen. [Raises arms in a gesture of triumph.]
T: Fourteen 5s.
Blair: I won!
T: Am I right—are there really 70 dots here?
Blair: Ah, okay. [In coordination with touching the first two tiles in sequence] 5, 10, ah. [Now in coordination with touching the first two tiles simultaneously, and continuing to touch all seven pairs in sequence] 10, 20, 30 … 70.
T: 70 dots. How many 5s?
Blair: How many 5s? Ah, 70 dots? Ah, 70. [Looking up to teacher.] How many cards?
T: How many 5s, yes.
Blair: How many 5s? Ah [looking at the tiles] … Fourteen.
T: Fourteen.
Blair: Fourteen 5s.

10.6.3 (Layer B): Local interpretations

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Calibrations in this task</th>
</tr>
</thead>
<tbody>
<tr>
<td>RNG</td>
<td>Advanced: Beyond 50.</td>
</tr>
<tr>
<td>ORN</td>
<td>Continued: N ( \cdot ) F; Switched between checking ( F ) and ( N ) a second time.</td>
</tr>
<tr>
<td>SET</td>
<td>Continued: Screening, with unscreening to check.</td>
</tr>
<tr>
<td>NTN</td>
<td>-</td>
</tr>
<tr>
<td>STR</td>
<td>Four comments shifting from speaking of “cards” to “5s”, which may have brought attention to structuring the units of 5.</td>
</tr>
</tbody>
</table>

Blair’s activity. Blair had three moments of confusion in this task, and overcame each. For his first answer of 140, which was incorrect, I presume he doubled 70, rather than 7. He self-corrected. After the tiles were unscreened, when checking how many 5s there were, he appeared confused about the units involved, counting the pairs of tiles as “10, 15, 20 …” before self-correcting the count to be “two, four, six ….” When he did confirm his answer was correct, his triumphant gestures and exclamation “I won!” show his excitement at the success of his new strategy. In contrast, when the teacher switched to asking about the number of dots, and then recapitulated the question of how many 5s, Blair appeared uncertain of the task: “How many 5s? Ah, 70 dots? Ah, 70. How many cards?” I interpret that he was uncertain of the term “5s” in the task, preferring to think in terms of the number of cards. Before answering, he looked at the tiles, and appeared to be counting them again. So he may have found the answer, 14, by counting tiles, rather than by his new strategy.

5.3.2.4 Task 10.6.4 (Layer A): Given 60 dots, find \( F \)
For 60 dots, he answered correctly 12 cards. Then, instead of unscreening, I asked What are you going to see under there?
Blair: Ah, they’re probably going to be in groups of two, three, or four
[while holding up two, three, then four fingers respectively].

T: I’ve still got them in groups of two. So what, how do you think they
are? Sketch what those groups of two are going to be like [gesturing
short strokes across the desk]. I’ve got 60 dots altogether.

Blair: 60 dots. So, two cards here [pointing with two fingers at the front of
the desk], that’s two, four, six [making two-fingered points at three
spots up the desk], so that’s about 30 there [gesturing up the same
section of desk]. Then you double that [stroking down the same
section of desk], so then you’ve got … twelve cards.

T: Mmm. [Removes screen.] Not how I had them, but let’s put them that
way. [Rearranges tiles as a row of three pairs, and a second row of
three pairs.] Like that?

Blair: Yeah, because, say if that wasn’t there [screening the second row of
tiles with one arm], that 30 [indicating the first row], then “foof”:
double [gesturing across the first row towards the second row].

There were further exchanges about the doubling gesture, and the twelve
cards. Then I asked:

T: How many cards when there’s just 30 dots? [Flashes and screens just
the first row of three pairs of tiles.]

Blair: Two, four, six [with a quick gesture of three points up the desk].

T: How many cards when there’s 60 dots? [Flashes both rows of tiles.]

Blair: 60 dots, ah … that’s what we’ve done? Ah … two, four, six—twelve.

T: Twelve.

Blair: Twelve.

T: Good. And how could you get that twelve just from the 60?

Blair: How can you get the twelve from the 60? You double the, you double
the- you take the zero off the end of the 60, you just leave the six
there, you double the six, then whatever answer by doubling that, is
the answer to your question.

10.6.4 (Layer B): Local interpretations

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Calibrations in this task</th>
</tr>
</thead>
<tbody>
<tr>
<td>RNG</td>
<td>Continued: Beyond 50 with momentary step back to high 5s.</td>
</tr>
<tr>
<td>ORN</td>
<td>Continued: $N \cdot F$</td>
</tr>
<tr>
<td>SET</td>
<td>Continued: Screening, with flashing and screening for further challenges</td>
</tr>
<tr>
<td>NTN</td>
<td>-</td>
</tr>
</tbody>
</table>
Blair’s activity. Blair answered the basic task fluently again. When asked to predict the arrangement of the screened tiles, he described a strongly structured arrangement. He described and gestured in pairs of tiles, and constructed a first row of three pairs as a unit, which he doubled to make a second row. In this visualised arrangement, he could distinguish and keep track of both $F$ and $T$: he knew the single row would be six 5s and 30, and the double row would be twelve 5s and 60. Furthermore, he could calculate $F$ by a multiplicative count of the visualised tiles, as in finding for 60 dots “two, four, six—twelve” tiles. On the other hand, at the end of the task, he could still describe his $2n \rightarrow F$ strategy, as in “take the zero off the end of the 60 … double the six”. Thus, at this point he juxtaposed two different multiplicative strategies for calculating $F$.

5.3.2.5 Task 10.6.5a (Layer A): Given 90 dots, find $F$, and find $T$

For 90 dots, he answered correctly: 18 cards. I confirmed the answer without unscreening, and then asked a new supplementary question.

T: How many 10s do I have?
Blair: How many 10s? Ah, 10, 20-[Begins again in coordination with raising fingers sequentially] 10, 20, 30, 40, 50-
T: No, don’t count them.
Blair: Oh no. Sh! [Looking at teacher.] How many 10s?
T: 90 dots.
Blair: 90 dots, how many 10s? There’d be two, four [in coordination with two-fingered points to two spots on the desk] … 40 … You’d have, ah, you’d have eight like there [spreading fingers of right hand on desk], and then the big blue ones would have another eight [turning to teacher and holding same fingers spread in air.] [Explanatory note: My teaching kit had only ten regular white 5-tiles, so for the last three tasks—70, 60, 90—to reach beyond 50, I also used large blue 5-dotted cards.] I’m guessin’ that’s how you laid it out. Is it?
T: Kinda [nodding] … But it’s not eight and eight … I’ve got 90 dots altogether-
Blair: No, no, no-
T: How many 10s?
Following this was a passage lasting 2½ minutes, during which Blair mistakenly said “90 cards”, which I corrected as 90 dots; he recapitulated there were 18 cards with five dots; he wondered whether he needed to find 18 divided by 9, but was unsure what 18 divided by 9 is; he asked if he could try writing; he also suggested 18 divided by 5; he eventually tried writing a division algorithm for 18 divided by 9, but was not sure how to do the algorithm; and finally concluded, “Then I’ve got 2. That doesn’t make any sense, I’m just gonna rub that out”.

10.6.5a (Layer B): Local interpretations

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Calibrations in this task</th>
</tr>
</thead>
<tbody>
<tr>
<td>RNG</td>
<td>Continued: Beyond 50</td>
</tr>
<tr>
<td>ORN</td>
<td>Continued: $N \cdot F$</td>
</tr>
<tr>
<td></td>
<td>New supplementary task to find $T$</td>
</tr>
<tr>
<td>SET</td>
<td>Continued: Screening</td>
</tr>
<tr>
<td>NTN</td>
<td>(Blair was allowed to try a written algorithm to calculate 18 divided by 9.)</td>
</tr>
<tr>
<td>STR</td>
<td>Challenged Blair to find $T$ without counting</td>
</tr>
</tbody>
</table>

Blair’s activity. Blair’s responses to this task revealed confusions about the three different units involved: 1s, 5s, and 10s. He found the number of 5s ($F$) fluently, as with the previous tasks in the segment. But when I challenged him to find the number of 10s ($T$) without counting, he spent over three minutes unable to find a solution. I suggest during this time he was unsure how to think about “10s” in this context. He tried to visualise the arrangement of tiles, saying “there’d be two, four” in coordination with two-fingered points to two spots on the desk, but was apparently counting the tiles, which would be 5s, rather than counting the pairs of tiles, which would be 10s. Also, he proposed an incorrect arrangement of eight and eight 5-tiles, rather than ten and eight. He tried to recapitulate what he knew—and slipped again on the terms “cards” and “dots”. Then he proposed calculating 18 divided by 9, which seemed unlikely to help him. Curiously, he did bring the number nine into his thinking here, nine being the actual answer. Where did this nine come from? Perhaps nine came from the sound of the number word “ninety”, or from a mental image of the numeral 90, or from some visualised recognition.
that there would be nine pairs of 5-tiles. Whichever way it was, he did not yet recognise that nine is the number of 10s in 90.

5.3.2.6 Task 10.6.5b (Layer A): 90-dot task continued, with tiles unscreened
Following the previous passage of three minutes without success, I unscreened the tiles.

T: Let me show you this again. [Unscreens the tiles, revealing the 5-tiles arranged as one row of five pairs, and one row of four pairs.] Tell me how many dots there are.
Blair: 10, 20, 30 … 90 [in coordination with pointing to each pair of 5-tiles].
   Oh, 90 cards.
T: 90 dots.
Blair: 90 dots.
T: How many 5s were there?
Blair: How many 5s were there? Er, nine-, how many 5s. Er, 10, 20, 30 … 90 [in coordination with two-fingered pointing to each pair of 5-tiles].
   There’s 90 dots. You said: how many cards.
T: [Nodding] How many 5s.
Blair: How many 5s. [Looking at the tiles] I’ve already done that.
T: Yeah, you have [screening the tiles again].
Blair: You just told me that. How many, how many 5s? There’s 90, [looking up] there’s 90 dots … there are eighteen 5s.
T: Eighteen 5s. [Unscreens the tiles.] Show me, show me that there’s eighteen 5s.
Blair: [Begins separating each pair of 5-tiles.]
T: [Interrupts, putting tiles back into pairs.] It’s all right to leave them there.
Blair: Oh yeah, yeah.
T: Count, count them for me.
Blair: Ah, 5, 10, 15, 20, 25, 30 [in coordination with pointing to each tile].
T: No, no, no. Don’t count the dots.
Blair: Oh!
T: Count 5s. That’s one 5 there [drags out the nearest 5-tile].
Blair: Oh! Yeah yeah yeah. Okay. Two, four, six … eighteen [in coordination with pointing to each pair of 5-tiles].
T: Eighteen. Good. We’ve got 90 dots, eighteen 5s. How many 10s?
   That’s one 10 right there [slapping a hand over a pair of 5-tiles].
Blair: [Looking at the tiles.] One, two, three four, [looking away] five, six, seven, eight. Eight, eight, eight, eight, eight.
T: Eight? Where?
Blair: [Looking back at the tiles.] Nine. How I got that? Five—four [in coordination with gesturing over the row of five pairs and the row of four pairs].

T: Five—four. Good.

... 

T: ... In the thick of all that, what we were saying is that there’s 90, how many 10s.

Blair: How many 10s.

T: And the answer is nine [looking at Blair].

Blair: Mm [looking back].

T: There’s nine 10s in 90.

Blair: Yes, but there are, t-, there are eighteen cards.

T: How many on each card?

Blair: Er, five dots.

T: So we can say, there’s eighteen 5s.

Blair: Eighteen 5s.

10.6.5b (Layer B): Local interpretations

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Calibrations in this task</th>
</tr>
</thead>
<tbody>
<tr>
<td>RNG</td>
<td>Continued: Beyond 50</td>
</tr>
<tr>
<td>ORN</td>
<td>Continued: $N$-$F&amp;T$</td>
</tr>
<tr>
<td>SET</td>
<td>Retreated: Visible 5-tiles</td>
</tr>
<tr>
<td>NTN</td>
<td>-</td>
</tr>
<tr>
<td>STR</td>
<td>Most of the teacher’s utterances were directed toward structuring the units. Some examples are highlighted here.</td>
</tr>
<tr>
<td></td>
<td>- The teacher clarified the names of the units, such as when Blair spoke of “90 cards” and the teacher corrected as 90 dots.</td>
</tr>
<tr>
<td></td>
<td>- The teacher asked Blair to demonstrate the values of $N$, $F$, and $T$ in the setting, as in Show me that there’s eighteen 5s.</td>
</tr>
<tr>
<td></td>
<td>- The teacher showed a single unit in the setting, as in: That’s one 5 there; That’s one 10 right there.</td>
</tr>
<tr>
<td></td>
<td>- The teacher re-presented answers for reflection: What we were saying is that there’s 90, how many 10s ... and the answer is nine [looking at Blair].</td>
</tr>
</tbody>
</table>

Blair’s activity. In this passage with the tiles unscreened, Blair revealed further difficulties with the units. In particular, he had difficulty interpreting the units of 5 and 10 in the context of the tiles. He spoke of 90 cards, rather than 90 dots. He could not recall how many 5s there were, counting the dots again, and eventually looking away from the tiles to recall his answer of
eighteen 5s. Asked to show the eighteen 5s in the tiles, he wanted to separate the pairs, and when he counted, he counted each tile as five rather than one.

When the teacher identified 10 with a pair of tiles, Blair initially miscounted the number of pairs as four and four, making eight. When he looked sufficiently carefully to see that the arrangement of pairs was five-four rather than four-four, he did recognise this as nine. “Nine. How I got that? Five—four”. Thus, his difficulty was not in recognising an arrangement of nine pairs, but in visualising the 18 tiles, and identifying the pairs as 10s.

At this point, he had found \( N, F, \) and \( T \): 90 dots, eighteen 5s, and nine 10s. In an attempt to prompt his recognition of the relationship between \( N \) and \( T \), I recapitulated to him his finding that \( \text{There's nine 10s in 90} \). He responded, “But there are eighteen cards”. He appeared to be still concerned with the challenge of clarifying the three different quantities, rather than recognising the relationships between the quantities.

5.3.2.7 Task 10.6.5c (Layer A): 90-dot task continued, with writing
I then moved to write in the workbook. I said that we had said that 90 was eighteen lots of 5, and also nine 10s, while writing “\( 90 \rightarrow 18 \times 5 \rightarrow 9 \times 10 \)” (see Figure 5.5). Blair expressed agreement, and wanted to copy the notation.

- **T:** Now I just wanna have a look at this for a moment. 90 … [indicating numeral 90 in book]
- **Blair:** Mm hm.
- **T:** … is nine 10s [indicating expression \( 9 \times 10 \)].
- **Blair:** Mm hm.
- **T:** Is that surprising?
- **Blair:** Ah. 90 is nine 10s. No it is not surprising, because obviously it is—90.
- **T:** Ah ha. Now, right now it’s obvious, but earlier on I was asking you—90, how many 10s? You went through all sorts of twists and turns.

...  
**T:** Why did you go through twists and turns?  
**Blair:** I didn’t have a piece of paper and a pencil.  
**T:** Ah? Maybe if we’d written 90 down [pointing to numeral], maybe that would have helped?  
**Blair:** Yes, that would have helped.  
**T:** Okay, let’s try that this time.
Figure 5.5  Task 10.6.5c. Arrow sentence: “We’ve got 90 … we said that’s eighteen lots of 5 … that’s also nine 10s”.

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Calibrations in this task</th>
</tr>
</thead>
<tbody>
<tr>
<td>RNG</td>
<td>Continued: Beyond 50</td>
</tr>
<tr>
<td>ORN</td>
<td>Continued: $N \cdot F &amp; T$</td>
</tr>
<tr>
<td>SET</td>
<td>-</td>
</tr>
<tr>
<td>NTN</td>
<td>Introduced: informal arrow sentence notation</td>
</tr>
<tr>
<td></td>
<td>Notation the focus of the passage</td>
</tr>
<tr>
<td>STR</td>
<td>Attention was brought to structuring the unit relations by returning to the reflective question:</td>
</tr>
<tr>
<td></td>
<td>90 ... is nine 10s ... is that surprising?</td>
</tr>
<tr>
<td></td>
<td>and asking about Blair’s thinking</td>
</tr>
<tr>
<td></td>
<td>Why did you go through twists and turns?</td>
</tr>
</tbody>
</table>

Blair’s activity. While Blair had struggled to name each of $N$, $F$ and $T$ in the previous passage 10.6.5b, he readily agreed to the values as they were notated in this passage. His desire to copy the notation into his own book may have indicated his sense that this was an important conclusion. When I put to Blair again that 90 is nine 10s, he finally appeared to recognise the relationship: “90 is nine 10s. No, it is not surprising, because obviously it is 90”. It may be, as he then reflected, that seeing the written numeral 90 helped his recognition.

5.3.3 Segment 10.6 (Layer C): Progressions in Blair’s activity

Blair still treated the tasks as being about the tiles, rather than bare numbers. For example, he counted the tiles to check his answers. He also spoke in terms of the number of “cards”, rather than the number of “5s”, as in his answers to 10.6.1, the 40-dot task—“Eight cards”—and to 10.6.2, the 50-dot task—“50 dots, that’s 10 cards”. In the 70-dot Task 10.6.3, when asked how many 5s, he appeared uncertain of the term “5s”, preferring to think in terms of the number of cards.

Blair showed some strong visual structuring of the tiles in pairs. In every instance of counting the tiles, he counted them in pairs, whether counting
dots, 5s, or in the final task, 10s. He often used a distinctive two-fingered point as he pointed to each pair. In the 60-dot Task 10.6.4, he had a clear visualisation that three pairs would be 30, and six pairs would be 60. Thus, his structuring of the tile setting appeared a potential resource for constructing units of 10, and coordinating three units: 10s, 5s, and 1s. However, in the 90-dot Task 10.6.5, when asked directly for $T$, he apparently did not have a clear visualisation of pairs of tiles, nor did he initiate counting pairs as units.

Blair successfully used his new strategy for $2n \rightarrow F$ in the five tasks of Segment 10.6. Furthermore, as indicated in comments such as “See it’s working” in the 50-dot task, and his triumphant arms in the 70-dot task, he regarded his strategy as a form of conjecture, and the success of the strategy as demonstrating its generality. While successful, his use of the strategy was not immediately robust. In Task 10.6.3, with 70 dots, he used the strategy and made an error of “140” before self-correcting to “14”. Then when the question of how many 5s was recapitulated, he was hesitant, and appeared to count tiles rather than use the strategy.

His $2n \rightarrow F$ strategy appeared closely tied to thinking of the numeral for $N$. For example, when asked to recapitulate how he could get 12 5s from 60 dots in Task 10.6.4, he still answered in terms of taking “the zero off the end of the 60, you just leave the six there, you double the six”.

By contrast, his strategy did not involve thinking of the number of 10s ($T$) involved, as revealed in the 90-dot task, when he could not even find the value for $T$. Likewise, his strategy did not seem to involve visualising the tiles. Indeed, after the tiles were unscreened in the 90-dot task, 10.6.5b, he was asked to find $F$ again, and it appeared that the tiles did not help him answer: he needed to look away from the tiles to recall his answer of 18. Thus, the strategy was limited: he did not yet understand the relationship between $N$ and $F$ in terms of $T$ or the tiles.

Blair had some difficulty distinguishing and relating the three different units involved: 1s, 5s, and 10s. A minor confusion occurred in the 70-dot Task 10.6.3: when checking how many 5s there were, he counted the pairs of tiles as “10, 15, 20 ….” before self-correcting the count to be “two, four, six ….” The striking confusions occurred in the 90-dot task, 10.6.5, with his protracted difficulties in thinking about 10s as units. Even once the tiles were
unscreened, he confused the counts of cards and dots, and struggled to organise a count of 5s and a count of 10s using the setting. He appeared occupied with the challenge of clarifying the three different units, rather than coordinating the relationships between them.

In summary, in Blair’s reasoning in Segment 10.6, he understood the task in terms of the tiles, and showed some strong visual structuring of the tiles, but also made some errors of visualisation. He was immediately successful with his new $2n \rightarrow F$ strategy, but did not relate the strategy to $T$, and showed more general difficulties with distinguishing and relating the three different units involved: 1s, 5s, and 10s.

5.3.4 Segment 10.6 (Layer D): Dimensions of instruction

5.3.4.1 RNG. Range
In Segment 10.6, the RNG was extended beyond 50, and this may have increased the challenge and the learning for Blair. The first two tasks were still in the high 5s range, and Blair was fluent. Then the RNG was extended to beyond 50, to source more decuples on which to try Blair’s new $2n \rightarrow F$ strategy. On the first task beyond 50, Task 10.6.3, given 70 dots find $F$, Blair made an initial error of 140, then self-corrected to 14. It is possible that the extension of the RNG perturbed his fluency with his strategy. In turn, by reflecting on and correcting his answer, he would have been strengthening his understanding of his strategy: clarifying the need to “take the zero off”, and realising the possibility of getting answers in the teens. Blair visualised a tile arrangement successfully for 60 dots, on Task 10.6.4, but not successfully for 90 dots, on Task 10.6.5. Again, it is possible that extending to the larger size of 90 contributed to the increased challenge for Blair. Thus, there is evidence that the extending of the RNG brought the tasks to the cutting edge of Blair’s mental calculations, and his visualisations.

5.3.4.2 ORN. Orientation
The ORN was manipulated in three main ways in this segment, each contributing to the learning: inverting the orientation of the tasks from the previous segment, augmenting the task to include finding $T$, and brief switches of orientation within two tasks. Each is described below.
While Segment 10.5 was generally with ORN $F \rightarrow N$, Segment 10.6 switched ORN to the inverse $N \rightarrow F$. The ORN was inverted to match Blair’s new $2n \rightarrow F$ strategy from the end of 10.5. With the 10.6 tasks in the inverted orientation, Blair established fluency and pride in his new strategy. He was also being challenged by the limits of his new strategy, in relating it to the tiles, and to $T$. Thus, the ORN inversion supported the key learning goals of fluency with multiplicative strategies, and investigating $N-F-T$ relations.

In the final 90-dot Task 10.6.5, the ORN was augmented to include finding $T$ as well as $F$, and this challenged Blair at the cutting edge of his understanding of $N-F-T$ relations. Note that the new question itself was not more challenging: normally, finding $T$ would be easier than finding $F$. Rather, the new question created complexity by involving a new unit—the 10s—in a task that already involved coordinating two units—the 1s and 5s. Certainly this complexity appeared to challenge Blair, who, having discovered a way to find $F$, appeared to lose a sense of how to find $T$. Then, persisting with the task, he eventually succeeded in finding $N$, $F$, and $T$, thus beginning to resolve the challenge. The augmented ORN revealed the weakness in Blair’s understanding, and then created the opportunity for him to resolve it.

Brief switches of ORN were used effectively in two tasks. The 70-dot Task 10.6.3 included switching between checking $N$ and $F$ again, and the 90-dot Task 10.6.5 included switching between checking $N$, $F$, and $T$. Blair was not fluent with these switching tasks; rather, he took thinking time and stumbled on what term and what count to use, yet he did generally succeed. I argue that these brief ORN shifts within tasks kept him on the cutting edge of his work in distinguishing the three different units.

5.3.4.3 SET. Setting

SET was a significant dimension of the instruction in Segment 10.6. I will discuss four ways the setting was used: as the site of meaning for the tasks; as a setting in which Blair could structure units of 1s, 5s, and 10s; as a setting that could be distanced by screening; and as a setting that could be distanced by negotiation of more formal terms.

The setting of the tiles was the site of meaning for the tasks in this segment. Blair’s new $2n \rightarrow F$ strategy was conceived to answer a question about the
tiles. The tasks in this segment were treated as trialling that strategy: to see if the number of tiles could be calculated without looking, and then to compare the calculated answer with the actual tiles when visible. Blair was excited, as each comparison with the tiles appeared to verify his strategy: “See, it’s working!” The tiles evidently served as the testable, anchoring reality for him, in accord with the RME context principle (Gravemeijer & Kindt, 2001).

The tiles were a setting in which Blair could structure the units of 1s, 5s, and 10s, and their relationships. In tasks 10.6.3, 10.6.4, and 10.6.5, after Blair had found $F$ using his strategy, the teacher unscreened the tiles and asked Blair to find both $F$ and $N$ in the setting: Are there really 70 dots here? These moments of recapitulation required Blair to distinguish the units of 1s and 5s in the tiles, and to find $F$ by structuring the tiles, rather than by his $2n \rightarrow F$ strategy. In visualising an arrangement for 60 dots in Task 10.6.4, Blair imagined two rows of three pairs. This visualisation involved treating the pairs as units; and then the row as a unit. Treating a pair of tiles as a unit is not equivalent to explicitly counting 10s as a unit, but it is a construction which likely supported Blair’s eventual success with counting pairs as units of 10. In the 90-dot task, especially in passage 10.6.5b, the setting was used to distinguish the units Blair was confusing. The teacher corrected a count of 90 cards as 90 dots. When Blair struggled to construct a count of 5s, the teacher specified No, no, no, don’t count the dots ... Count 5s. That’s one 5 there—dragging out the nearest 5-tile—after which Blair succeeded in finding $F$. Similarly, “one 10” was identified with a pair of tiles, after which Blair succeeded in finding $T$. Blair was not only structuring the units in the tiles, he was structuring the action of counting. For the 90-dot task, he ended up using the same action of two-fingered pointing at each pair of tiles to accomplish three different counts: 10, 20, 30 ... 90; 2, 4, 6 ... 18; and 1, 2, 3 ... 9. Thus, in his pointing gesture, he could treat the pairs as different units: as ten 1s, as two 5s, and as one 10. If he could feel how each of these counts involved the same pointing action, this may have increased his awareness of the relations between 1s, 5s, and 10s. Thus, the negotiations with the tile setting supported Blair to distinguish and structure the units involved.

The availability of the tile setting was manipulated strategically, using the screen. The tiles could be distanced by screening, and made more available
by unscreening. Every task in the segment involved adjustments in screening the tiles. The basic adjustment of unscreening after Blair answered each task allowed the verification of his strategy. By Task 10.6.5a, the teacher did not unscreen the tiles and have Blair check: Blair appeared satisfied with his answer of 18 without such checking. Thus, through the screening and unscreening of the tile setting, Blair apparently extended the realm of his common sense a little way beyond the visible tiles, to increasingly include his new strategy. At the same time, when the How many 10s? question appeared to stretch beyond Blair’s common sense, the tiles could be unscreened again in 10.6.5b to re-establish an understanding of the units of 1s, 5s, and 10s in the context of the tiles, as described above. Screening and flashing were also used in Task 10.6.4, with 60 dots, to encourage Blair to visualise the tiles, and to clarify his visual structuring of the tiles. In Task 10.6.5a, with 90 dots, Blair again tried to visualise the screened tiles; he came close, but did not succeed. Thus, visualising the tiles, and calculating $F$ and $T$, appeared to be at Blair’s cutting edge, and manipulating the distance of the tile setting was an important instructional technique for adjusting the pitch of the tasks to Blair’s cutting edge.

The tile setting was also distanced by negotiating a shift to less setting-dependent terms. Blair spoke of finding the number of “cards”. In this segment, the teacher began to use the term “5s” in place of “cards”, and later introduced the term “10s”. Using these more formal terms was not straightforward for Blair; instead, we can observe a subtle negotiation. For example, in Task 10.6.3 with 70 dots:

T: 70 dots. How many 5s?
Blair: How many 5s? Ah, 70 dots? Ah, 70. [Looking up to teacher.] How many cards?
T: How many 5s, yes.
Blair: How many 5s? Ah [looking at the tiles] … Fourteen.
T: Fourteen.
Blair: Fourteen 5s.

In Task 10.6.5b with 90 dots:

T: How many 5s were there?
Blair: How many 5s were there? Er, nine-, how many 5s. Er, 10, 20, 30 … 90. There’s 90 dots. You said: how many cards.
T: [Nodding] How many 5s.
Blair: How many 5s. [Looking at the tiles] I’ve already done that.
T: Yeah, you have …
And later in 10.6.5b:
T: There’s nine 10s in 90.
Blair: Yes, but there are, t-, there are eighteen cards.
T: How many on each card?
Blair: Er, five dots.
T: So we can say, there’s eighteen 5s.
Blair: Eighteen 5s.

Thus, the tile setting was an important site for the negotiation of more formal terms.

5.3.4.4 NTN. Notation
The notating of Blair’s $2n \rightarrow F$ strategy may have supported his clarity with the strategy. When Blair conceived his strategy at the end of Segment 10.5, the teacher notated it informally (see Figure 6.4), and Blair copied the notation into another book saying “I feel really good taking notes”. In the five tasks in this Segment 10.6, Blair quickly became fluent with calculating using this strategy, and was able to describe it again clearly in Task 10.6.4. Indeed, he referred to this strategy again in later lessons. For example, in the first task in the later Segment 12.4, posed how many 5s to make 70, he answered: “Fourteen. It works! … I took the 0 off and doubled the 7”. I suggest the initial instructional attention to notating the strategy may have supported Blair in establishing his fluency and clarity.

Notation was not used again until the end of Segment 10.6. Here again, the use of notation appeared to consolidate Blair’s new findings. In the 90-dot Task 10.6.5, Blair had struggled to find $T$, and then to recall any of $N$, $F$, and $T$, and to use the new terms “5s” and “10s”. When the teacher finally wrote the $N$, $F$, and $T$ values together in an arrow sentence, using the terms 5s and 10s (see Figure 5.5), Blair readily agreed with the statements, and wanted to copy out the written conclusion. Seeing the written numeral 90 may also have helped him recognise the relationship of 90 ($N$) and nine 10s ($T$).

The arrow sentence became the beginning of the subsequent sequence of notated tasks, which I have labelled Segment 10.7. The ongoing influence of the arrow notation is discussed in Section 5.4 on 10.7.
5.3.4.5 STR. Attention to structuring and strategies

Much of the instruction in this segment was devoted to drawing attention to multiplicative structuring and multiplicative strategies. I will discuss four aspects of this dimension of the instruction.

The whole segment was treated as a trial of Blair’s new $2n \rightarrow F$ strategy. Let’s try it a few times, began the teacher, and already after the second task Blair responded “See, it’s working”. With this instructional time and attention, Blair developed fluency and clarity with his new strategy, a breakthrough in using multiplicative reasoning for division tasks in the high 5s range and beyond.

The instruction repeatedly challenged Blair to clarify the units involved in the tasks. As discussed in the analysis of Blair’s reasoning, while he fluently found $F$, he struggled to recapitulate counts of $N$ and $F$ alongside each other in the setting of the tiles. In each of Tasks 10.6.3, 10.6.4, and 10.6.5, the instruction challenged Blair to recapitulate these counts with the tiles, working at this cutting edge in his understanding of the units of 1s and 5s. In a related way, the negotiations introducing the term “5s” continued to focus attention on clarifying the units.

In Task 10.6.4, a challenge to visualise the screened tile arrangement brought productive attention to structuring the tiles. Blair responded with rich multiplicative structuring involving unitising pairs and rows, doubling, and tracking $N$ and $F$, for both 30 dots and 60 dots. The teacher also asked Blair to articulate his strategy for finding $F$ again: And how could you get that twelve just from the 60? This brought attention to consolidating the strategy, and furthermore it brought attention to the emergence of two ways of finding $F$: by visualising 6 pairs of tiles, and by the $2n \rightarrow F$ strategy. Blair did not yet appear to connect these two ways. Thus, the persistent instructional attention to STR revealed the successes and the edge in Blair’s multiplicative reasoning here. The attention given to visualising in this task may also have led Blair to try visualising a structured arrangement of tiles in the next task, Task 10.6.5a, without prompting.

The instruction brought attention to unit relations by pursuing the new question of finding $T$, How many 10s? Until Task 10.6.5, Blair had related to
10s implicitly through his paired structuring of tiles, and his counting of 1s by ten, and of 5s by two. In Task 10.6.5a, the teacher challenged Blair for the first time to find $T$ alongside $F$, and when he began to count 10s on his fingers, the teacher pressed the challenge to find $T$ without counting. Blair did not answer successfully. The instruction then devoted seven minutes to pursuing and reviewing this task. By introducing the *How many 10s?* question, the instruction brought Blair to the cutting edge of his knowledge of these units, and initiated attention to the whole system of $N$-$F$-$T$ relations, which became the focus of the next lesson segment and segments of several subsequent lessons.

### 5.4 Segment 10.7. Varied Orientations With 5-Tiles

#### 5.4.1 Overview of Segment 10.7

In Segment 10.7, I continued the lesson with similar tasks to the previous segment, and introduced an extra element: Blair writing arrow sentences for each task. The arrow sentences were aligned below the sentence introduced in the previous task on 90 dots, generating an impromptu table of results. Figure 5.6 shows the final table as it appeared at the end of the segment. I also changed the orientation of some of the tasks.

The task type was *varied orientations with 5-tiles*. Five tasks were posed, summarised as follows. Given 80 dots ($N$), find the number of tens ($T$) and number of fives ($F$); and likewise given 60 dots ($N$), find $T$ and $F$. Given four 10s ($T$), find $F$ and $N$; and likewise given two 10s ($T$), find $F$ and $N$. Finally, given four 5s ($F$), find $T$ and $N$. He succeeded on each task. Following these five tasks, I asked why there was a doubling relationship between $T$ and $F$, referring to the numbers in the list of arrow sentences, and his explanation revealed further insight into this multiplicative relationship.

The instructional progression in the segment is particularly rich in small adjustments on all five dimensions. Aspects of interest include: continuing with tasks like the difficult final task of 10.6, the use of adjustments in range and orientation to find new challenges for Blair, the progression of increasing involvement with the notation, the related progression with distancing the setting, and the shift in the final task to reflect on a general $F$-$T$ relationship.
5.4.2 Segment 10.7: Task-by-task account (Layer A) and interpretations (Layer B)

5.4.2.1 Task 10.7.1 (Layer A): Given 80 dots, find \( T \) and \( F \)

I began by handing the pen to Blair and asking him to write down 80 (see Figure 5.7a). Then I screened the tiles, arranged eight pairs of 5-tiles, announced I had 80, and asked *How many 10s do I have?* Blair looked at the page for 12 seconds, during which time he marked an arrow following the 80. Then pointing to the expression on the previous line “18 \( \times \) 5s” he asked “They were the 10s for the 90 weren’t they...?” I pointed to the same expression, saying it was the 5s, then pointed to the expression “9 \( \times \) 10s”, saying it was the 10s. Blair responded “Okay, okay”. I reiterated the question *How many 10s?* Blair looked at the page, answered “Eight 10s”, and looked up at me. I nodded, gestured to the space in the new line corresponding to the expression 9 \( \times \) 10s, and suggested he write his answer there to match. He wrote “8 \( \times \) 10s” (see Figure 5.7b).
Figure 5.7  Task 10.7.1. (a) Beginning with 80; (b) Writing in 8 × 10s aligned with the previous arrow sentence above; and (c) Writing in 16 × 5s to complete the arrow sentence.

a)

\[ 90 \rightarrow 18 \times 5s \rightarrow 9 \times 10s \]

\[ 80 \]

b)

\[ 90 \rightarrow 18 \times 5s \rightarrow 9 \times 10s \]

\[ 80 \rightarrow \]

\[ 8 \times 10s \]

c)

\[ 90 \rightarrow 18 \times 5s \rightarrow 9 \times 10s \]

\[ 80 \rightarrow 16 \times 5s \rightarrow 8 \times 10s \]

Note: These figures are scanned from the workbook. Parts of the final notation have been blanked out in (a) and (b) to reveal the order that the parts were written.

Then I said, *Let’s just check.*

T:  Here we go [unscreening the tiles], this is what a 10 looks like
    [placing a hand on nearest pair of tiles.]

Blair:  [Laughs.]

T:  There’s a 10. Are there eight of them?

Blair:  [In coordination with pointing to each pair of tiles in the first row]
    One, two, three, four, [then pointing across to the second row] yep,
    there’s eight.

T:  All right [replaces screen].

Blair:  Eight 10s.
T: Now, how many 5s are there?
Blair: [Immediately.] There’d be sixteen 5s.
T: Ah. [Unscreens the tiles.] Where were the sixteen 5s?
Blair: [Finger hovering over the nearest tile] Ah. [Then in coordination with pointing to each of the 16 tiles in succession] One, two, three … sixteen. [Moves immediately to write $16 \times 5s$, shown in Figure 5.7c.]
T: Very good.

10.7.1 (Layer B): Local interpretations

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Calibrations in this task</th>
</tr>
</thead>
<tbody>
<tr>
<td>RNG</td>
<td>Continued: Beyond 50</td>
</tr>
<tr>
<td>ORN</td>
<td>Continued: $N \rightarrow T&amp;F$</td>
</tr>
<tr>
<td>SET</td>
<td>Screening, with unscreening to check</td>
</tr>
<tr>
<td>NTN</td>
<td>Continued: arrow sentence notation Teacher indicated to student where to record the parts of the sentence.</td>
</tr>
<tr>
<td>STR</td>
<td>Attention was drawn to structuring by asking Blair to demonstrate the values of $T$ and $F$ in the setting, as in There’s a 10. Are there eight of them?</td>
</tr>
</tbody>
</table>

Blair’s activity. The first sub-task was $N \rightarrow T$. In the previous task, 10.6.5 with 90 dots, finding $T$ from $N$ had proved a protracted task for Blair, and he had not succeeded. In this task, 10.7.1, Blair still took some time, but he did succeed. Prior to answering, he had an exchange orienting to where the 10s were shown in the previous arrow sentence. Once he had clarified where the 10s were, he found his answer for the number of 10s. I suggest he had constructed a relationship in the previous sentence between the numeral 90, and the 9 of $9 \times 10$s; and then used the same relationship to go from the numeral 80 to the answer of eight 10s. If so, he used a form of the relation $n \rightarrow T$, but he was dependent on the arrow sentence to see that relation. Thus, he had made progress from the previous task, but he was far from fluent, and his strategy was dependent on using the notation.

For the sub-task of $N \rightarrow F$, Blair found $F$ fluently. Most likely, he used the same $2n \rightarrow F$ strategy as in the previous segment 10.6. His reasoning may also have involved connections to the arrow sentences and the tile arrangement, having just looked at them.
5.4.2.2 Task 10.7.2 (Layer A): Given 60 dots, find \( T \) and \( F \)

For the next task, I arranged six pairs of tiles behind the screen, said I have 60, and asked Blair to write the number down. I asked how many 10s, then how many 5s; in both cases he immediately answered correctly, without looking up from the page, and wrote the corresponding expressions in place. He did not check the tiles.

10.7.2 (Layer B): Local interpretations

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Calibrations in this task</th>
</tr>
</thead>
<tbody>
<tr>
<td>RNG</td>
<td>Continued: Beyond 50</td>
</tr>
<tr>
<td>ORN</td>
<td>Continued: ( N \rightarrow T &amp; F )</td>
</tr>
<tr>
<td>SET</td>
<td>Continued screening.</td>
</tr>
<tr>
<td></td>
<td>No longer unscreening to check.</td>
</tr>
<tr>
<td>NTN</td>
<td>Continued: arrow sentence notation</td>
</tr>
<tr>
<td></td>
<td>Student wrote the notation.</td>
</tr>
<tr>
<td>STR</td>
<td>-</td>
</tr>
</tbody>
</table>

Blair’s activity. \( N \rightarrow T \) was fluent. Given his attention to the workbook, I suggest he was again reasoning in terms of the arrow sentence and the \( n \rightarrow T \) relation he had found in the previous task. That this relation agrees with basic facts he had known before—that 60 has six 10s—likely supported his sudden fluency and certitude. For \( N \rightarrow F \), his fluency suggests he continued to use the \( 2n \rightarrow F \) strategy.

5.4.2.3 Task 10.7.3 (Layer A): Given four 10s, find \( F \) and \( N \)

Without arranging tiles behind the screen, I began the task saying Okay, here’s a different kind of question. This time, I’m gonna tell you: I’ve got four 10s. On the next line of the workbook, I wrote “4 × 10s” in the appropriate position for the arrow sentence. I then asked how many 5s. Blair took eight seconds to answer correctly eight 5s. He then asked me to wait, and took a further six seconds to confirm his answer. I asked

\[ T: \] \text{How did you just check that?}

\[ \text{Blair: } \] \text{How did I just check that? I looked here [pointing to the previous line in the workbook for the 60-dot task—see Figure 5.6]. Six times six, ah, six times two [pointing to the 6] is twelve, right, there [pointing to the 12]. I thought I had to put a one on them. Like four times four, ah, four, ah, two times four, or four times two, two times four equals eight, not eighteen.}
Blair wrote “8 \times 5s \rightarrow” into the arrow sentence. I then used a red pen to make two annotations: in the previous line, an arrow from the 6 to the 12, labelled “\times 2”, and in the current line, an equivalent arrow from the 4 to the 8, shown in Figure 5.6. Finally, I asked And how many would I have altogether in this? pointing to the place for N in the current line. Blair answered “40”, and wrote 40 into place.

### 10.7.3 (Layer B): Local interpretations

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Calibrations in this task</th>
</tr>
</thead>
<tbody>
<tr>
<td>RNG</td>
<td>Retreated: High 5s</td>
</tr>
<tr>
<td>ORN</td>
<td>Shifted: T\rightarrow F&amp;N</td>
</tr>
<tr>
<td>SET</td>
<td>Advanced: bare</td>
</tr>
</tbody>
</table>
| NTN       | Continued: arrow sentence notation  
            Begun by teacher, completed by student. |
| STR       | Attention to strategies with the question: 
            How did you just check that?  
            Attention to structuring with the annotations of the “times two” relationships Blair then described. |

**Blair’s activity.** Sub-task $T \rightarrow F$ was his first attempt at this orientation. He took time to answer, and further time to check and confirm his answer. I cannot determine how he first found the answer of eight 5s, however he did explain how he checked his answer. He saw a relation in the previous arrow sentence—double the $T$ value of 6 gives the $F$ value of 12, which I code as $2T \rightarrow F$—and used the same relation for the current sentence—doubling 4 to get 8. He also described checking a competing conjecture, of needing to “put a one on them”, presumably as the previous values of $F$ in the table have a 1 in front, like 12 and 16. This competing conjecture shows that the doubling relation was not yet taken for granted by him.

For the sub-task of finding $N$, he found 40 fluently, presumably from $T$ of four 10s.

### 5.4.2.4 Task 10.7.4 (Layer A): Given two 10s, find $F$ and $N$

Again without arranging tiles behind the screen, I posed the task *What if I told you I had two 10s?* Blair promptly gave the correct answers of 20, and four
5s. I asked him to Write that in all the right places, please, and he completed the arrow sentence.

10.7.4 (Layer B): Local interpretations

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Calibrations in this task</th>
</tr>
</thead>
<tbody>
<tr>
<td>RNG</td>
<td>Retreated: Low 5s</td>
</tr>
<tr>
<td>ORN</td>
<td>Continued: T→F&amp;N</td>
</tr>
<tr>
<td>SET</td>
<td>Continued: bare</td>
</tr>
<tr>
<td>NTN</td>
<td>Continued: arrow sentence notation</td>
</tr>
<tr>
<td>Written by student.</td>
<td></td>
</tr>
<tr>
<td>STR</td>
<td>-</td>
</tr>
</tbody>
</table>

Blair’s activity. T→F was fluent. He may have used the 2T→F relation described in the previous task. Alternatively, in this low range, he may have recalled the answer as a known fact. T→N was also fluent.

5.4.2.5 Task 10.7.5 (Layer A): Given 14 5s, find T and N
I arranged seven pairs of tiles behind the screen. Warning again that this would be a different kind of question, I posed fourteen 5s, and indicated the place in the new line of the workbook, where Blair wrote “14 × 5s”.

T: How many 10s do I have?
Blair: [Pause of 8 seconds.] Seven. You’d have seven 10s.
T: Write that over there [indicating place in the line].
Blair: [Writes in 7 × 10s.]
T: How many would that be?
Blair: 14 … 14… [beginning to look at the previous lines in the table.]
T: [Screening the previous lines.] You’ve got seven 10s.
Blair: I’ve got seven 10s. 70. [Writes 70 into the arrow sentence.]

I suggested we check. Unscreening the tiles, I asked Have I got 70 dots? Blair turned to the tiles, and in coordination with pointing to each tile said “5, 10, 15, … 70”, and confirmed “Yep”. Next, referring back to the arrow sentence, he said “And 14 5s”. Then in coordination with pointing to each tile, he said “1, 2, 3 … 14”. Referring back to the arrow sentence a second time, he said “Err” then in coordination with pointing to each pair of tiles, he said “1, 2, 3 … 7”. He finished by saying “Yes. Everything is correct”.

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**10.7.5 (Layer B): Local interpretations**

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Calibrations in this task</th>
</tr>
</thead>
<tbody>
<tr>
<td>RNG</td>
<td>Advanced: Beyond 50</td>
</tr>
</tbody>
</table>
| ORN       | Shifted: $F\rightarrow T \& N$  
A significant inversion of orientation. |
| SET       | Retreated: screening, with unscreening to check |
| NTN       | Continued: arrow sentence notation  
Written by student. |
| STR       | -                       |

**Blair’s activity.** For the new sub-task $F\rightarrow T$, Blair answered successfully, taking some time but with no uncertainty. This appears to me as a striking breakthrough. Only 20 minutes earlier in Segment 10.5 he had been challenged to keep track of 10s when incrementing and decrementing by 5s. Here, he readily calculated that 14 5s is seven 10s. This indicates that he used some version of the relation $\frac{1}{2}F\rightarrow T$, inverting the relation $2T\rightarrow F$, which he used in 10.7.3. To recognise this inverted relation, he could have drawn on the relations in the arrow sentence, or on a visualisation of the screened 5-tile setting, or a more abstract understanding of the relations.

For the sub-task of finding $N$, Blair stalled, and began to look at previous lines of the table. I screened the table, and prompted him with *You’ve got seven 10s*. He then answered 70 readily. His stalling is evidence that this new orientation was not all obvious to him. With the prompt, he found $N$ from $T$, so I did not assess whether he could find $N$ directly from $F$.

When the tiles were unscreened Blair took initiative in checking his written values for each of $N$, $F$, and $T$ against the tiles, and he fluently coordinated the three different forms of counting in the tiles: counting by 5s, counting 5s, and counting pairs of 5s as 10s.

**5.4.2.6 Task 10.7.6 (Layer A): Explain the relationship between $T$ and $F$**

Finally, I asked: *Why is it?* Pointing to the arrow sentence about 60, I continued *When there’s six 10s, there are twice as many 5s*, then pointing to the next sentence about 40, *There’s four 10s, there’s twice as many 5s*, and I continued to point out this relationship for the next two sentences. Blair began to exclaim “I know, I know”.

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Blair: Ten, tens … you can fit two 5s into … one 10, two 5s; one, two. So it’s basically doubling that number. So … you know.

T: Good, good. I like that: You can fit two 5s into each 10.

Blair then wrote a note, “You can fit 2 fives in 1 10” beside “1 × 10 = 2 × 5”, shown in Figure 5.8.

Figure 5.8 Task 10.7.6. Blair’s notation for $2T \rightarrow F$ insight: “You can fit two 5s into one 10”.

10.7.6 (Layer B): Local interpretations

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Calibrations in this task</th>
</tr>
</thead>
<tbody>
<tr>
<td>RNG</td>
<td>A generalising task, addressing both high 5s and beyond 50.</td>
</tr>
<tr>
<td>ORN</td>
<td>Shifted: $T \rightarrow F$</td>
</tr>
<tr>
<td>SET</td>
<td>Advanced: Bare numbers</td>
</tr>
<tr>
<td>NTN</td>
<td>Used record of arrow sentences to frame the task Student recorded his conclusion</td>
</tr>
<tr>
<td>STR</td>
<td>Attention to structuring was the central purpose of this final task: drawing attention to the multiplicative relationship between $T$ and $F$ recurring in each sentence, and seeking an explanation</td>
</tr>
</tbody>
</table>

**Blair’s activity.** Blair’s response, heralded with exclamations, is convincing of an insight into the general relation involved here. He explained in terms of the multiplicative relationship between the units: that “Tens … you can fit two 5s into”. I find the use of the verb “fit” compelling: there is a strong sense that he recognised a proportional relationship between the two units. I take his locution “One 10, two 5s” to be another expression of how any unit of 10 could be considered as two 5s. He linked this relationship of the units to the relation of the $T$ and $F$: “so it’s basically doubling that number”, that is, the $F$ value will be double the $T$ value. I have coded this relation $2T \rightarrow F$. In summary, he understood how the proportions of the units—two 5s in every 10—implied the proportions of the total numbers of units—$F$ is double $T$. His initiative in writing his explanation down in his workbook suggests he recognised that his insight was significant.
5.4.3 Segment 10.7 (Layer C): Progressions in Blair’s activity

I discuss three aspects of Blair’s activity in Segment 10.7: his negotiation of different orientations of the $N$-$F$-$T$ relations, his use of the new notation, and his use of the 5-tile setting.

5.4.3.1 Negotiating different orientations of the $N$-$F$-$T$ relations

This segment posed sub-tasks involving $N$, $F$, and $T$ in all six different possible orientations. Below, I summarise how Blair managed each of these sub-task orientations.

For $N \rightarrow F$, in tasks in Segment 10.6 he had been successful, and in 10.7.1 and 10.7.2 he continued with fluency, probably continuing to use a form of his $2n \rightarrow F$ strategy, possibly supported by the notation.

For $N \rightarrow T$, in Task 10.6.5 with 90 dots he had been unsuccessful, in 10.7.1 he was hesitant but successful, and in 10.7.2 he was fluent, apparently learning to use a form of the relation $n \rightarrow T$ seen in the notation.

For $T \rightarrow F$, in 10.7.3 he took time, checked a competing conjecture, saw a $2T \rightarrow F$ relation in the arrow sentence, and was successful; in 10.7.4 he was fluent.

With $T \rightarrow N$, he was fluent in Tasks 10.7.3 and 10.7.4. While finding the converse $N \rightarrow T$ had proved problematic for him only three tasks previously, on the 10.6.5 90 dots task, finding $T \rightarrow N$ remained straightforward.

With $F \rightarrow T$, the most radical change in ORN, in 10.7.5, he was successful, with certitude. He may have used the relation $\frac{1}{2}F \rightarrow T$ by inverting the relation $2T \rightarrow F$.

With $F \rightarrow N$, in Task 10.7.5 he was hesitant; after a prompt he found $T \rightarrow N$, but did not appear to go directly from $F$ to find $N$.

I want to highlight four aspects of Blair’s activity regarding the $N$-$F$-$T$ relations. Firstly, I argue that he was working at the cutting edge of his understanding of the $N$-$F$-$T$ relations. While with $N \rightarrow F$ and $T \rightarrow N$ orientations
he was fluent, with $F\rightarrow T$ he was newly successful, with $N\rightarrow T$ and $T\rightarrow F$ he took time at first, becoming more fluent on his second task with each, and with $F\rightarrow N$ he was hesitant and inconclusive. Thus, he was needing to think, then generally achieving success.

Secondly, to solve the tasks in this segment, he did not use skip-counting; rather, he sought multiplicative relations between $N$, $F$, and $T$. This was a major advance on his reasoning in Segment 8.8, where he persistently sought to use skip-counting to solve similar tasks. Given one of the instructional aims of the high 5s topic was for Blair to structure these $N$-$F$-$T$ relations, it seems an essential breakthrough that he began to seek these relations.

Thirdly, Blair not only sought multiplicative relations, he succeeded in using multiplicative relations for most orientations in the $N$-$F$-$T$ system. In particular, he appeared to use an inverse relation to solve $F\rightarrow T$, which is central to the sense of these all being one system of multiplicative relations. His success appears as a significant breakthrough in his multiplicative computation.

Fourthly, beyond succeeding with calculations using these relations, he articulated general reasoning about these relations. In the final Task 10.7.6, he gave an explanation of the $2T\rightarrow F$ relation, based on the proportional relationship of the units of 5 and 10. At this point he had achieved an instructional aim: he could coordinate these two composite units.

5.4.3.2 Using the notation
A feature of Blair’s activity in the segment was his attention to the arrow sentence notation (see Figure 5.6). He looked at the workbook while thinking about each task, apparently consulting previous arrow sentences as well as the current sentence. From the first task, 10.7.1, he readily took up the role of recording his answers. In his attention to where he recorded each part of the sentence, he evidently understood there were different parts in the notation, and followed the principle of aligning the three parts of each arrow sentence underneath the parts of the previous sentence.

The notation quickly came to hold the tasks for the teacher and student, in the sense that each arrow sentence was a whole, in which any one of the three
parts could be given and the task was then to find the other two parts. For example, at the end of Task 10.7.3, realising I hadn’t asked about \( N \), I could point to the empty place for \( N \) in the current sentence and ask *And how many would I have altogether in this?* In Task 10.7.4, after Blair had given fluent verbal answers, I could say *Write that in all the right places, please* and he did. In Task 10.7.5, with 14 5s, after I initiated checking 70 dots in the tile setting, Blair took initiative to check the other two parts of the final arrow sentence against the tiles: were there 14 5s, and were there seven 10s? When these two parts of the sentence were checked, he said “Yes. Everything is correct”. Thus, he treated the arrow sentence as representing the complete task, and he kept track of the parts of the task by using the parts of the arrow sentence.

As remarked above, Blair sought relationships between \( N \), \( F \), and \( T \). In particular, he sought relationships between the parts in the arrow sentences, and he expected the relationships he found to hold across different arrow sentences. In Task 10.7.1, with 80 dots, when finding how many 10s, he wanted to know where the 10s were in the previous arrow sentence: once he was orientated to the parts in the notation, he solved the task. Likewise, in Task 10.7.3, with four 10s, he checked his answer for the number of 5s by finding the \( 2T \rightarrow F \) relation between values in the previous arrow sentences. In tasks 10.7.2 and 10.7.4, he answered fluently while looking at the arrow sentences: my contention is that he quickly used the same relations in the notation he had found in the previous tasks. I also suspect that, in Task 10.7.5 with fourteen 5s, looking at the notation helped him use the same relations in inverse. I suggest he had begun to treat each arrow sentence as a coherent system of invertible relationships, though he was not yet fluent or robust with these relationships. The final reflection task was posed in terms of the relationship between \( T \) and \( F \) in each arrow sentence, of which he made new insightful sense. Thus, the arrow sentences clearly helped him organise his reasoning about the relationships between \( N \), \( F \), and \( T \) within each task, and in general.

While the notation provided a context for Blair to find important mathematical relationships, those relationships were not self-evident in the notation. I describe three episodes of Blair interacting with the notation,
revealing that the relationships were not self-evident. In Task 10.7.3, with four 10s, Blair considered a possible pattern of the $F$ number getting a 1 in front, which proved a misleading pattern. After Task 10.7.4 he noticed a pattern in the succession of $N$ numbers at the left end of the arrow sentences: 80, 60, 40, 20 (see Figure 5.6). This pattern actually appeared by chance, from the numbers I happened to select for each task, but he seemed interested at first in the potential mathematical significance of this pattern. In Task 10.7.5 with fourteen 5s, while he appeared to fluently recognise the inverted relation to find $T$, he did not initially recognise which relation would help him find $N$. Thus, the $N$-$F$-$T$ relationships were not self-evident in the notation; rather, finding those relationships required problem-solving, reasoning and learning on Blair’s part.

5.4.3.3 Using the 5-tile setting

In Task 10.7.1 with 80 dots, he used the tiles successfully to check both of his answers: he counted pairs as 10s, and counted singles as 5s. In counting the eight 10s, he used the organised configuration of the tiles to support his multiplicative reasoning by counting the four pairs of tiles in one row, then pointing to the second row saying “yep there’s eight”, apparently reasoning that the second row would double the count, from four to eight. This fluency shows improvement from the previous task, 10.6.5 with 90 dots, when he had difficulty organising a count of 10s. In counting the sixteen 5s, he counted the tiles one by one. This was the first time in the lesson that he counted single tiles. In Segment 10.6 he had successfully counted 5s in pairs, but had then encountered difficulty counting 10s and counting the total dots. He may have switched to counting 5s singly in this task, as part of organising his thinking to better distinguish the different units.

He did not use the tile setting in Tasks 10.7.2, 10.7.3, and 10.7.4. As described above, the arrow sentence notation came to hold the task. Nevertheless, the task remained meaningfully about the tiles too: when the tiles were reintroduced in Task 10.7.5, he related to them successfully. He fluently navigated how to count each different unit in the tile setting: to count 70 dots he pointed to each tile while skip-counting by 5s; to count fourteen 5s, he counted the tiles one by one; to count seven 10s he counted each pair of tiles. Compare this to the previous segment, 10.6, where he encountered some
difficulties with counting in the tile setting. For example, in Task 10.6.3 with 70 dots, he confused how to count the tiles, and stumbled when switching to counting the dots; and especially in Task 10.6.5 with 90 dots, when the tiles were finally unscreened, he had several difficulties organising his counting of the three different units. By comparison, his activity in Task 10.7.5 appeared as a new level of understanding and clarity in distinguishing the three units in the setting of the tiles.

The tiles were not involved in the final generalisation task, Task 10.7.6. He both realised and explained his insight about the relationship of $T$ and $F$ without reference to the tile setting. Recall his insight at the end of the first segment, 10.5, which was embedded in the tile setting: “If you said there are 30, in 5-dotted cards, how many cards were there? … six cards. You take the zero off the 30, then you just double the three”. Here, 11 tasks and 22 minutes later, he expressed his new insight without the tiles: “You can fit two 5s into one 10. So it’s basically doubling that number”. He had arrived at a clearer, more general, invertible relationship, removed from the tile setting.

5.4.3.4 Summary of Blair’s activity in 10.7
To summarise Blair’s activity in Segment 10.7, he had increasingly fluent success calculating different orientations in the $N$-$F$-$T$ relations in the context of the arrow sentence notation. The notation appeared to support his success in negotiating the rapid changes in ORN, and connecting the different sub-tasks as instances of the same whole system of relations. He learned to coordinate the three different parts of the notation, and the corresponding parts of the tile setting. His understanding of the tasks increasingly relied on the notation, while his reasoning became increasingly independent of the tile setting. Finally, he reached a new general insight into the invertible multiplicative relationship between $F$ and $T$, founded in the proportional relationship between the units of 5 and 10.

Blair’s activity with the tile setting and the arrow sentence notation supported his progress. His problem-solving with both the setting and the notation consistently involved clarifying where the three different units were: where were the 10s, where were the 5s, where was the whole number. Furthermore, the problem-solving involved organising relations between these units: how
the counting of 5s related to the counting of 10s in the tile setting; how the number of 5s related to the number of 10s in an arrow sentence. I contend that this intensive mental organising of these units in the context of the tiles and of the notation supported his increasing fluency and his final insight that two 5s in every 10 implies $2T \rightarrow F$.

Overall, he appeared to be consistently at the cutting edge of his reasoning about 5s, to be intensively attending to multiplicative relations around 5s, and to be progressing in his fluency and insight with these relations. Hence, again, this segment is a fruitful case to investigate the instruction: how did the instruction maintain the tasks at Blair’s cutting edge, maintain his attention on multiplicative relations, and support his increasing success? The next section on the instruction in this segment pursues this investigation.

5.4.4 Segment 10.7 (Layer D): Dimensions of instruction
As for the previous two segments, I will analyse each of the five dimensions separately. I chart the progressions made in each dimension, and suggest how these progressions supported Blair’s reasoning and learning, as analysed in the previous section on Layer C.

5.4.4.1 RNG. Range
In Segment 10.7, instruction shifted the numbers across the whole range up to 100. Tasks 10.7.1 and 10.7.2 continued in the range beyond 50; 10.7.3 retreated to the high 5s; 10.7.4 retreated to the low 5s; and 10.7.5 advanced again beyond 50. Blair’s fluency in 10.7.4 may have been supported by his familiarity with the low 5s. Overall, it appeared that Blair achieved some fluency across the whole range to 100. The shifts on this dimension provided the opportunity for Blair to consolidate his new fluency across the extended range.

5.4.4.2 ORN. Orientation
In Segment 10.7, changing ORN became a feature in the progression of tasks. As summarised in the previous section on Blair’s activity, the instruction rotated through three main orientations, posing almost all orientations of the relations between the three values $N$, $F$, and $T$. Furthermore, each shift in ORN responded to Blair achieving some fluency on the previous task. Consider the shifts in ORN across the five tasks of 10.7, shown in Table 5.1.
Table 5.1  Shifts from ORN of previous task corresponding with achieving fluency on previous task

<table>
<thead>
<tr>
<th>Task</th>
<th>Fluency on previous task</th>
<th>ORN</th>
<th>ORN shifted from previous task</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.6.5</td>
<td>-</td>
<td>N→T,F</td>
<td>-</td>
</tr>
<tr>
<td>10.7.1</td>
<td>No</td>
<td>N→T,F</td>
<td>No</td>
</tr>
<tr>
<td>10.7.2</td>
<td>No</td>
<td>N→T,F</td>
<td>No</td>
</tr>
<tr>
<td>10.7.3</td>
<td>Yes</td>
<td>T→N,F</td>
<td>Yes</td>
</tr>
<tr>
<td>10.7.4</td>
<td>No</td>
<td>T→N,F</td>
<td>No</td>
</tr>
<tr>
<td>10.7.5</td>
<td>Yes</td>
<td>F→T,N</td>
<td>Yes</td>
</tr>
</tbody>
</table>

Thus, the change in ORN enabled the instruction to pitch to Blair’s cutting edge: to stay with an ORN when he was still having to think hard about it; to challenge with a new ORN when he appeared more fluent with the current ORN.

Furthermore, as argued in the previous section on Blair’s activity (Layer C), Blair’s experience in negotiating the relations between $N$, $F$, and $T$ from different orientations in rapid succession probably supported his final insight into the general relation between $F$ and $T$. The rapid changing of ORN was a dimension of instructional progression that both kept Blair at his cutting edge, and supported his deepening insight into multiplicative relations.

5.4.4.3  SET. Setting

In the use of the tile setting in Segment 10.7, the instruction made a progression of distancing the setting, then returned to the setting in the penultimate task. In Task 10.7.1, the instruction continued the practice of setting up the tiles behind the screen, and unscreening to check answers. In Task 10.7.2, the tiles were set up, but the instruction did not pursue the checking practice following Blair’s fluent answers. For Task 10.7.3, the tiles were not set up, but the same language of “I’ve got four 10s” was used. For Task 10.7.4, the tiles were not set up, and the language shifted to What if I told you I had two 10s?, situating the task still further from the tile setting. The final pattern-explaining Task 10.7.6 was posed without reference to the tile setting. Blair succeeded in making sense of the task, and articulating a more general insight about the $F-T$ relation, independent of the tiles. I argue that this progression of distancing along SET supported the developments in Blair’s reasoning of increasing independence from the tile setting and increasing fluency in the context of the notation.
For Task 10.7.5, the instruction returned to the practice of setting up the tiles behind the screen, and unscreening to check the answers. Blair responded by demonstrating a new level of clarity and fluency in counting the three units in the context of the tiles. I suggest that this brief return to the setting enabled Blair to consolidate his new understanding and fluency.

5.4.4.4 NTN. Notation

In the previous section on Layer C, I argued that the arrow sentence notation supported Blair’s learning in this segment. I now analyse how the instruction pursued a progression in the development of the notation which supported his engagement with it. The progression in the notation had several aspects.

Firstly, I draw attention to the initial decision to pursue using the notation. The notation had only been introduced to record the results of the previous Task 10.6.5 with 90 dots, when Blair was finding it difficult to calculate and recall these results. The decision to continue to use the notation was a spontaneous experiment, which proved productive.

Secondly, the instruction nurtured a transition from the teacher writing, to the student writing. In the first arrow sentence, Task 10.6.5 with 90 dots, the teacher wrote the whole sentence. In the second sentence, for Task 10.7.1 with 80 dots, the teacher handed the pen to the student, indicating where to write each element to align with the first sentence. In the fourth sentence, for Task 10.7.3 with four 10s, with a change in task ORN, the teacher wrote the given part “4×10s”, then the student wrote the other two parts. By the fifth sentence, for Task 10.7.4 with 40 dots, the teacher could say Write that in all the right places, please, and the notating was accomplished independently by the student. In the final pattern-explaining Task 10.7.6, Blair took initiative to write down his insight, which suggests the independent agency he felt with recording at this point.

Thirdly, I began to screen the notation. In Task 10.7.5 with fourteen 5s, I screened from view the previous arrow sentences. He had appeared to be using previous sentences to help him solve the tasks. Screening the previous sentences might demand that he recall the reasoning he had been using with the notation, without seeing the written record. Thus screening the notation
could serve as screening the tile setting does, to challenge the student to organise his reasoning with more independence.

Fourthly, in deciding to continue using the notation, I decided to align the parts of the subsequent arrow sentences. This alignment required attention to the parts at each new sentence, especially when the ORN changed. The instruction fostered attention to parts as in the following examples: Task 10.7.1, responding to Blair’s question “They were the 10s for the 90 weren’t they…?”; Task 10.7.3, annotating an arrow between the 5s and 10s parts; Task 10.7.3, linking the question And how many would I have altogether in this? with the gesture of pointing to the place for the N part in the current sentence. I suggest the instructional attention to the parts supported Blair’s attention to the parts, and the alignment of the parts supported his search for relationships between the parts, which (as argued in the previous section) became central to his success and insight.

Fifthly, I drew attention to Blair’s search for relationships in the notation. For example, in Task 10.7.3 with four 10s, when Blair appeared to be thinking about the notation, I asked about his thinking. His answer led to his indication of a doubling relationship in the notation, and the teacher notating this relationship. In the final task, Task 10.7.6, I could use the record of the previous sentences to point to each instance of doubling, and leave to Blair the challenge of recognising and articulating the pattern. Thus, the instruction drew attention to his interest in the notation, and introduced the significant role of notation as a record to reflect on patterns over multiple instances.

Thus, over the six tasks of the segment, the instruction developed Blair’s engagement in the notation in the following five ways, through: continuing to use the notation, attending to aligning the parts of the notation, transitioning to independent writing, screening the notation, and encouraging his search for relationships in the notation. I see these as subtle aspects of a basic dimension of instructional progression in NTN, which proved to be powerful in supporting Blair’s learning in this segment.

5.4.4.5 STR. Attention to structuring and strategies
There were two STR passages in the segment. In Task 10.7.3, the instruction brought attention to STR by enquiring about Blair’s strategy and then
annotating the doubling relationship he identified between $T$ and $F$. The final task, 10.7.6 was essentially an STR task, asking Blair to explain the $2T \rightarrow F$ relationship he was using, and giving Blair’s subsequent insight positive feedback *Good, good. I like that: You can fit two 5s into each 10*. Making the final task explicitly address a relationship, and generalising across all the previous tasks, emphasised the importance being given to structuring and strategies: the final task framed the whole sequence of tasks as being about realising these $N$-$F$-$T$ relationships. In this sense, the segment advanced to a deeper emphasis in the STR dimension of instruction.
Chapter 6 – Lesson 10 Sequence, Layers C, D, E, and F: Analysis of Instructional Progressions Through the Sequence

Just wait, just wait. Let me figure this out. (Blair)

This is the second of the two chapters analysing the Lesson 10 Sequence. Where the previous chapter analysed Layers A, B, C, and D in each of Segments 10.5, 10.6, and 10.7 separately, this chapter analyses Layer C briefly, and especially Layers D, E, and F, across the whole sequence. The detailed work of the previous chapter, analysing task-by-task and segment-by-segment, now bears fruit, as I can draw on that material to develop a rich coherent account of the five mathematical dimensions of progression, and of the interwoven multidimensional progression.

The first section of this chapter briefly addresses analysis Layer C. I give an overview of the development of Blair’s reasoning and activity across the three segments, as a context for analysing the instruction. The second section of the chapter addresses analysis Layer D. I synthesise an account of the progressions in each of the five dimensions, across all three segments. The third section of the chapter addresses analysis Layer E. By tracking the progression of two or more dimensions alongside one another, I describe three different patterns of interaction between dimensions. Finally, the fourth section of the chapter addresses analysis Layer F. I develop a distinctive, concise account of the Lesson 10 Sequence, which I call the multidimensional account. This account describes each task in terms of the adjustments made in each of the five dimensions, tracking their progressions and interactions. I see this account as the culmination of the previous layers of analysis. Reflecting on this account, I will propose that the instructional progression can be characterised as an interwoven calibration across all five dimensions, and that this interwoven calibration was significant in supporting Blair’s reasoning and learning of high 5s.
Recall that the research aims of this thesis are to:

1. Identify key dimensions of instructional progression.
2. Describe the progressions in each dimension, and
3. Characterise the instructional progression, using the dimensions.

By the end of the chapter, I can offer meaningful responses to the three research aims, drawing on this single, intensely studied Lesson 10 Sequence.

6.1 Lesson 10 Sequence Layer C: Overview of Blair’s Progress

It is fascinating to recognise what changed in Blair’s knowledge and reasoning about 5s over the course of the three segments of the Lesson 10 Sequence. In Lesson 8 he had been determined to use counting-based computation strategies. While he clearly already knew something about two 5s making 10 at the beginning of Segment 10.5, he knew much more about this relationship by the end of Segment 10.7. In Segment 10.5, in the context of visible incrementing and decrementing 5-tiles, he readily counted two 5s as one 10. But he made his calculations in each task by skip-counting; he did not see the multiplicative relationship between \( F \) and \( N \). With tiles screened, he could not reckon the \( N \) directly from \( F \), instead keeping separate counts of each. Then, when he had an insight about a multiplicative strategy from \( N \) to \( F \) at the end of Segment 10.5 (“Take the zero off the 30, then just double the 3”), it was couched in the tiles, was only in one direction, and showed no understanding of the underlying relationship of the units of 5 and 10. In Segment 10.6 he solved several tasks using this new strategy, and reiterated the strategy (“Take the zero off the end of the 60, … double the 6”). But he continued to struggle to keep distinct the units of 1s, 5s, and 10s in his calculations, and had prolonged difficulty finding how many 10s in 90. By the end of 10.7, however, he had solved tasks in different inversions using multiplicative reasoning, and had an insight of a multiplicative relationship between \( F \) and \( T \), founded on the proportional relationship between 5 and 10 (“You can fit two 5s into one 10 … So it’s basically doubling that number”). He was more facile in organising the units of 1s, 5s, and 10s. He was also beginning to reason more independently of the tile setting, relying more on the informal arrow sentence notation.
6.2 Lesson 10 Sequence Layer D: Describing the Progressions in Each Dimension

In the previous chapter, I analysed three lesson segments closely. In each segment, I developed a Layer D analysis of each of the five dimensions, highlighting key moments of interest. I will now try to synthesise this close analysis into a more coherent portrait of the significance and progression of each dimension over all three segments. These accounts of each dimension can be complemented by tracking the progression of each dimension down the coloured columns of the Lesson 10 Analysis Chart (Figure 5.1). This analysis contributes to answering the first two research aims: to identify the key dimensions, and to describe how they each progress.

6.2.1 RNG: Progressions in the range dimension in Lesson 10

Over the course of these three segments, I have identified three different ranges involved: the low 5s (1×5, 2×5, 3×5, 4×5, 5×5), the high 5s (6×5, 7×5, 8×5, 9×5, 10×5), and beyond 50 (55 up to 100). Blair could calculate in low 5s as known facts, but could not calculate in high 5s as known facts, and the products beyond 50 appeared even less familiar to him. These differences in his knowledge meant that adjusting tasks between these three ranges had a significant effect on the level of challenge for him. As he established increased fluency and understanding in Segment 10.7, the adjustments between the ranges may have become less significant for him.

Tracing down the red RNG column of the Lesson 10 Analysis Chart, the instruction over the three segments made a basic progression from low 5s, to high 5s, to beyond 50, with only a couple of brief returns to the lower ranges. The low 5s were used mainly for the first task of Segment 10.5, as an introduction to the task type and setting. The high 5s became the focus, seeking insights into multiplicative relations and strategies in this range. The third range of beyond 50 was introduced to furnish further examples for tasks. This third range brought some extra challenge—as in Task 10.6.3, challenging his calculation for 70 dots, and in 10.6.5 challenging his visualisation of 90 dots. Importantly, the third range also led to structuring of more general relations between multiples of 5s and 10s—as in Task 10.7.6. One brief return to a lower range (in Task 10.6.4) supported a consolidation of a relationship being structured (60 as double 30). Other returns to lower
ranges (in Segment 10.7) provided the opportunity for Blair to consolidate new fluency across an extended range.

Another important distinction in RNG evolved, between even multiples of 5—the decuples—and odd multiples of 5. Blair’s difficulty in Task 10.5.4 with nine 5s suggested he may have found odd multiples more challenging than even multiples. Then, as the insights that arose pertained only to even multiples, all subsequent tasks involved only even multiples. It was left to later lessons to extend the RNG to include odd multiples.

6.2.2 ORN: Progressions in the dimension of orientation

Switching between orientations proved constructive. The progression in ORN is shown in the yellow column of the Lesson 10 Analysis Chart. In the task type of Segment 10.5, incrementing and decrementing with 5-tiles, the increment or decrement was always given, and the instruction could switch the ORN between tracking N and tracking F. The switching revealed how Blair was tracking N and F independently. At the same time, I have conjectured that the regular switching contributed to Blair’s realising a relationship between N and F, with his $2n \rightarrow F$ insight in Task 10.5.6 (“Take the zero off the 30, then just double the 3”). In the task types of Segments 10.6 and 10.7, multiplication and division with screened 5-tiles, three unit counts became involved: N, F, and T. Now the ORN could switch between which one of the three values—N, F, T—was given, and also switch between which of the other two values was to be found.

Broadly, the instructional progression went from maintaining one ORN over a sequence of tasks, as in Segment 10.6, to frequently switching ORN, as in Segment 10.7 (see down the yellow ORN column in the Chart). Maintaining one ORN appeared to support Blair in consolidation of a strategy aligned with that orientation. Frequent switching of ORN pitched the instruction at the cutting edge of Blair’s knowledge of the orientations, and supported deeper insight into the system of $N-F-T$ relationships.

Tasks in one orientation can involve learning about other orientations. Blair’s $2n \rightarrow F$ insight (Task 10.5.6) was essentially in the inverse orientation to the prevailing orientation in the segment. The instruction could respond to the
insight by shifting to the inverse orientation for the subsequent tasks in Segment 10.6.

On one occasion, introducing a new orientation proved a signal moment in the instructional progression: Task 10.6.5a with 90 dots, when switching ORN to finding $T$ without counting. This new orientation initially presented an insurmountable challenge for Blair, leading to retreats on other dimensions. The resolution of this challenge led to the introduction of new notation, the productive sequence of tasks incorporating the new orientation (Segment 10.7), and the greater fluency and insight at the end of the lesson.

Minor shifts in ORN were frequently involved as sub-tasks within one task. In several tasks, after the tiles were unscreened, the orientation shifted between checking two or three different counts. I have argued that these shifts supported Blair in clarifying the different units, and structuring the relationships between these different counts.

6.2.3 SET: Progressions in the dimension of setting
Adjustments in the SET dimension were significant in the instructional progression. The adjustments are tracked down the green SET column in the Lesson 10 Analysis Chart. I summarise the use of the setting under four headings:

- The setting as a context for reasoning about tasks;
- Structuring in the setting;
- Distancing the setting;
- Uses of screening and unscreening.

6.2.3.1 SET: Setting as a context for reasoning about the tasks
The instruction introduced and used the 5-tile setting as a context in which Blair could reason about the tasks. As described in the Layer C analyses, the setting had a significant role in Blair’s reasoning in the Lesson 10 Sequence. The tasks in 10.5, 10.6, and most of 10.7 were posed with reference to the 5-tile setting, and Blair understood the tasks in terms of the tiles. His language referred to the tiles. His reasoning typically involved the tiles. When uncertain of an answer, as in 10.5.4 (eight 5s and another 5) and more strikingly in 10.6.5 (90, find $T$), a return to visible tiles helped him make
sense again of the task. When trialling a new strategy, as in the tasks of 10.6, and in Task 10.7.5, a return to visible tiles served to verify his strategy.

6.2.3.2 SET: Structuring in the setting
The 5-tiles were a setting in which Blair could structure the units of 1s, 5s, 10s, and their relationships. Early structuring, in Task 10.5.1, involved organising the 5-tiles in pairs, and counting them in pairs, and this became taken for granted through all three segments. Activity in the visible tile setting supported Blair with his challenge of distinguishing the units of 1s, 5s, and 10s, and to count $N$, $F$, and later $T$ alongside each other, as when the teacher corrected a count of 90 cards as “90 dots”, and when, while he struggled to construct a count of 5s, the teacher reminded him *Don’t count the dots … count 5s … that’s one 5 there* indicating a 5-tile. In resolving Task 10.6.5, the 90-dot task, he used the same action of two-fingered pointing at each of the nine pairs of tiles to accomplish three different counts: counting $N$ 10, 20, 30 … 90; counting $F$ 2, 4, 6 … 18; and counting $T$ 1, 2, 3 … 9. His difficulty with these counts in 10.6.5b contrasted with his fluency with the same counts five tasks later in 10.7.5, which suggested his improved coordination of these three units. Activity in imagining the screened tiles also supported Blair in structuring, as in 10.6.4, with his construction of an arrangement of 60 dots as two rows of three paired 5-tiles. His important $2n \rightarrow F$ insight at 10.5.6 (“Take the zero off the 30, then just double the 3”) and his later insight explaining this relation at 10.7.6 (“You can fit two 5s into one 10”) both appeared to arise from this structuring work with the tiles.

6.2.3.3 SET: Distancing the setting
The SET dimension progressed broadly from more visible setting and setting dependence in early 10.5, through to unused setting and setting independence in later 10.7, as indicated down the green SET column of the Chart. Most of the instruction in these three segments was somewhere in the middle of this progression, but there were frequent subtle adjustments made, using screening, visualising, and formality of language. A range of subtleties in the progression included:

- posing a task with visible tiles;
- posing a task with visible tiles, then screening them;
• posing a task screened, but flashing the tiles before the student has answered;
• posing a task screened, then unscreening after the student has answered, to check the answer;
• posing a task screened, then asking what the tiles will look like, then unscreening;
• posing a task screened, but not unscreening afterwards;
• posing a task screened, but shifting to less setting-dependent terms, such as shifting from “cards” to “5s”;
• posing a task as though it were about tiles, but not actually setting up the tiles; and
• posing a task without reference to tiles.

A common basic rhythm in SET was posing a task with setting screened, then retreating to unscreened to check the answer, then advancing to screen again for the next task. Varieties of this practice recurred in all three segments. More broadly, there could be a sense of distancing over a sequence of tasks. For example, the first four tasks of 10.6 had a sense of increasing distance by culminating in not unscreening to check, instead challenging Blair to visualise the task arrangement under the screen (What are you going to see under there?). In similar vein, the tasks of 10.7 were progressively distanced from the setting. However, in other instances a retreat on the SET dimension proved effective, such as to resolve the impasse of the How many 10s? task 10.6.5, and to revisit checking all the unit counts in 10.7.5 (“Everything is correct”). Thus, while there was a broad progression in SET over the three segments, there was a great deal of advancing and retreating within the instructional progression.

6.2.3.4 SET: Uses of screening and unscreening
The analysis revealed several different uses of screening and unscreening of the tile setting in the instructional progression. Screening commonly advanced the challenge of a task. Most tasks in 10.5 and 10.6 involved screening in this way, when Blair’s reasoning appeared closely dependent on the tiles. Screening could also essentially change the task. In 10.5, with the task type of incrementing and decrementing 5-tiles, when tiles were visible in
Task 10.5.1 the student could answer tasks by counting tiles without awareness of the increment, whereas once the tiles were screened, the reasoning needed to involve awareness of the increment. Screening also appeared to bring the student’s attention to visualising the tiles, as in 10.5.4 when Blair suggested the tiles “wouldn’t be even”. In some tasks this was made explicit by asking the student to describe the tile arrangement behind the screen, as in 10.6.4, *What are you going to see under there?* which proved an instance of rich structuring.

Unscreening allowed the student to verify an answer. This supported the instructional principle of child checking (Wright, Martland, Stafford, et al., 2006, p. 35). Most tasks in the Lesson 10 Sequence used unscreening to successful effect in this way. The unscreening could provide the student with immediate feedback on his own reasoning. For example, in 10.5.4, when Blair expected uneven tiles, the unscreening confirmed for him the uneven arrangement, while revealing his answer was incorrect. In 10.6, when Blair construed the tasks as testing his strategy, the unscreening could offer confirmation of his expectations: “See, it’s working!”

The variations in the progression of screening and unscreening can lead to different approaches to solving a task. In Task 10.6.4, finding $F$ for 60 dots, Blair solved first with tiles screened, by visualising a structured arrangement of 12 tiles. After the tiles were unscreened, he explained how he could find 12 in a different way, using the $2n\rightarrow F$ strategy, “Take the zero off the end of the 60,…double the six”. Overall, a combination of screening and unscreening was frequently powerful in maintaining the instruction at the student’s cutting edge, and keeping the instructional attention on the structuring of multiplicative relations.

6.2.4 **NTN: Progressions in the dimension of notation**

The main development of notation was in Segment 10.7, where it proved to be powerful in supporting Blair’s learning (see the blue NTN column in the Lesson 10 Analysis Chart). I observed an instructional progression of increasing engagement with the notation. A sense developed of the notation holding the task for the student and the teacher, which was new for us. This became a feature of the instructional tasks in subsequent lessons.
The Layer D analysis for Segment 10.7 described five aspects of the progression in NTN. The first aspect was the instructional decision to pursue the notation in the first task, 10.7.1, which followed an experiment with arrow sentence notation in resolving the previous challenging task, 10.6.5. Thus, instruction can introduce and continue notation on the NTN dimension, according to the responses of the student.

A central aspect of the progression in NTN in 10.7 was a transition from the teacher writing to the student writing. This can be described as a progression increasing the student’s independence with the notation. The progression can include independence *from* the notation, challenging the student to solve the tasks without using the notation. Subtle shifts in this transition included:

- Teacher writes whole notation sentence;
- Student writes, with teacher indicating where to write each part of the sentence;
- Teacher writes one part, leaving student to complete the sentence;
- Teacher asks student to write the sentence;
- Teacher screens previous lines of notation, challenging student to use notation with independence from seeing earlier instances; and
- Student takes own initiative to write the sentence.

Also important in the NTN progression was attending to Blair’s search for relationships in the notation. This was supported particularly by repeated attention to aligning the parts of the respective arrow sentences. This attention culminated in the final task, 10.7.6, where I could use the record of sentences to inquire about a key doubling pattern, *Why is it ... when there’s six 10s, there are twice as many 5s?*

The instructional progression in the earlier segments, 10.5 and 10.6, made use of one-off instances of notating. These also had a role in supporting recognition or clarification of Blair’s reasoning. At 10.5.1, notating Blair’s calculation of $6 \times 5 = 30$ gave him a note he could refer to for a later instance of $6 \times 5$, supporting the rehearsal of this fact. Notating his new “drop the zero, and double it” insight at 10.5.6 appeared to consolidate this articulation.
of the strategy. The arrow sentence in 10.6.5c, after the *How many 10s?* difficulty, enabled Blair to clarify and reflect on the resolution of the task.

6.2.5 STR: Progressions in the dimension of structuring and strategies
Adjustments in the STR dimension were significant in the instructional progression. I discuss four aspects of the STR dimension:

- Different types of STR instruction;
- Pitching the instruction at the student’s cutting edge;
- Pitching instruction toward the instructional aims; and
- Teacher’s attention supports student’s attention.

6.2.5.1 STR: Different types of STR instruction
One aspect is to identify different types of STR instruction. I distinguish four different types of STR instruction in the Lesson 10 Sequence, with examples drawn from the analysis.

- Enquiring about the student’s reasoning, and affirming his reasoning:
  - 10.5.1 *How can you see 30?* ... *Very good, seeing those 10s.*
  - 10.5.6 *Let’s write this down, this is important.* After Blair described his new $2n \rightarrow F$ insight.
  - 10.7.3 *How did you just check that?*

- Prompting the student to recall or consolidate his earlier reasoning:
  - 10.6.1 *Let’s try it a few times,* that is, try his new $2n \rightarrow F$ strategy.
  - 10.6.4 *And how could you get that 12 just from the 60?* A prompt to recall his new strategy.

- Challenging the student to improve his structuring or strategy:
  - 10.6.4 *What are you going to see under there?* Challenge to predict and describe the arrangement of tiles.
  - 10.6.5a *How many 10s do I have?* ... *No, don’t count them,* restricting what computation strategy is allowed.
  - 10.6.5c *90 ... is nine 10s ... is that surprising?* ... *Why did you go through twists and turns?* Challenging Blair to clarify his thinking on the previous task.
• Challenging the student to explain a relationship or strategy:
  
  o 10.7.6 Why is it... When there’s six 10s, there are twice as many 5s
    ... There’s four 10s, there’s twice as many ...

Recognising these four types helps the research task of describing the STR dimension. Also, there is the potential for instructional progression through these different types. In this lesson, these four types appeared in the order given. Enquiries about, and affirmations of, the student’s reasoning appeared from the beginning, and recurred through the three segments. Prompts to recall earlier reasoning appeared in Segment 10.6, after his new strategy had appeared at the end of Segment 10.5. Challenges to improve the reasoning around this new strategy appeared soon after. The final task of the lesson shifted to explaining the strategy. The progression through these four types has a sense of progressing through increasing mathematical sophistication, from describing, to recalling, to improving, to explaining mathematical relations or calculations. There was also potential significance in bringing the instructional progression to a conclusion with the STR advance to the pattern-explaining task. This may have created a sense that this task, this N-F-T relationship, was the main goal of the lesson, deepening the instructional attention given to structuring.

6.2.5.2 STR: Pitching instruction at the student's cutting edge
The analysis of the three Lesson 10 segments describes several episodes where a shift to STR appeared to bring the instruction to the student’s cutting edge. In 10.6, the STR adjustment of treating the tasks as a trial of Blair’s new strategy had Blair challenged, and mostly successful. The new STR challenge of visualising the tile arrangement at 10.6.4 challenged Blair to a new multiplicative structuring connecting 30 and 60. The repeated STR attention to distinguishing the different units of 1s and 5s kept Blair on the edge of his facility with these units, and the STR introduction of finding a new unit count, T, proved a reach beyond his cutting edge, while further STR attention was part of resolving this challenge. The final task, 10.7.6, challenged Blair to a new level of insight in the N-F-T relations. These episodes are convincing evidence that shifts in STR can be effective in pitching instruction to the cutting edge of the student’s knowledge.
6.2.5.3 STR: Pitching instruction toward the instructional aims

Furthermore, the same STR episodes show that STR was important for pitching the instruction toward some of the target mathematics. In this case, the instructional aims included developing multiplicative structuring and multiplicative strategies for basic facts, so it is no surprise that attention to STR attended to these aims. Nevertheless, it is impressive how consistently Blair’s activity remained focused on aspects of structuring multiplicative units and relations, and developing new multiplicative strategies, throughout these segments. This consistency of attention to the target mathematics did not come from a single introductory instruction or task. Rather, the several different STR episodes each brought attention in different ways: attention to how Blair counted the tiles, how he arranged the tiles, his new strategy, how he distinguished units, how he related units, how he understood the system of unit relations, and so on.

6.2.5.4 STR: Teacher's attention supports student's attention

I conjecture that the regular instructional attention to STR supported Blair’s own attention to structuring the $N$-$F$-$T$ relations over the course of these segments. There are several examples where, after the teacher’s STR attention to an aspect of Blair’s reasoning, the reasoning continued prominently in subsequent tasks. In Task 10.5.1, I drew attention to his initiative in counting the 5-tiles in pairs; and in subsequent tasks, he continued to pair 5-tiles, almost always using a double-fingered point in his counting. Also in Task 10.5.1, I drew STR attention to recording his result for $6\times5$, and in the later task, 10.5.5, he recalled this result to solve a six 5s task. In Task 10.5.6 I recorded and affirmed his new $2n\rightarrow F$ strategy, and his recollection of this strategy became prominent in subsequent tasks, and indeed in later lessons, as we will see in the next chapter. In instances like this, I understand the STR attention as raising the student’s awareness of his own structuring and strategies, which in turn can consolidate this reasoning.

There are also several examples where Blair took initiative with his own attention to structuring and strategies. In Task 10.5.1, after answering my question of how he saw 30, he then took initiative to suggest another way of structuring the tiles. In Task 10.5.6, his new $2n\rightarrow F$ insight ("Take the zero..."
off the 30, then just double the three”) came without prompt from his own initiative in noticing the pattern, and was announced with excitement as an important discovery. In 10.6.4 I investigated how he imagined the arrangement of tiles for 60 dots; then in the following task, 10.6.5, when struggling with finding $T$ for 90 dots, he took the initiative to try to imagine the arrangement of tiles. In Task 10.7.1, he initiated asking how the units were aligned in the previous arrow sentence (“They were the 10s for the 90 weren’t they … ?”), and similarly in 10.7.3 he initiated checking his answer against the pattern in the previous sentences (“I looked here”). In Task 10.7.5, after earlier tasks involved STR attention to counting all units in the visible tiles, he now took initiative with this practice, checking three different counts against the written answers, and announcing “Yes. Everything is correct”. There is a pattern throughout the lesson of the student attending to the aspects of structuring and strategies highlighted by the teacher’s regular attention to STR.

6.3 Lesson 10 Sequence Layer E: Characteristic Interactions Between Dimensions

The previous section addressed analysis Layer D, analysing progressions in each of the five dimensions separately. This section addresses Layer E, seeking to identify and describe interactions between dimensions. Below I describe three interactions:

- coordinating progressions in SET and NTN;
- complementarity between ORN and SET; and
- responding with STR to adjustments in other dimensions.

6.3.1 Dimension interactions: Coordinating progressions in SET and NTN

The progressions in SET and NTN had a notable relationship, where advances in the two dimensions could be intertwined. In Segments 10.5 and 10.6, an ongoing notation was not used, and the tasks were based in the visible and screened 5-tiles setting. At Task 10.6.5c, NTN advanced to introduce an arrow sentence notation as a recording of results about the tile setting. Then, through Segment 10.7, as the NTN developed the use of the arrow sentence notation, the 5-tiles appeared to become less prominent in Blair’s thinking, and SET was increasingly distanced to bare numbers. It
appears that NTN was first advanced in the context of the tile setting, then in turn SET was advanced to bare numbers through the increasing engagement with the notation.

More subtly, within some tasks of Segment 10.7, the instruction moved between the setting and notation, such that developments in settings and notation were mutually supportive. Consider Task 10.7.1. I began by asking Blair to write 80 in place in the arrow sentence (NTN). Then I arranged the screened tiles and posed the task of 80, how many 10s (SET). Blair asked where the 10s were in the previous arrow sentence, which I clarified for him (NTN). After he arrived at his answer, I unscreened the setting to check, beginning by indicating a pair of tiles and saying *This is what a 10 looks like* (SET). After checking the counts of 10s and 5s in the tiles (SET), Blair moved to write the answers in the arrow sentence (NTN). In summary, as the task was posed in the notation, it was posed alongside in the setting; as Blair tried to clarify 10 in the notation, 10 was clarified alongside in the setting; and as he confirmed his answer in the setting, he wrote his answer in the notation. This kind of micro-progression of the instruction alternating between SET and NTN may have supported his increasing competence with both in this segment, culminating in handling a similar alternation of setting and notation fluently in Task 10.7.5 (“Everything is correct”.)

### 6.3.2 Dimension interactions: Complementarity between ORN and SET

The 5-tile setting provided a context in which questions in different orientations could be posed about the same situation. In Segment 10.5, most tasks posed a situation of screened tiles, then asked two orientations: finding $F$ and finding $N$. In Segment 10.6, the main orientation was $N \rightarrow F$, but once the tiles were unscreened for checking, the instruction could ask to find both $F$ and $N$ again. So still using tiles for the SET dimension, and regularly unscreening them, could support the ORN dimension to advance to switching orientations.

At the same time, the switching in ORN may have supported Blair’s structuring of the tile setting. One of Blair’s main challenges in these segments was clarifying the different units and their counts. The regular
switching of ORN in the instruction challenged Blair to keep clarifying which unit he was counting in the setting: 5s, 10s, or the total dots.

A third way to describe the interaction between these dimensions in these segments is that they were complementary in supporting Blair’s structuring of $N$-$F$-$T$ relations. The switching of ORN challenged Blair to think through these relations in different directions: from $N$ to $F$, from $F$ to $T$, and so on. Meanwhile, the retreat to visible tiles and visualising screened tiles created situations where all those directions could be checked alongside each other. By the end of 10.7, after significant switches in ORN and returns to visible SET, Blair did appear to realise, at some level, that the tasks were different orientations of the same situation, so the approaches to solving the different orientations could be connected.

6.3.3 Dimension interactions: Responding with STR to adjustments in other dimensions

The moments of STR instruction often arose in response to an adjustment in one of the other four dimensions. I list some examples. In Task 10.5.4 (40 and another 5) RNG adjusted by introducing an odd multiple. Following Blair’s response, the instruction brought STR attention to a distinction between odd and even arrangements of tiles. In Task 10.6.4 (60 dots), instead of unscreening to check, SET kept the tiles screened and then posed the STR challenge to visualise the tile arrangement. In Task 10.6.5 (90, find $T$), after ORN switched to finding a different unit count, there was STR involvement in seeking to distinguish the units in the tile setting: \emph{How many dots?}... \emph{How many 5s?}... \emph{How many 10s?}\ In the early tasks of 10.7, after NTN introduced the new arrow sentence notation, STR instruction drew attention to the parts of the notation. In each case, an adjustment in a dimension created an opportunity for structuring relations. Instruction then attended to these opportunities by having STR interact with the other dimension.

6.4 Lesson 10 Sequence Layer F: Multidimensional Analysis of the Instructional Progression

Now I turn to Layer F, the final layer in the analysis of the Lesson 10 Sequence. This layer builds on all the prior layers, seeking to understand how the five dimensions are coordinated to create the instructional progression. I first develop what I call a \emph{multidimensional account} of the instructional
progression through the three segments. This is a concise account of the instruction, expressed as a series of adjustments along the five dimensions. Following the multidimensional account, I investigate the texture of the instructional progression, as viewed in the account. The analysis contributes to the third and final research aim: to characterise the instructional progression in terms of the multiple dimensions.

6.4.1 Developing a multidimensional account of instructional progression

The analysis has observed a variety of ways that each of the five dimensions can be calibrated to affect the instruction. My purpose now is to develop a coherent account of how the instruction progressed task by task in terms of the calibrations across all five dimensions. I want the account to suggest how these calibrations were coordinated to create a responsive instruction which adjusted the pitch to the changes in the student’s cutting edge and kept working on the mathematics of interest.

This multidimensional account is the culmination of the layered analysis, and it draws on all the previous layers. Drawing on Layers B and C, I take for granted here the interpretation of Blair’s activity in each task, and the significance of his activity in terms of the mathematical issues he was learning about in these segments. Drawing on Layers B and D, I take for granted the interpretation of how each dimension was retreated, advanced, or otherwise adjusted in each task, and from Layer E, I obtain suggestions regarding ways the dimensions interacted. By establishing all this material in the previous layers, I can keep this account focused on revealing the instructional progression as an interwoven calibration of the dimensions, responsive to Blair’s cutting edge. The multidimensional account is actually brief, but this brevity is achieved only by drawing on the lengthy prior analyses.

6.4.2 Multidimensional account of the instructional progression in Lesson 10

Task 10.5.1 (one 5-tile, two 5-tiles … six 5-tiles) was pitched as an introductory task, with RNG in low 5s and a SET of visible tiles. Following Blair’s fluent response, an STR follow-up question, How can you see 30? prompted structuring of the tiles in pairs, and NTN recorded a result. Task 10.5.2 (another 5 → 35) increased the challenge by advancing RNG to the
high 5s and advancing SET to screened tiles. The increased level of challenge for Blair on 10.5.2 and subsequent tasks confirmed that the high 5s was an appropriate topic at the cutting edge of Blair’s knowledge. Task 10.5.3 (another 5 → 40: eight 5s) increased the challenge further by advancing ORN with an extra unit count \( F \), then sought to verify Blair’s answers with a retreat in SET, the temporary unscreening of the tiles. In Task 10.5.4 (another 5 → 45: nine 5s) Blair revealed confusion in distinguishing units of 5 and 10. An STR comment about uneven tiles drew attention to structuring, and for the next two tasks (take three 5s → 30; add two 5s → 40), the RNG, ORN, and SET dimensions were kept unchanged. Blair achieved success again, and in Task 10.5.6 realised his 2n→F insight about relations in these tasks: “You take the zero off the 30, then you just double the three”. Following his insight, instruction involved STR and NTN to clarify and affirm his reasoning: Let’s write this down, this is important.

Task 10.6.1 (40 dots N\(\mapsto\)F) was posed to align with trialling Blair’s new strategy (Let’s try it a few times), which required maintaining the same RNG and SET, but shifting the ORN to N\(\mapsto\)F. Task 10.6.2 (50 dots N\(\mapsto\)F) maintained the same dimension levels. After Blair indicated increased confidence and fluency (“See, it’s working!”), Task 10.6.3 (70 dots N\(\mapsto\)F) increased the challenge by advancing the RNG beyond 50. When Blair was less fluent, a sub-task (unscreening: Check) sought clarification by retreating SET to visible tiles, shifting ORN to track \( N \) as well as \( F \), and bringing attention to STR, which challenged Blair. Task 10.6.4 (60 dots N\(\mapsto\)F) returned to the previous dimension levels by advancing SET to screened tiles again. Following Blair’s fluent answer, a follow-up question What are you going to see under there? increased the challenge with an STR focus involving an advance in SET, asking Blair to describe the tile layout while still screened. When Blair demonstrated strong structuring of 60 dots, sub-tasks How many cards for 30? ... for 60? sought to rehearse his calculations using a brief retreat to high 5s and a return to beyond 50 in the RNG, and a subtle retreat to flashing and a return to screening tiles in SET. Responding to Blair’s continued fluency, an STR sub-task How could you get that 12 just from 60? sought to rehearse his 2n→F strategy.
Task 10.6.5a (90 dots $N \rightarrow T$) maintained the same pitch as the previous two tasks, and Blair again responded fluently, so a new challenge was posed, *How many tens? Don’t count*, advancing ORN to find a new unit count $T$ and making an STR restriction on computation strategy. Blair struggled long and could not succeed with the new challenge, so Task 10.6.5b eased the challenge by retreating SET to visible tiles, and sought clarity with ORN and STR questions about all three unit counts $N$, $F$, and $T$. Blair eventually succeeded. To consolidate his answers, Task 10.6.5c introduced NTN with an arrow sentence, and a further STR question (*90 is nine 10s. Is that surprising?*), which brought further resolution.

Task 10.7.1 (90 dots $N \rightarrow T&F$) returned to the original challenge of 10.6.5a, by continuing the new ORN of finding $T$, and returning SET to screened tiles. Blair struggled again, but then succeeded with a strategy involving the new notation. Follow-up (unscreening: *Check*) sought self-verification of his answers by retreating SET to visible tiles, and posing STR questions. Task 10.7.2 (60 dots $N \rightarrow T&F$) continued with ORN of finding $T$ and SET with screened tiles; this time Blair responded fluently, completing the notation and ignoring the tiles. Task 10.7.3 (4 10s $T \rightarrow F&N$) changed the challenge by shifting the ORN to $T \rightarrow F&N$. At the same time, responding to Blair’s shift of attention away from the tiles and toward the notation, the SET was advanced to not using the tiles, and the NTN of arrow sentences was continued. Blair succeeded. To clarify his strategy, an STR question was posed *How did you just check that?*, and NTN was used to annotate the arrow sentences with his answer about a $2T \rightarrow F$ pattern (“I looked here”). Task 10.7.4 (two 10s $T \rightarrow F&N$) continued the same line, maintaining the ORN, SET and NTN, while retreating the RNG, and Blair succeeded fluently. Task 10.7.5 (14 5s $F \rightarrow T&N$) changed the challenge again, trying a third and most difficult inverted ORN of $F \rightarrow T&N$, and returning the RNG to beyond 50. When Blair succeeded again, apparently with a new strategy of $\frac{1}{2}F \rightarrow T$, a follow-up (unscreening: *Check*) sought to consolidate his understanding by retreating the SET to visible tiles for self-verification of all answers. After Blair responded with a new fluency in handling all three units and coordinating between the setting and the notation (“Yes, everything is correct”), Task 10.7.6 set a challenge of
generalising from the previous tasks, by posing an STR question *Why is it?* with a new ORN and an increased attention to NTN, while advancing the SET to bare numbers again. Blair’s insightful answer (“You can fit two 5s into one 10”) was affirmed and clarified using NTN.

### 6.4.3 Characterising the multidimensional instructional progression

The multidimensional account expresses the instructional progression in terms of adjustments across the five dimensions. I contend it stands as a coherent, substantial account of the instructional progression. The account suggests a compelling characterisation of the instructional progression as:

> a strategic interwoven calibration of the five dimensions, toward the instructional aims of keeping at the learner’s cutting edge and attending to the mathematics of interest.

In this section, I explain this characterisation.

The multidimensional account above describes the calibration of five dimensions at each task. This description proves sufficient to portray all the main action in the instructional progression. Almost all instructional adjustments are included, because they can be recognised as an adjustment in some of these five dimensions. The progression from one task to the next can be accounted for as a combination of advancing on some dimensions, retreating on some dimensions, and maintaining the pitch of the remaining dimensions. The instructional progression can be effectively described in terms of adjustments in the five dimensions.

A basic characteristic to observe is that most tasks involved an adjustment in at least one of the four dimensions RNG, ORN, SET, and NTN. At the same time, few tasks involved an adjustment in three or four dimensions simultaneously. Instead, the instruction largely progressed with an adjustment in the pitch of one or two dimensions at each task or sub-task.

Also, the adjustments tended to be relatively small. The earlier Layer D section on the progressions within each dimension described the subtle shifts possible. For example, RNG could make small shifts from high 5s to beyond 50; SET from flashing to screening; NTN from teacher writing all to student writing some. Viewed from further distance, all three segments could be considered as working on a limited range of very similar tasks. Nevertheless,
the few and small adjustments made in each task appeared to have a significant impact on Blair’s activity.

While the instructional progression can be expressed in terms of adjustments on only five dimensions, and those adjustments are typically few and small, nevertheless, the progression appears as a rich, subtle interweaving. There is little that appears as routine or standard in the progression of the adjustments. The adjustments involve a whole mix of different single dimensions and pairs of dimensions. The instructional progression overall appears as a subtle interweaving of adjustments in RNG, ORN, SET, and NTN, including many temporary retreats in SET, punctuated with frequent attention to STR.

At each task and sub-task, the coordination of adjustments across the five dimensions pitched the instruction in response to Blair’s strategies and success. As the student succeeded, selected dimensions were advanced, increasing the challenge. When the student did not succeed, selected dimensions were retreated. Adjustments in one dimension could be coordinated with, or complementary to, adjustments in another dimension. Over the course of the three segments, a whole range of subtly different mixtures of dimension levels were posed. Through this subtle responsiveness of the instruction, the student appeared to be kept at his cutting edge throughout. The adjustments in each dimension also served to direct the tasks toward the key instructional aim of developing \( N\text{-}F\text{-}T \) relations and strategies. Blair was consistently attending to the tasks and structuring of most relevance, consistently thinking hard, and generally succeeding. The calibrating instruction powerfully supported Blair’s progress.

Thus, I contend that the instructional progression can be characterised by the interwoven calibration across the five dimensions, and this characterisation illuminates how the instruction can be responsive to the student’s activity. This characterisation is the main offering of my thesis.

6.5 Chapter Summary

In this chapter analysing the Lesson 10 Sequence, I have given:

- a brief Layer C account of Blair’s progress;
- a Layer D account of the progression in each of the five dimensions;
• a Layer E account of three interactions between dimensions; and
• a Layer F account of the multidimensional instructional progression.

With this analysis, I have already achieved much for the research aims. Responding to the first aim, I can identify that the five dimensions of RNG, ORN, SET, NTN, and STR are the five main dimensions in this Lesson 10 Sequence. Responding to the second aim, I can summarise the significant progressions within each of those dimensions. Responding to the third aim, I can characterise the instructional progression as a strategic, interwoven calibration of the five dimensions. In this case, the calibrated instruction has powerfully supported Blair’s progress with the main learning aims of developing relations among \( N \), \( F \), and \( T \), and developing strategies for computing one from another.

I now want to pursue a similar analysis beyond this single lesson, to consider the whole High 5s Sequence.
Chapter 7 – High 5s Sequence, Layers A, B, and C: Analysis of Each Segment and of Progressions in Student Reasoning

Can you give me another like this, so I can figure it out? (Blair)

I turn now from the nested case of the Lesson 10 Sequence to the longer case of the High 5s Sequence. This Chapter 7 is the first of the two chapters analysing the High 5s Sequence. The purpose of these two chapters is to illuminate the role of the five dimensions in the instructional progression through the sequence.

The High 5s Sequence is the sequence of all the main lesson segments with Blair that focused on the high 5s topic, comprising segments from eight lessons: Lessons 8, 10, 11, 12, 14, 15, 16, and 17 (see Figure 4.1 showing Blair’s High 5s Sequence within the 24 lessons of Term B). Blair made significant progress over the High 5s Sequence, with the insights that arose in the three Lesson 10 segments developing over the rest of the sequence into a fair facility with all high 5s tasks. The clear progress in a compact sequence motivates the analysis of the case.

The analysis of the High 5s Sequence is organised again into the six layers:

- Layer A: Observation;
- Layer B: Local interpretation;
- Layer C: Progressions in student activity;
- Layer D: Progressions in each dimension of instruction;
- Layer E: Interactions between dimensions;
- Layer F: Multidimensional progressions in instruction.

For the analysis of the Lesson 10 Sequence in Chapter 5, I addressed Layers A, B, C, and D for each segment in turn to keep track of a very detailed analysis. For the present analysis of the High 5s Sequence, the organisation of the layers is more straightforward, working through each layer once for the
whole sequence. This chapter, Chapter 7, addresses Layers A, B, and C, while Chapter 8 addresses Layers D, E, and F.

This chapter is in three main sections. A brief first section gives an outline of the High 5s Sequence and introduces the major analysis chart which summarises the sequence. The second section develops the Layers A and B analysis, describing all the episodes and moments of interest for the subsequent layers in the case study. The third section is the Layer C analysis of progressions in Blair’s reasoning.

7.1 High 5s Sequence: Outline and Chart

7.1.1 Outline of the High 5s Sequence

Before launching into the analysis, I offer the following outline of the story of the High 5s Sequence. Initially, Blair did not have an efficient multiplicative strategy for multiplication or division of high 5s. The first lesson segment in the sequence, Segment 8.8, began with a setting of 5-tiles, but quickly progressed to a sequence of bare number tasks in the range beyond 50: given even number of 5s \( F \), calculate the number of 10s \( T \). Blair had difficulty with the tasks, and did not appear to recognise a relationship between \( F \) and \( T \). In the three segments from Lesson 10, as I have now recounted in detail, the instruction returned to the 5-tiles setting, and made a slower progression toward bare number tasks and an informal notation of arrow sentences, with varying task orientations. Blair had a series of confusions and insights about the \( N-F-T \) relations, and by the final tasks achieved some facility with \( 2T \rightarrow F \) and \( \frac{1}{2}F \rightarrow T \) strategies. The two segments of Lesson 11 appeared to recapitulate the progressions of the Lesson 10 segments: the instruction progressing from the 5-tile setting toward bare numbers and informal notation, and Blair rediscovering the \( 2T \rightarrow F \) and \( \frac{1}{2}F \rightarrow T \) strategies. In place of the arrow sentences, I introduced a form I called parallel expressions notation, which became central to the instruction in subsequent lessons. The three segments of Lesson 12 recapitulated some of the same progressions again, but this time without the 5-tiles. The lesson centred around a protracted challenge for Blair to recall the \( \frac{1}{2}F \rightarrow T \) reasoning, which will become a key episode in the analysis. Segments 14.3 and 15.3 posed tasks now familiar from Lessons 11 and 12, to which Blair now responded with fluency. Three
further segments in Lesson 15 progressed to newer tasks: successfully transitioning to conventional number sentence notation, justifying the \textit{N-F-T} strategies for even multiples of 5, and devising alternative strategies for an odd multiple of 5. In Lessons 16 and 17, the final two lessons of the sequence, the instruction progressed to mixing high 5s tasks with high 3s and 4s tasks.

Four of these lessons had extensive passages on high 5s comprising three or four segments: Lessons 10, 11, 12, and 15. Most of the key episodes in the analysis are from these four lessons.

Through the sequence overall, the range dimension of the instruction progressed from being focused mostly on even multiples of 5 through to mixed multiples of 3s, 4s, and 5s. The setting progressed from the 5-tile setting through to bare number tasks; and the notation progressed from verbal tasks through informal notations to conventional number sentences. Blair progressed from weak to strong knowledge of multiplicative \textit{N-F-T} relations, and he progressed from skip-counting-based strategies to efficient multiplicative strategies and increasingly automatised basic facts. However, as observed in the Lesson 10 Sequence analysis, these progressions in the instruction and in the learning were not straightforward; rather, they were recursive and convoluted.

### 7.1.2 High 5s Analysis Chart

The large High 5s Analysis Chart (Figure 7.1) summarises the Layer A and B analyses of each segment in the sequence. This chart becomes an important reference when following the analysis of the progressions in Blair's activity and the progressions in instruction across the segments. The chart is organised in the same way as the Lesson 10 Analysis Chart.
<table>
<thead>
<tr>
<th>Layer A Analysis: Observations</th>
<th>Layer B Analysis: Local Interceptions</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Task type and tasks</strong></td>
<td><strong>RNG</strong></td>
</tr>
<tr>
<td>Multiplication with 5-tiles → base numbers</td>
<td>Blair’s responses</td>
</tr>
<tr>
<td>6-5s, 6-5s.</td>
<td>✓</td>
</tr>
<tr>
<td>Bare, recording in a table: 2 × 6 = 1 × 10 = 10</td>
<td>HI</td>
</tr>
<tr>
<td>4 × 6, 6 × 6, 10 × 6</td>
<td>HI</td>
</tr>
<tr>
<td>16 × 5, 22 × 6, 46 × 6.</td>
<td>HI</td>
</tr>
<tr>
<td>What’s the relationship between 12 × 6?</td>
<td>HI</td>
</tr>
<tr>
<td>“I don’t know” “Two times” “18, 8, times 7.”</td>
<td>HI</td>
</tr>
<tr>
<td>Increasing &amp; decrementing with 5-tiles</td>
<td>Blair’s responses</td>
</tr>
<tr>
<td>Visible: 3-5, less 5s, less 5s.</td>
<td>HI</td>
</tr>
<tr>
<td>Screened: 75s (55) + 40 (60) + 40 (60)</td>
<td>HI</td>
</tr>
<tr>
<td>“take these 5s (60) + then 5s (60).”</td>
<td>HI</td>
</tr>
<tr>
<td>Of interest: Let's write this down, this is important.</td>
<td>HI</td>
</tr>
<tr>
<td>Division with screened 5-tiles (Given N find M)</td>
<td>Blair’s responses</td>
</tr>
<tr>
<td>40, 50, 70</td>
<td>HI</td>
</tr>
<tr>
<td>“What are you going to see under them?...”</td>
<td>HI</td>
</tr>
<tr>
<td>20, “How many?” 30, “Don’t count.”</td>
<td>HI</td>
</tr>
<tr>
<td>Uncovering: How many digits? 5s? 10s? Write down in sentence: 60 → 16 × 5s → 9 × 10s.</td>
<td>HI</td>
</tr>
<tr>
<td>Viable orientations with screened 5-tiles</td>
<td>Blair’s responses</td>
</tr>
<tr>
<td>Recorded in a table: 80 → 16 × 5s → 8 × 10s.</td>
<td>HI</td>
</tr>
<tr>
<td>80, 80 find T5; 45, 15s (find FAN); 110 find BAN.</td>
<td>HI</td>
</tr>
<tr>
<td>Why is K &amp; P (Working about orientations in tables)?</td>
<td></td>
</tr>
<tr>
<td>Multiplication with 5-tiles (Given F find M)</td>
<td>Blair’s responses</td>
</tr>
<tr>
<td>Visible: 2-5s, 5-5s, 5-5s, 5-10s, 5-9s.</td>
<td>HI</td>
</tr>
<tr>
<td>Screened 5s</td>
<td>HI</td>
</tr>
<tr>
<td>2 × 5s, 4 × 5s, 6 × 5s, 10 × 5s, 9 × 5s.</td>
<td>HI</td>
</tr>
<tr>
<td>4 × 5s, 8 × 5s, 6 × 10s, 10 × 10s, 5 × 10s, 9 × 10s.</td>
<td>HI</td>
</tr>
<tr>
<td>8 × 5s “Two, four, six, eight, you’ve got four 10s.”</td>
<td>HI</td>
</tr>
<tr>
<td>“23” Corrected after shown pairs of 5-tiles as 10.</td>
<td>HI</td>
</tr>
<tr>
<td>80 “Duplicate 1...16 5s...”</td>
<td>HI</td>
</tr>
<tr>
<td>“If I have 54, I’ve got 7...3 times 10.”</td>
<td>HI</td>
</tr>
<tr>
<td>Division with bare numbers (Given N find F &amp; P)</td>
<td>Blair’s responses</td>
</tr>
<tr>
<td>Parallel expressions. 40 = find F; 70, 40, 60.</td>
<td>HI</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Division with bare numbers (Given F &amp; N)</td>
<td>Blair’s responses</td>
</tr>
<tr>
<td>Posed and recorded in number sentences.</td>
<td>HI</td>
</tr>
<tr>
<td>6 × 5s. Stated. “In the ten's, it’s 55.”</td>
<td>HI</td>
</tr>
<tr>
<td>5 × 10s. “Corrected by 5s, keeping track on fingers.”</td>
<td>HI</td>
</tr>
<tr>
<td>5 × 10s. Corrected by 5s, O’ “5s, 5s.” “Add 5.”</td>
<td>HI</td>
</tr>
<tr>
<td>Or proposed on 10s × 5s = 125s.</td>
<td>HI</td>
</tr>
<tr>
<td>15 × 6s.</td>
<td>HI</td>
</tr>
<tr>
<td>24 × 5s. Can you get there without adding?</td>
<td>HI</td>
</tr>
<tr>
<td>Find T7. Last row you see something...</td>
<td>HI</td>
</tr>
<tr>
<td>Multiplication with bare numbers (F + T)</td>
<td>Blair’s responses</td>
</tr>
<tr>
<td>Acrostic writing in parallel exp notation. We’ve got more...we’ve got 16 of these (5-16s).</td>
<td>HI</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Parallel exp: 15 × 5s, 14 × 5s.</td>
<td>HI</td>
</tr>
<tr>
<td>Screened notation: 10 × 5s.</td>
<td>HI</td>
</tr>
<tr>
<td>&quot;Just wait, let me do it this way&quot; put into parallel expression. Still confused about F. “Once times 10...?”</td>
<td>HI</td>
</tr>
<tr>
<td>Find N via 6 × 5s = 30 + 50 = 80 + 10.</td>
<td>HI</td>
</tr>
<tr>
<td>&quot;Why does your own strategy?&quot;</td>
<td>HI</td>
</tr>
<tr>
<td>&quot;Oh! I forgot about my own strategy!&quot;</td>
<td>HI</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Layer A Analysis: Observations</td>
<td>Task type and tasks</td>
</tr>
<tr>
<td>-------------------------------</td>
<td>---------------------</td>
</tr>
<tr>
<td>14.3 Varied orientations, 5-6 tiles → bare numbers</td>
<td>Visible: 6 tiles, find N &amp; T; Screened: 5 tiles, find T &amp; T</td>
</tr>
<tr>
<td></td>
<td>Bare: 5 tiles, find P</td>
</tr>
<tr>
<td>15.3 Varied orientations with bare numbers</td>
<td>Screened: 5 tiles, find N &amp; N; Bare: 6 tiles, find N, P</td>
</tr>
<tr>
<td></td>
<td>Bare: 5 tiles, find P, Q</td>
</tr>
<tr>
<td>15.4 Varied orientations with bare numbers</td>
<td>Posted and recalled in number sentences: 10x^2, 11x^2</td>
</tr>
<tr>
<td></td>
<td>Vf: How would we write that normally?</td>
</tr>
<tr>
<td></td>
<td>bare:</td>
</tr>
<tr>
<td>15.5 Discussion of strategy</td>
<td>Referring to solving 40x^2 by doubling 4:</td>
</tr>
<tr>
<td></td>
<td>Why does that work?</td>
</tr>
<tr>
<td></td>
<td>Explained the inverse of EF - X, but not the requested strategy 2x - EF.</td>
</tr>
<tr>
<td>15.6 Multiplication with bare numbers</td>
<td>5x^2 passed an number sentence.</td>
</tr>
<tr>
<td></td>
<td>Using a closer multiple.</td>
</tr>
<tr>
<td></td>
<td>Notated strategies with arrow sentences.</td>
</tr>
<tr>
<td>16.3 Multiplication with screened 5-tiles</td>
<td>Student to write in number sentences.</td>
</tr>
<tr>
<td></td>
<td>Bare: 6 tiles, find N</td>
</tr>
<tr>
<td>15.4 Varied orientations with a fixed table</td>
<td>Including High 5s, 4s, and 6s</td>
</tr>
<tr>
<td></td>
<td>Recorded in number sentences.</td>
</tr>
<tr>
<td>16.4 Varied orientations with a fixed table</td>
<td>Including High 5s, 4s, and 6s</td>
</tr>
<tr>
<td></td>
<td>Recorded in number sentences.</td>
</tr>
<tr>
<td>17.1 Varied orientations with a fixed table</td>
<td>Including High 5s, 4s and 6s</td>
</tr>
<tr>
<td></td>
<td>Recorded in number sentences.</td>
</tr>
<tr>
<td></td>
<td>15 tiles, including: 6x = 30</td>
</tr>
<tr>
<td>17.2 Varied orientations with various 5-tiles</td>
<td>Two tasks, including: 45 with 5x6 tiles, how many tiles?</td>
</tr>
<tr>
<td></td>
<td>v ‘It isn’t 45... by 3 equals 9.’</td>
</tr>
</tbody>
</table>

Figure 7.1 continued
7.2 High 5s Sequence Layers A and B: Account of Each Segment
In the earlier analysis of the Lesson 10 Sequence, I gave a Layer A and B analysis of each task. For the present analysis of the whole High 5s Sequence, I am less detailed. I give an account of what happened in each segment, rather than each task, summarising the tasks and responses, and expanding on passages of instruction that are of particular interest to the later layers of analysis. This account includes briefly what was done and said—Layer A—and some local interpretation of that activity—Layer B—regarding adjustments in the dimensions, and the student’s reasoning. By putting sufficient detail here, I can be more streamlined in my references to the data in the later analysis sections.

7.2.1 High 5s Sequence: Lesson 8
In Segment 8.8 I showed how 5-tiles could be placed in pairs, which Blair recognised as making 10s. In this paired 5-tile setting, he recognised that six 5s makes 30, and eight 5s makes 40. “So, if you ask me 6 times 5, I would say 30 straight away, because now you’ve just shown me this”. I recorded these early findings as a table of number sentences (see Figure 7.2a) showing the value of $F$, $T$, and $N$ in the form:

\[ 2 \times 5 = 1 \times 10 = 10 \]

I then posed three bare number tasks—$16 \times 5$, $22 \times 5$, $26 \times 5$—giving an even $F$ and asking to find $T$ and $N$, and continued recording the results (see Figure 7.2b). For each task, Blair sought to find the number of 10s by counting up from an earlier line in the table. For example, for 16 5s, he began from 10 5s being 5 10s, then for every two 5s he incremented another 10, so 12 5s was 6 10s, 14 5s was 7 10s, 16 5s was 8 10s, which made 80. This strategy involved a coordination of counting three different units: the 5s, the 10s, and the 1s. For the higher multiples, he had difficulty managing this coordination. Blair also noticed the sequences going down the $F$ and $T$ columns in the table. For example, he pointed to the space below $5 \times 10$ saying “That would be $6 \times 10$”.

When asked how many 5s would be needed for 6 times 10, he answered 12, explaining by indicating the sequence of $T$ values: “It’s a pattern: 2, 4, 6, 8, 10” (see Figure 7.2b).
Figure 7.2  Segment 8.8. (a) First five number sentences; and (b) Final annotated table of number sentences.

Thus, Blair showed awareness of the additive sequence within the recorded $F$ and $T$ values, but not of a proportional relation between $F$ and $T$. I challenged Blair to find $T$ without counting up, and later to find a relation between $F$ and $T$ in each row of the table, but he could not see a relation. At one point in the discussion I asked directly, pointing to the $F$ and $T$ values in the table:

T: What’s the relationship between 12, 6; 14, 7; 16, 8?
Blair: I don’t know.
Later he did recognise a relation between these numbers as “They’re all two times”, which is in inverse orientation $T \cdot F$. I then pressed for the relation in the orientation of the tasks, $F \cdot T$, but he could not make this multiplicative inversion:

T: And going back the other way?

At the end of the lesson segment, I posed a final task of 46 times 5, and through discussion recorded a doubling and halving relation between $F$ and $T$ in this case.

In summary, in this first lesson segment devoted to multiples of 5, Blair had an initial insight about pairing 5s to make 10s, in the context of the visible paired 5-tile setting, and he also recognised additive sequences in the notation. However, with tasks advancing RNG to a higher range, SET to bare numbers, and NTN to formal number sentences, and with persistent STR challenges, he could not recognise the proportional relationship from $F$ to $T$.

### 7.2.2 High 5s Sequence: Lesson 10

Lesson 10 was an extended lesson, with a long passage of 33 minutes addressing high 5s, which I have divided into three long segments: 10.5, 10.6, and 10.7. These three segments of Lesson 10 have of course already been described and analysed in detail in the previous two chapters. I include a brief account of them again here, to keep this account of the sequence complete and coherent, and to put the key moments from Lesson 10 in the context of the longer sequence.

Segment 10.5 involved the task type incrementing and decrementing with 5-tiles. Blair was to keep track of the total $N$, and the number of 5s, $F$. After an introductory task in the low 5s range with visible tiles, five tasks were posed in the high 5s with tiles screened, and unscreening was used to check answers. Blair found the tasks challenging, but he succeeded, except for an interesting error at 10.5.4, eight 5s and another 5, which he answered as 50. At the end of the segment, Blair had an insight about the relation $2n \rightarrow F$:

Blair: If you said there are 30, in 5-dotted cards, how many cards were there? ... Six cards. You take the zero off the 30, then you just double the three.
This was his first recognition of a multiplicative $N$-$F$-$T$ relation, and I stopped to record and clarify his proposal.

In Segment 10.6, I posed five tasks of division with screened 5-tiles, using even multiples and the orientation $N \times F$ to trial his new strategy. He recalled and used his strategy successfully, “See, it’s working!” He also, when challenged to visualise the screened tiles, showed some strong multiplicative structuring. However, when checking answers with the visible tiles, he lacked fluency counting 1s and 5s. In Task 10.6.5 with 90 dots, I posed a supplementary task to find $T$ without counting, and Blair got stuck. While he had made a breakthrough in finding $F$, he had lost a sense of how to find $T$. A seven-minute episode ensued, involving Blair’s halting efforts to solve the task, unscreening of the tiles, a resolution, and notation of the resolution with an arrow sentence.

In Segment 10.7, I continued with similar tasks, but varied the unknowns among $N$, $F$, and $T$. I continued to record the results with arrow sentences, and Blair paid increasing attention to the notation, identifying a doubling relation between $T$ and $F$ values, and using the relation to check his answers. I also continued to pose with screened 5-tiles for the first two tasks, but the tiles were abandoned by the third task. Blair succeeded on the tasks, showing increasing fluency with strategies using $2n \rightarrow F$ and $2T \rightarrow F$. For the fifth task, in inverse orientation, he apparently used the inverse relation $\frac{1}{2} F \rightarrow T$. With a return to visible tiles to check the answer, he coordinated fluent counts of 1s, 5s, and 10s, declaring “Yes, everything is correct”. In a final task, I challenged Blair to explain the doubling relations between $T$ and $F$ in the notation, Why is it? and he reached a further $2T \rightarrow F$ insight, which we recorded in his workbook:

Blair: You can fit two 5s into … one 10, two 5s … So it’s basically doubling that number.

Thus, by the end of the three Lesson 10 segments, Blair had overcome his initial difficulty in finding $T$ alongside $F$ and $N$, he was using multiplicative $N$-$F$-$T$ relations in different inversions, and he could explain the relations in terms of the proportions of the units of 5 and 10. He had also shifted from a focus on the 5-tile setting to a focus on the informal notation.
Lesson 11 was an extended lesson, with 12 minutes addressing high 5s, which I have divided into two segments: 11.3 and 11.4.

Segment 11.3 used task type *multiplication with 5-tiles*, that is, given \( F \) find \( N \), in the range of low and high 5s. There was an introductory task in visible 5-tiles, incrementing low 5s up to six 5s. The next task posed eight 5s with screened tiles, which Blair incorrectly answered 80. After I showed six 5s with visible tiles again and reposed the eight 5s task, he correctly answered 40. Then followed a series of five even multiples of 5, with screened 5-tiles, and Blair was successful. Finally, there was a series of mixed even and odd multiples of 5. Blair was successful with all except the last, nine 5s.

Blair: [Pause] That’s like 10 off 50. Ah, 40. No, ah. What’d you say?
T: Nine 5s. I like that thinking, coming from 50.
Blair: [Pause.] It’s 40! 40.
T: [Unscreens tiles, showing four pairs of 5-tiles and one more 5-tile.]
Blair: [looking at tiles.] 45. 45, see I told you it was 45.

I then made some comments on the challenge of working with 5s and 10s, concluding *Sometimes we gotta think about 10s, sometimes think about 5s*. A likely explanation for Blair’s error is that he recognised nine 5s would be one unit less than ten 5s, but he chose the wrong unit: he thought one 10 less, rather than one 5 less. However, when the tiles were unscreened, Blair could recognise the unpaired 5-tile making 5 more than a decuple, and was able to distinguish again between 5s and 10s, in the context of the tiles. This episode revealed Blair connecting nine 5s to ten 5s, which was an important insight in the development of his multiplicative strategies. At the same time, the episode revealed the challenge for Blair of distinguishing the units of 5 and 10.

In Segment 11.4 I introduced an informal notation I called *parallel expressions notation* to record the results (see Figure 7.3). This was a notation I had planned for this lesson to improve on the arrow sentence notation of 10.7 by aligning the expression for \( T \times 10 \) below the parallel expression for \( F \times 5 \). I thought the alignment might support attention to the relations between the values in these expressions, which Blair had already recognised in the arrow sentence notation. I demonstrated the notation to record the first task, and supported Blair to complete the notation for subsequent tasks, which he managed with increasing independence.
I posed seven tasks, with varied orientations: six 5s \((F\rightarrow T&N)\); eight 5s \((F\rightarrow T&N)\); four 5s \((F\rightarrow T&N)\); five 10s \((T\rightarrow N&F)\); 80 \((N\rightarrow T&F)\); 60 \((N\rightarrow T&F)\); fourteen 5s \((F\rightarrow T&N)\). The range was even multiples of 5, extending from high 5s to beyond 50. I continued to pose with screened 5-tiles, but these were increasingly ignored by Blair, who after the fourth task stopped looking at the unscreened tiles to check his answers. Instead, he related increasingly to the new notation.

The second task, given eight 5s find \(T\) and \(N\), he solved by counting up 2s in eight:

   Blair: Every two 5s equals one 10. So: two, four, six, eight (in conjunction with raising four fingers): you’ve got four. Four 10s.

This was an impressive coordination of units, to count 2s in eight as a count of 10s in eight 5s. Nevertheless, this was less fluent than his solving of eight 5s earlier in Segment 11.3, and less sophisticated than the strategy \(\frac{1}{2}F\rightarrow T\).
which he had used with the same orientation at the end of Segment 10.7. For the task of five 10s find \( F \), with a shifted orientation, he initially answered incorrectly 25. In response, I briefly revealed one pair of 5-tiles from behind the screen, saying *Here’s one of my 10s there. I’ve got five of them. How many 5s?* He then corrected his answer.

The last three tasks he answered increasingly fluently, using \( N-F-T \) strategies. For the task 80 find \( T \) and \( F \), I helped him write 80 in the correct position for the parallel expressions notation, saying *Big number here*, and he promptly wrote \( T \) in position “So 10 times 8 right there”. He then found \( F \) saying “Duplicate eight … sixteen 5s”, and wrote \( 5 \times 16 \) in position, his explanation of duplicating eight suggesting he used the \( 2n \rightarrow F \) strategy developed in Lesson 10. Finally, he annotated his writing with an arrow from the \( 10 \times \) expression to the \( 5 \times \) expression, labelling the arrow “doubles” (see Figure 7.4), and drew similar arrows for the previous expressions. The arrows suggested he recognised \( 2T \rightarrow F \) as a general relation, in the context of the parallel expressions notation.

**Figure 7.4** Segment 11.4: Final three tasks, showing Blair’s annotation of an arrow labelled “doubles” between parallel expressions.
For the next task, 60 \((N \cdot T \& F)\) in the same orientation, he immediately wrote 60 into position, saying “that there, 60 right there”, then answered \(T\) of 6, then anticipating the remaining part of the notation he calculated \(F\) of 12, and promptly wrote the \(T\) and \(F\) values in place in the notation. After completing the notation, he remarked “I did that wrong” in reference to the alignment of the writing, and promptly adjusted the lower expression from “\(6 \times 10\)” to “\(10 \times 6\)”, which made the \(T\) of 6 correspond with the \(F\) of 12 above. Figure 7.4 above shows the corrected notation. I interpret his activity as showing awareness of relations between the parts of the notation, and organising his solution of the task by working through the parts of the notation.

The final task fourteen 5s \((F \cdot T \& N)\) was in inverse orientation, matching the orientation of the initial tasks in the segment; but this time he did solve it using the inverse multiplicative strategy \(\frac{1}{2}F \rightarrow T\):

Blair: If I halve fourteen, I’ve got seven, so then I’ve got 70 [writing 70 in place]. So that means, 7 times 10 [Writing 10 \(\times\) 7 in place].

Again, his reasoning appeared closely connected with working through the parts of the parallel expressions notation. I argue that as he expected the \(F\) in the notation to be double the \(T\), so he could reason the \(T\) to be half of \(F\) as an inverse relation.

Thus, Segment 11.4 was somewhat like Segment 10.7, with a task type of varied ORN, a SET of screened 5-tiles increasingly ignored, and an informal NTN increasingly involved in organising his reasoning. And, as was the case in Segment 10.7, Blair had some initial challenges, but finished by consolidating his awareness of the multiplicative \(N \cdot F \rightarrow T\) relations in two different inversions.

### 7.2.4 High 5s Sequence: Lesson 12

Lesson 12 was an extended lesson, and included 30 minutes addressing high 5s, which I have divided into three segments: 12.4, 12.5, and 12.6. In these segments, I posed tasks in bare numbers, without 5-tiles.

In Segment 12.4, I posed five tasks in the orientation \(N \rightarrow F\). The given \(N\) values were 70, 40, 60, 35, 95. The first three tasks, with even multiples, he
solved using $2n \rightarrow F$, and recorded the results with the parallel expressions notation introduced in the previous lesson. The last two tasks, with odd multiples, he solved using neighbouring multiple strategies: for 35, he found $F$ via the number of 5s in 40; for 95 he found $F$ via the number of 5s in 100. This was a pleasing new fluency with multiplicative strategies for both even and odd multiples.

In Segment 12.5, I switched to the inverse orientation $F \times N$. I posed six tasks: $6 \times 5$, $9 \times 5$, $12 \times 5$, $15 \times 5$, $18 \times 5$, $24 \times 5$. The tasks were just posed and recorded in number sentences, not in parallel expressions notation. For the first task $6 \times 5$ he gave stalled and confused answers: “What is 6 times 5? 12. No. Umm … I was looking at that, that didn’t help me … er, 6 times 5 ah, 6 times 5. 2 times 15. 20”. I picked up the set of 5-tiles to re-pose the task with the tiles. Before I could lay the tiles out, he said “No no no no no no!” apparently rejecting the assistance. He promptly skip-counted six 5s, keeping track on his fingers, and answered correctly. The mere suggestion of the 5-tile setting appeared to help Blair make sense of the task as a multiple of 5s.

Blair was successful on the remaining tasks, however he did not use the multiplicative strategy $\frac{1}{2}F \rightarrow T$. At first he used skip-counting by 5. For $9 \times 5$, after he used skip-counting by 5s, I challenged him to find a strategy without counting from 1. He proposed “Six 5s, and then we just have to add on another 15”. This is a partial products strategy, which was an improvement on skip-counting and on counting-on by 5s. Still, I challenged him to find a closer multiple than six 5s, and he proposed using ten 5s, which I deemed a further improvement. He continued to seek for closer multiples in subsequent tasks, solving $12 \times 5$ via $10 \times 5 + 2 \times 5$, and $15 \times 5$ via $12 \times 5 + 2 \times 5 + 5$.

In the fifth task, I began to challenge him to find a different strategy. *I’m looking to see whether you can get there without adding from somewhere else. This is my little, my little agenda here.* This challenge took different forms, including recalling 5-tiles, explicitly asking him to find $T$, and suggesting he had found different strategies before:

---

T: Last week, you saw something, and I want to help you use what you saw …

Blair: But to use the thing that I saw, I need to have like the answers first.
T: Mm.
Blair: Like 60 [pointing to an earlier task in workbook], you double that for 12 times five.
T: Exactly, well I’m trying to now see if you can—
Blair: Switch.
T: —do it back the other way [gesturing turning hands past each other].

Apparently, Blair could recall his $2n \rightarrow F$ strategy, and he recognised that the current $F \rightarrow N$ tasks were in the wrong orientation for that strategy.

The extended search for a better strategy reached a climax and resolution with a task $16 \times 5 F \rightarrow T$, which I have since coded as the first task of Segment 12.6. For this task, Blair suggested recalling parallel expressions notation, which I helped him set up:

Blair: Just wait, let me do it this way. So … [begins to write in parallel expressions notation] how’d you do it?…
T: Oh, that’s a very good idea. In that case, that doesn’t belong there. [Erases some numerals, re-writes the notation.] This belongs here. The question is that, and that [underlining the gaps for T and N].
Blair: OK. That’s better, that’s a bit easier now.

However, he still struggled to find $T$, trying to find how many 10s in 16, rather than in 16 5s.

Blair: Now, so 16 times 5. One times 10. Cos that’s 30, there … one times 10, that’s 10 ahh …
T: Why is it one times 10?
Blair: Cos you’ve got the one there [pointing to the digit 1 in the written 16], you can only fit one 10 in that…

I brought out 5-tiles to support his reasoning:

T: All right. It’s not that we’ve got 16 dots. We’ve got a lot more than 16 dots. We’ve got 16 of these [holding up a 5-tile]. 16 5s.
Blair: Ahh.

After this intervention, Blair was able to reason correctly about 16 5s:

Blair: Wait, there’d be 50, then that would be 5 tens …
T: There would be from the fifty, but there’s also the 6 [pointing to the 6 of 16].
Blair: There’s also the 6. Just wait, I’ve got to do some working … I’ll do it in my head. 5 times 6 – 5, 10, 15, 20, 25, 30, so 30. So 30, then 30 plus 50 equals 80, so then there’s eight 10s there.
That is, he went back to skip-counting by 5s to calculate $6 \cdot 5s$, then he found $N$ as

$$6 \cdot 5s + 10 \cdot 5s \rightarrow 30 + 50 \rightarrow 80,$$

and finally from $N$ being 80 he determined that $T$ was 8. While he reached a correct answer, he did not yet use proportional reasoning from $F$ to $T$. However, he then wrote his answers into the parallel expressions notation, and looking at the completed notation (see Figure 7.5a), he had an insight. Figure 7.5b shows a video still of his moment of realisation.

Blair: So then there’s eight 10s there [writing $8 \times 10$ in place]. And then … I’ll just put 80 here [writing 80 in place] … Um … I get it now, I get it. [Reading notation again] 16 5s … Oh, what?! Oh, why didn’t I figure that? Oh! Ah! I could have just taken that off, and halved that down. Oh! I forgot about my own strategy!

Figure 7.5 Segment 12.6 task 16 × 5: (a) Completed parallel expressions notation; and (b) Blair’s moment of realisation after reading the notation “Oh! I forgot about my own strategy!”
I contend that, when he saw the set of three $N \rightarrow F \rightarrow T$ values in place in the notation, he could recall his reasoning of the $2T \rightarrow F$ relation—taking off the 0 of 80 and doubling 8 to make 16—and in turn he could recall the inverse relation $\frac{1}{2}F \rightarrow T$—halving 16 to make 8.

Following this resolution of the $16 \times 5$ task, I posed three more tasks in the same orientation, two written in parallel expressions notation and the last as a standard number sentence. Blair solved each of them fluently using $\frac{1}{2}F \rightarrow T$.

In summary, Segments 12.4, 12.5 and 12.6 had Blair using a $2n \rightarrow F$ strategy, and developing his use of neighbouring multiples and partial products strategies, in bare number contexts. Through 12.5 and 12.6 he had a protracted challenge to recall the inverse $\frac{1}{2}F \rightarrow T$ reasoning, a challenge that was resolved following returns to the 5-tile SET and the parallel expressions NTN. The later Layer C and D analysis will draw significantly on this protracted episode, tracing the instructional adjustments around his confusion and eventual success.

### 7.2.5 High 5s Sequence: Lesson 14

Lesson 14 included one brief segment on high 5s, Segment 14.3. I posed four tasks: six 5s ($F \rightarrow T&N$); eight 5s ($F \rightarrow T&N$); 60 ($N \rightarrow T&F$); and 80 ($N \rightarrow F$). Thus, the tasks were with varied ORN, with the RNG of even multiples in both high 5s and beyond 50.

The first task was with visible paired 5-tiles. Blair solved the task immediately, saying “Five 5s equals 25, six 5s equals 30” so apparently not using the reasoning of three 10s. For the remaining tasks, I asked him to record his responses in parallel expressions notation. He stopped looking at the tiles, and I abandoned them by the last task. He solved the tasks fluently by writing in the notation and using the multiplicative strategies, both $\frac{1}{2}F \rightarrow T$ and $2T \rightarrow F$. For example:

```
T: What about if there are eight 5s, 8 times 5? [Begins to arrange four pairs of tiles.]
Blair: [Not looking at the tiles, writing the notation] Ah, four 10s.
T: Mm hm, and what’s the total?
Blair: [After an exchange about neatening his parentheses] 40.
```
After this brief successful segment, I concluded that he was consolidating his \(N-F-T\) strategies in the context of the parallel expressions notation, and that next lesson I would try to transition to a conventional notation.

### 7.2.6 High 5s Sequence: Lesson 15

Lesson 15 was the fourth and last extended lesson in the High 5s Sequence, with a 15-minute passage addressing high 5s, which I have divided into four segments: 15.3, 15.4, 15.5, and 15.6.

Segment 15.3 was similar to Segment 14.3. I posed four tasks in varying orientations, with even multiples: six 5s (find \(N&T\)); eight 5s (find \(N\)); 60 (find \(F\)); 90 (find \(F&T\)). I used screened 5-tiles for the first task, then abandoned the tiles as Blair took no notice of them. For the first task of six 5s, Blair answered 30 immediately. Then to find \(T\), he wanted to use the parallel expressions notation, saying “Just wait, let me write this down”. As with 14.3, he solved the tasks fluently by writing in the notation and using the \(N-F-T\) strategies, both \(\frac{1}{2}F\to T\) and \(2T\to F\). He also reflected on the strategies: “You see, for doing this type of thing, this strategy, you need an answer like 60, 40, or 30; or like four 10s or whatever”. I interpret his reflection as showing he now treated the \(N-F-T\) strategies, in different orientations, as a single system, and had awareness that these strategies were limited to the decuples, that is, the even multiples of 5. I also understand that for Blair at this point, the \(N-F-T\) strategies were embedded in the parallel expressions notation: he sought to write the notation in order to answer the tasks.

In Segment 15.4, I continued with bare number tasks, and made a transition to recording with conventional number sentences. I first posed \(10\times 5\) and \(14\times 5\) with written expressions. He solved both readily, and described solving the latter using a partial product strategy, which I notated in his workbook:

\[
10\times 5 + 4\times 5 \rightarrow 50 + 20 \rightarrow 70.
\]

Next, to pose tasks in division orientation, I referred to the parallel expressions notation for an earlier task from 15.3—given 60 find \(F\)—and asked Blair how he would write the task.
T: Now these ones here [indicating earlier 12×5=60 task], I said, I said there’s 60, you had to figure out how many 5s. How would we write that question normally?

Blair: If you said that you’ve got 60, um. Here [indicating new number sentences, see Figure 7.6a], I would say, you write 60 there [pointing beneath the 70 of previous number sentence “14×5 = 70.”]

T: [Writes 60 in position.] Mm hm.

Blair: So there … and write the equal there [pointing to the left of 60.]

T: [T writes =]

Blair: … and then

T: [Completes sentence as __×5=60] Like that?

Blair: Yeah.

Thus, Blair explained translating the parallel expressions notation into the number sentence notation by writing the task as a missing multiplier sentence. I notated a second task the same way: __×5=80. He solved both tasks fluently, apparently using 2n→F.

I then suggested I could notate the previous task using the division sign.

T: Now. I could also write it this way. I could write 80 divided by 5

[writes 80 ÷ 5 = (see Figure 7.6b)]

Blair: Whaat? Hmm?

T: How many 5s in 80, 80 divided by 5.

Blair: … OK … 80 divided by 5 equals 16.

T: [writes in 16.]

Blair was hesitant about the division notation. Nevertheless, when I posed two further tasks in this form, 30 ÷ 5 and 50 ÷ 5, he solved them successfully. Finally, I handed the pen to him to notate the task 40 divided by 5, which he wrote and solved correctly. Asked how he got the answer 8, he pointed to some earlier parallel expressions notation from a different task and replied “I did that, I did, like, I took 40, times it by 2—8”. He appeared to use “I did that” to mean doing the associated 2n→F strategy.

Figure 7.6 Segment 15.4. (a) Notating the task 60 ... how many 5s? as a missing multiplier sentence aligned under the previous number sentence 14×5=70; and (b) Notating the task __×5=80 using a division sign.
Thus the instruction made a graded progression to standard notation over seven tasks. For the division tasks, Blair continued to use the multiplicative $N$-$F$-$T$ strategies in the bare number standard notation. This was a major accomplishment, at least temporarily, of an instructional aim.

Following the success of Segment 15.4, in Segment 15.5 I paused from posing arithmetic tasks, and challenged Blair to explain why his $2n \rightarrow F$ strategy worked: *Why does that trick work?* Blair could not readily offer an explanation, and a protracted episode of discussion ensued, described below.

I offered 5-tiles to help. He managed to explain the $\frac{1}{2}F \rightarrow n$ relation: why, given eight 5s, we can halve eight to four to find $N$ of 40.

:     Here’s eight 5s. [Shows eight 5-tiles in pairs.] Do they make 40?  
Blair:  Yes.  
:     Why?  
Blair:  Cos two 5s, every two 5s equals a 10, and you’ve got eight cards, and with those eight cards, you divide them into two, into groups.  
:     Groups of two.  
Blair:  Four groups of two. Now when you put them into groups of two, you count up one group of cards making 10, then you would know that all the rest equal 10. Then you count up how many groups of two you have, then, so one, two, three, four [counting the four pairs of 5-tiles], you know that one group of two makes 10, you’ve got four, you put a zero on the end of four, you’ve got 40.

However, he could not explain the inverse, and had some confusion with the 5-tiles in the attempt.

:     Now this question was going the other way, this question gave you 40, said 40 [slapping a hand down to cover all the 40 dots] how many 5s? Why does it work to take that four, and double it and make eight …  
Blair:  You’ve got 40 dots. But you’ve got like eight cards. You just have to halve it [pulls all paired 5-tiles apart in a single gesture, making two groups of four unpaired tiles]. I don’t know how that works. But if you halve it, you’ve got four cards, 40 dots, on 5-dot cards. So then you divide them into two groups of four. Making that groups of eight … no no no. Putting them into groups of 40. I don’t know.

I was concerned that the paired 5-tiles might be a confusing setting for this explanation, and decided to offer my own explanation for $2n \rightarrow F$ for the task $40 \div 5$ without using the tiles. Some way into my explanation, I referred to the parallel expressions notation.
T: If we have four 10s, how many 5s do we have?
Blair: … Eight 5s.
T: Eight 5s. So the number of 5s is double the number of 10s?
Blair: Yes.
T: That’s why you can double that [pointing to the 4 of 40, in $40 \div 5$,
Figure 7.7a]. Cos 40 means four 10s. Which is why we’ve been doing
this here [pointing to earlier parallel expressions notation of
$8 \cdot 5 \rightarrow 40$, see Figure 7.7b]. 40 means four 10s [pointing to $4 \cdot 10$],
and four 10s means eight 5s [pointing to $8 \cdot 5$].
Blair: Mm. So it goes from up there, to there [drawing an arrow from 40 to
$4 \cdot 10$] then from there to there [drawing an arrow from $4 \cdot 10$ to $8 \cdot 5$]. Is
that what you’re saying?
T: That is exactly what I’m saying.

After I recapitulated this explanation, he exclaimed “Oh, it’s like a cycle” and
he notated the explanation with a cycle of arrows from $8 \cdot 5$ to 40 to $4 \cdot 10$ and
back, indicating some understanding of the relations (see Figure 7.7c).

Figure 7.7 Segment 15.5. (a) The original task $40 \div 5 = 8$; (b) The parallel
expressions notation, with arrows added by Blair; and (c) The
cycle notation by Blair.

Finally, after this discussion of strategies, I returned to an arithmetic task,
which I have labelled Segment 15.6. I posed a bare number task with an odd
multiple: $5 \times 9$. He solved it first via:

$$5 \times 5 + 4 \times 5 \rightarrow 25 + 20 \rightarrow 45,$$

a partial products strategy. I asked him to find a closer multiple, and he
solved it via:

$$5 \times 10 - 5 \rightarrow 50 - 5 \rightarrow 45,$$

a neighbouring multiples strategy. I notated both strategies.

In summary, over the four Lesson 15 segments, Blair demonstrated fluency
with his $N$-$F$-$T$ strategies for even multiples in the context of the parallel
expressions notation, he transitioned successfully to expressing the tasks in
conventional notation, he engaged in explanations about the $N$-$F$-$T$ strategies,
and he solved a bare number task with an odd multiple, using effective multiplicative strategies. The Layer C analysis will track the significance of each of these developments in Blair’s learning, and the Layer D analysis will track the instructional adjustments in each of the dimensions involved in supporting these developments.

7.2.7 High 5s Sequence: Lesson 16

Lesson 16 had two segments attending to high 5s: 16.3 and 16.4. Segment 16.3 posed three multiplication tasks in high 5s with screened 5-tiles: eight 5s, six 5s, and seven 5s, and Blair was successful on each. For the first task of eight 5s, Blair solved it as two 5s less than ten 5s, and I notated his strategy. Then I challenged him to find a \(N\rightarrow F\rightarrow T\) relationship:

- **T:** Now I’m wanting to see if you can think of another way. There’s eight 5s: how many 10s are there?

- **Blair:** There’s eight 10s.

- **T:** There’s eight 5s [pointing to numeral 5 in written expression \(8 \cdot 5\)], how many 10s are there? I’ve got eight of these guys [holding up one 5-tile above the screen].

- **Blair:** Ohhh. There’s four 10s.

- **T:** Four 10s. So how many dots altogether?

- **Blair:** 40.

I interpret that when the instruction recalled the visible 5-tile SET, Blair recalled how to distinguish 5s from 10s, and recalled the \(\frac{1}{2}F\rightarrow T\) relation.

For the second task, six 5s, he answered 30, then wrote in a form of parallel expressions notation relating \(6 \cdot 5\) to \(3 \cdot 10\) (see Figure 7.8a), which appeared as a spontaneous return to the notation of the previous lessons to demonstrate his reasoning. For the third task of seven 5s, he answered 35, then asked his own extension question “And how many tens?” and dictated his own written answer: “Three dot 10s plus one, no plus five” (see Figure 7.8b). Note in these last two tasks Blair’s initiative to relate the tasks to the number of tens \(T\), and the continued challenge for Blair of expressing the different units, shifting from calling the extra 5 “plus one” to “plus five”.
Segment 16.4 involved the task type *various orientations with a fixed table of multiplications*, which had been established in the lessons of the previous week for high 3s and 4s products. A table of candidate multiplications was given, including the high 3s and high 4s with the high 5s, expressed as a subset of the multiplications chart:

<table>
<thead>
<tr>
<th>3×6</th>
<th>3×7</th>
<th>3×8</th>
<th>3×9</th>
</tr>
</thead>
<tbody>
<tr>
<td>4×6</td>
<td>4×7</td>
<td>4×8</td>
<td>4×9</td>
</tr>
<tr>
<td>5×6</td>
<td>5×7</td>
<td>5×8</td>
<td>5×9</td>
</tr>
</tbody>
</table>

This was the first time the instruction addressed these products mixed together. There were three high 5s tasks in the segment: 5 times 7, 30 divided by 5, and 45 divided by 9. Blair successfully answered each task. For the task 45 divided by 9 he appeared to consult the table of candidates, and confirmed his answer of 5 for himself through checking the multiplication “Yeah. Nine 5s equals 45”.

### 7.2.8 High 5s Sequence: Lesson 17

Lesson 17 had two segments attending to high 5s: 17.1 and 17.2. Segment 17.1 continued the task type of varied orientations with a fixed table. The table of candidates was again the high 3s, 4s, and 5s multiplications. The first task was in multiplication orientation: *What is 3 times 7?* The next three tasks were posed as missing factors, for example *What times 7 equals 28?* The remaining ten tasks were posed as divisions, for example *What is 27 divided by 3?* Blair added a step to these division tasks, seeking to anticipate the divisor, as well as determine the quotient: for example, when posed the incomplete question “32 divided by …” he responded “… by 4 equals 8”.

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*Figure 7.8* Segment 16.4. (a) Blair’s notation for six 5s, with a form of parallel expressions to relate to $3 \cdot 10$; and (b) Blair’s notation for seven 5s, showing $3 \cdot 10 + 5$. 
Three tasks involved high 5s. 6 times what equals 30? was answered as a known fact. 40 divided by__ he solved first as “you could do 4 times 10”, but then after looking at the table of candidates for 6 seconds, he proposed “5 times 8” which he then wrote as “40 ÷ 5 = 8”. It is not clear whether 5 × 8 = 40 was a known fact for Blair at this point, but it was readily recognised among the table of candidates.

The task 35 divided by__ was not answered fluently.

Blair: 35 divided by … 6? No, divided by 5 equals 6. [Writes 35 ÷ 5 = 6] Is that it?
T: 35 divided by 5 does equal something, but it doesn’t equal 6.
Blair: Oh. [Erasing 6.] But six 5s equals 35?
T: Six 5s is … [beginning to lay out 5-tiles, which are ignored by Blair.]
Blair: No it does not. It’s eight. 35 divided by 5 equals 8. [writing in 8.]
T: Here’s six 5s [shows three pairs of 5-tiles.]
Blair: [looking at the 5-tiles.] That’s 30. [Looking back to his written number sentence.] 35…no it’s 7!
T: [placing seventh 5-tile.] Aha!
Blair: [Erasing the 8 and writing in 7.] Oh OK. I got fooled.

In this passage, Blair was at first uncertain and incorrect; once the teacher introduced the paired 5-tile setting for six 5s, Blair was able to clarify both that six 5s was 30, and that 35 would be seven 5s.

Also of interest, for the task 27 divided by__ he responded “Just wait. Let me figure this out. It can’t be 5 … by 3 equals 9”. Thus he recognised that 27 was not a 5s product, another aspect of knowledge of high 5s.

Segment 17.2 used a task type Varied orientations with varied n-tiles, which had been established in the lessons of the previous two weeks for high 3s and 4s products. One task involved high 5s. The task was posed as Some 5s [getting 5-tiles] and I’m going to put some down to make 45. How many do I need to put down? Blair responded “45 divided by 5 equals … 9” and wrote this as a number sentence. Asked how he got the answer, he replied “I minused it off 10 times 5”. Note three aspects of this response. Firstly, his fluency in converting the verbal task to “divided by” language and formal written division. In the context of these familiar tasks at least, this conversion now appeared transparent for Blair. Secondly, his use of multiplication to calculate division. This calculation was neither a repeated subtraction, nor a
known fact. Thirdly, the multiplication strategy was an efficient and sophisticated one, using a neighbouring multiple, and compensating the appropriate unit of one 5.

Thus, by Lesson 17, it appeared that the relations $6 \times 5 = 30$ and $8 \times 5 = 40$ presented as bare number tasks, in division as well as multiplication orientations, were fluent or known facts. The relation $7 \times 5 = 35$ in division orientation was not a known fact as a bare number task, but in a setting of 5-tiles, both $7 \times 5 = 35$ and $9 \times 5 = 45$ could be calculated using an efficient strategy relating to a neighbouring even multiple.

That completes the Layer A and B account of the High 5s Sequence.

7.3 High 5s Sequence Layer C: Progression in Blair's Reasoning

The previous section has given a segment-by-segment Layer A and B account of the High 5s Sequence of segments. The current section gives the Layer C analysis, tracing through the segments significant threads in the development of Blair’s reasoning about high 5s. My purpose is to show what the different threads were, and what the significant challenges and breakthroughs were in each thread. I also want to suggest the texture of Blair’s development through those threads. I will then be able to draw on this rich Layer C picture of the learning progressions in the subsequent Layer D, E, and F analyses of the instructional progressions. This Layer C analysis will enable a recognition of the significance of Blair’s reasoning in each episode: why I am interested in moments when, for example, he subtracted 10 instead of 5, or did not use a multiplicative strategy, or wrote some notation before using a strategy. In turn, I will be able to recognise the significance of instructional adjustments in those moments: for example, when I brought out the 5-tiles again, or challenged Blair to find another strategy, or refined his use of some notation.

The appreciation of his learning forms a basis for an appreciation of how the instruction responded to and supported the learning.

I will describe six threads in the development of Blair’s reasoning:

- distinguishing the three units 1s, 5s, and 10s;
- structuring $N-F-T$ relations and related strategies for multiples of 5;
- developing strategies for odd multiples of 5;
• developing strategies for division with 5s multiples;
• developing independence from the setting;
• relating to notation.

I will trace the progression in each of these threads through episodes in the High 5s Sequence.

### 7.3.1 Distinguishing the three units (1s, 5s, and 10s)

The high 5s tasks involved three multiplicative units: 1s numbered to \(N\), 5s numbered to \(F\), and 10s numbered to \(T\). Blair’s central challenge was to coordinate these units, and know the multiplicative \(N–F–T\) relations between them. A more basic challenge was merely to distinguish these units: to not count a 10 instead of a 5; to not treat the value of \(F\) as the value of the whole \(N\), and so on. I trace here his progress in distinguishing the units over the High 5s Sequence.

Segment 8.8 posed tasks \(F\rightarrow T\&N\). Blair’s successful counting strategies showed he could keep distinct the three different unit counts in the context of the table of number sentences being used.

However, in Lesson 10, he had some difficulties distinguishing the three units. In Task 10.5.4—eight 5s and another 5—he calculated the total incorrectly as 50. It is possible that he added to 40 a unit of 10, rather than a unit of 5. In Task 10.6.5—given 90 dots find \(F\) and \(T\)—Blair was unable to find \(T\) without counting, an episode highlighted in the analysis of Lesson 10. During the extended exchanges for this task, both with setting screened, and with setting unscreened, he had difficulty keeping clear the three units and the unit counts. For example, he referred to 90 cards, rather than 90 dots; he couldn’t recall his answer for \(F\); when trying to count 5s he counted each tile as five instead of one; and he miscounted the 10s in the tiles. It could be that, as he was trying more sophisticated multiplicative strategies relating \(F\) and \(N\) in this lesson, there was more potential for confusion between 5s and 10s. After the difficulties of 10.6.5, Blair managed the different units successfully in Segment 10.7, and by the final task he could calculate each of \(N\), \(F\), and \(T\), and demonstrate their distinct counts in the tile setting, without confusion (“Yes, everything is correct”).
In Segment 11.3, Blair had one task which revealed the continued potential for confusion of units. He calculated nine 5s as “10 off 50”, apparently thinking of one unit less than 50, but confusing the unit as 10 rather than 5, similar to the confusion proposed above for Task 10.5.4—eight 5s and another 5. In Segment 11.4, in the context of the new parallel expressions notation, Blair attained fluency in coordinating the three units and unit counts.

Segment 12.4 posed similar tasks to the troublesome 10.6.5 *How many 10s?* which Blair now handled without difficulty. However, in 12.5 and 12.6, I posed tasks in a different ORN $F \rightarrow T$, and confusion about 5s and 10s arose again. For the task 24 5s find $T$, he suggested incorrectly “There’d be two 10s”. Likewise in the next task, sixteen 5s find $T$, he said:

Blair: Now, so 16 times 5. One times 10. Cos that’s 30, there … the 16 … one times 10, that’s 10, ahh …

T: Why is it one times 10?

Blair: Cos you’ve got the one there [pointing to the digit 1 in the written 16], you can only fit one 10 in that.

He was apparently thinking of the number of 10s in 16, rather than the number of 10s in 16 5s. Through the subsequent instruction, involving a return to 5-tiles and to parallel expressions notation, he resolved this confusion, and was able to answer the remaining tasks of 12.6 successfully.

In later segments, Blair became fluent in his distinguishing of the units and unit counts. Segments 14.3 and 15.3 posed tasks with varying ORN similar to Segments 10.7, 11.4, and 12.5, and Blair continued the fluency that had appeared so new in 10.7. Segment 15.4 shifted the same kind of arithmetic to formal notation, and still the fluency in negotiating the different units persisted.

Hints of the challenge of distinguishing units were recalled in Segment 16.3, which posed multiplication tasks with screened 5-tiles. In one task I posed *There’s eight 5s, how many 10s are there?* to which Blair replied “There’s eight 10s”. After I sought to clarify what a 5 was, *I’ve got eight of these guys*, he corrected himself “Oh! There’s four 10s”. For the later task of seven 5s, Blair solved the task fluently as 35, but when explaining his thinking said “Three 10s plus one, no plus five”. Presumably the “plus one” here arose.
from thinking plus one unit of 5. His ready self-correction confirmed his consolidated awareness and fluency in negotiating this potential confusion.

### 7.3.2 Structuring $N$-$F$-$T$ relations and strategies

Alongside this development in distinguishing between the three different units, Blair developed his awareness of the multiplicative relationships between the units, and multiplicative strategies for deriving $F$ given $N$, and $N$ given $F$. The development of multiplicative relations and strategies was one of the main instructional aims of the high 5s topic, and of the intervention overall, so I have observed the drama of the development in some detail.

In assessments and lessons prior to Lesson 8, Blair had reported finding high 5s products using a strategy of counting-on by 5s from a lower multiple: for example, finding eight 5s as

\[ 5 \times 5 \rightarrow 25 \rightarrow 30 \rightarrow 35 \rightarrow 40. \]

In Segment 8.8, the first segment in the High 5s Sequence, Blair’s counting strategies showed an awareness of counting two 5s as one 10, an important foundational relationship. However, he needed to count each 10 successively. His strategies appear as a form of counting-on by 5s, or by pairs of 5s. For example, to find sixteen 5s, he reasoned:

10 5s is 5 10s
20 5s is 10 10s
30 5s is 15 10s
40 5s is 20 10s
Thus, he did not recognise a proportional relationship directly from $F$ to $T$, as $\frac{1}{2}F \rightarrow T$. When pressed to find a relationship between the numbers recorded in the table, he eventually recognised a doubling relation between some written $T$ and $F$ values, but when asked for a relation back the other way he could not invert multiplicatively: “Minus. Minus 8, minus 7, minus 6”. There was no indication that he recognised a multiplicative relation between the abstract units of 5s and 10s.

The three segments of Lesson 10, as analysed in the Lesson 10 chapters, featured significant moments of both insight and confusion in the $N$-$F$-$T$ relations. At the end of Segment 10.5, Blair had an initial insight into the relation $2n \rightarrow F$: “You take the zero off the 30, then you just double the three”. This was the first time Blair noticed a proportional relationship directly between $N$ and $F$. He clearly expected the relationship to generalise to other cases, but it was still a limited insight: it appeared closely tied to numerals
and the 5-tiles setting, without awareness of inversion, without awareness of the relation to $T$, and without an explanation. While he proceeded to have success using this new strategy for four tasks of 10.6, when Task 10.6.5 introduced finding $T$, he could find no way to relate $N$ or $F$ to $T$.

During the final segment, 10.7, he improved from the limitations of the original insight. His strategies became less dependent on the tile setting. He developed a relation of $T$ to $F$, solving tasks using $2T \rightarrow F$. He solved one task using the inverse relation $\frac{1}{2}F \rightarrow T$. Finally, he found an explanation of these relations in terms of iterable composite units of units: “You can fit two 5s into one 10 … so it’s basically doubling that number”. Thus, within one long lesson, he had moved from having no multiplicative strategy for high 5s to having an invertible strategy which he could explain in terms of the proportions of the units.

The two segments of Lesson 11 appeared to recapitulate the progress of Lesson 10, rediscovering the new $N\,F\,T$ based strategies. Initially, Blair used counting-on type strategies, including counting two 5s as one 10, without recalling the new strategies. One striking example was when he calculated eight 5s thus: “Every two 5s equals one 10. So: two, four, six, eight (in conjunction with raising four fingers): you’ve got four. Four 10s”. This was an impressive coordination of composite units of units, but he needed to count the twos in eight one at a time, rather than using the more abstract simplicity of halving eight. However, in later tasks he found again the $N\,F\,T$ strategies. For example, for the task Eight 10s find $F$ he used $2T \rightarrow F$: “Duplicate eight”. For the task Fourteen 5s find $T$, he used $\frac{1}{2}F \rightarrow T$: “If I halve fourteen, I’ve got seven … 7 times 10”. He also annotated the tasks recorded in his notebook to indicate the doubling relationship from $T$ values to $F$ values, as shown in Figure 7.4.

The three segments of Lesson 12 recapitulated the discovery of $N\,F\,T$ strategies again, this time in a bare number context. In Segment 12.4, with tasks in the division orientation, he used his new $2n \rightarrow F$ strategy from the outset. However, in Segment 12.5 with six tasks in the multiplication orientation, he did not use the inverse strategy $\frac{1}{2}F \rightarrow T$, mostly returning to his strategy of counting-on by 5s. The struggle to recall the $N\,F\,T$ strategy was
protracted. With appeals from the teacher to find another strategy, not using counting or adding but using \( T \), he had difficulties. For the task 24 5s how many 10s, he answered “two”, then he answered “Two 5s, that’s 40 … 48”—apparently recognising a doubling relation, but unable to handle the inversion in that relation. When pressed to use the insight he’d found in previous lessons, he said “But to use the thing that I saw, I need to have like the answers first. Like 60, you double that for 12 \( \times 5 \)”, showing an awareness of his \( 2n \rightarrow F \) strategy, and an awareness that I was posing tasks in the wrong orientation. Affirmations that he could use that strategy in the reverse direction were not immediately sufficient for him to recall \( \frac{1}{2}F \rightarrow T \). In 12.6, task 16 \( \times 5 \), he returned to recording his answers in the parallel expressions notation, and then realised a \( \frac{1}{2}F \rightarrow T \) strategy. “Oh, why didn’t I figure that, oh! I could have just taken that off, and halved that down. Ohh! I forgot about my own strategy”. His exclamations suggested he linked this strategy to his original \( 2n \rightarrow F \) strategy. He then used \( \frac{1}{2}F \rightarrow T \) fluently for the final three bare number tasks.

Lessons 14 and 15 were the first lessons where he returned promptly to using the \( N-F-T \) strategies, and the parallel expressions notation appeared central to his success. With tasks presented in the notation, he solved tasks in any orientation fluently using \( 2n \rightarrow F \) and \( \frac{1}{2}F \rightarrow n \). Furthermore, in Lesson 15 he showed an increasing insight into the whole system of \( N-F-T \) strategies. He commented that “You see, for doing this type of thing, this strategy, you need an answer like 60, 40, or 30, or like four 10s or whatever”. The comment showed he now treated these strategies together as a single system. In the context of the parallel expressions notation at least, the orientation of a task had become transparent to him; they all involved the same set of relations. He also recognised that these strategies only worked with multiples of 10, showing a higher level of awareness about the strategies. When asked, he could not explain why his \( 2n \rightarrow F \) strategy worked. However, he demonstrated an understanding of the relationships between 5s and 10s involved in the strategy, and he could explain why the inverse strategy \( \frac{1}{2}F \rightarrow n \) worked.
In Segment 15.4, Blair continued to use the \(N-F-T\) strategies as I transitioned to presenting tasks in standard number sentence notation. He also solved one higher multiple, \(14 \times 5\), using a new partial products strategy:

\[
14 \times 5 \rightarrow 10 \times 5 + 4 \times 5 \rightarrow 50 + 20 \rightarrow 70.
\]

This showed a subtle but significant advance from counting-on by 5s or 10s, to a properly multiplicative handling of the units of units.

By Lessons 16 and 17, Blair was using a variety of good mental strategies on high 5s tasks. I posed high 5s tasks in settings of both screened n-tiles and formal number sentences, mixed amongst tasks of high 3s and 4s, in both multiplication and division orientations. He was not yet fluent on all tasks, but he generally used appropriate strategies indicating an understanding of multiplicative number relationships. His strategies included \(\frac{1}{2}F \rightarrow n\), partial products, using a neighbouring multiple, and using known multiplications to solve division. For example, in Segment 16.3 he solved seven 5s via \(3 \times 10 + 5\). The contrast with earlier strategies is striking. Here, he related to the nearest decuple 30, rather than to 25 as he used to; he coordinated units of 5 and 10 simultaneously; and he saw three 10s at once, rather than needing to count them one by one.

In summary, Blair began Lesson 8 with counting-on by 5s from 25 as his main strategy for high 5s tasks. By Lesson 17 he was using a range of effective multiplicative strategies. The major breakthroughs were in recognising the relationships \(2n \rightarrow F\) and \(\frac{1}{2}F \rightarrow T\). The partial products strategy was also a breakthrough. These relationships were not discovered once and for all. Rather, they were rediscovered several times, in different lessons and different contexts. They were still not established as routine strategies by the end of Lesson 17. Along the way he had some significant insights into the \(N-F-T\) relations involved in the strategies, but he still lacked a comprehensive understanding of \(N-F-T\) relations.

### 7.3.3 Developing strategies for odd multiples of 5

A neat thread of learning can be traced in his handling of odd multiples of 5. This small thread reveals the familiar features of his learning progression: potential confusions, moments of insight, insights not achieved once and for all, yet still an increasing familiarity and fluency. In the target range of the
high 5s, the odd multiples just amount to two facts—7×5 and 9×5—along with their turn-arounds and inversions. Initially, Blair’s strategy with odd multiples of 5, as with all high 5s, was to count on by 5s from 25.

The early lesson segments on high 5s were mostly with even multiples of 5. Tasks on odd multiples appeared as stranger territory, not amenable to the emerging strategies based on N-F-T relations. Blair needed to find new ways to structure odd multiples of 5. Two moments in Lessons 10 and 11, discussed in the earlier section on distinguishing units, show how odd multiples were susceptible to a confusion of units. In Task 10.5.4, posing eight 5s and another 5 with screened 5-tiles, Blair recognised that the tiles “wouldn’t be even”, but calculated the total incorrectly to be 50, perhaps adding to 40 one unit of 10, rather than one unit of 5. In Segment 11.3, he calculated 9 5s as “10 less than 50”, apparently thinking of one unit less than 50, but again confusing the unit as 10 rather than 5.

While the units were confused, relating nine 5s to 50 here was an important new awareness. Prior to this task, Blair had always structured nine 5s as a count of 5s beyond five 5s. Structuring nine 5s in relation to the higher multiple ten 5s suggests Blair had a stronger awareness of nine 5s within a set of 5s multiples, including multiples above as well as below. This structuring of nine 5s enabled a neighbouring multiple strategy, which is particularly appropriate for odd multiples of 5, where a neighbouring even multiple may be more familiar.

In the following lesson, Lesson 12, Blair adapted the neighbouring multiple strategy to solve two tasks with odd multiples of 5 in division orientation. For 35, he found F via the number of 5s in 40. For 95, he found F via the number of 5s in 100. However, for nine 5s in a multiplication orientation, he tried a new strategy. He used a partial product:

\[ 9 \times 5 \rightarrow 6 \times 5 + 3 \times 5 \rightarrow 30 + 15 \rightarrow 45. \]

As well as indicating a growing sophistication in his handling of multiplicative units, as discussed above, this was an important new strategy for handling an odd multiple.

In subsequent lessons, tasks involving odd multiples of 5 were generally solved in one of these two ways: a partial product strategy, or a neighbouring
multiple strategy. In Segment 15.6, he solved $5 \times 9$ as a partial product again, but a different partial product:

$$5 \times 9 \rightarrow 5 \times 5 + 4 \times 5 \rightarrow 25 + 20 \rightarrow 45.$$  

After the teacher explicitly invited him to think of relating $5 \times 9$ to a closer multiple, he solved using a neighbouring multiple strategy:

$$5 \times 9 \rightarrow 5 \times 10 - 5 \rightarrow 50 - 5 \rightarrow 45.$$  

In Segment 17.2, he solved 45 find $F$, posed with screened 5-tiles, via the neighbouring multiple strategy:

$$45 \rightarrow 10 \times 5 - 5 \rightarrow 9 \times 5$$

This was the first time he spontaneously and successfully used the neighbouring multiple strategy for a task about a $9 \times 5$ fact.

Two orientations of $7 \times 5$ in Lessons 16 and 17 revealed his inconsistency at this point. In Segment 16.3 he explained seven $5$s as $3 \times 10 + 5$, a form of neighbouring multiple strategy, using a strong structuring of an odd multiple of 5 as a combination of 10s and 5s. However, in the 17.1 tasks mixing high 5s with high 3s and 4s, he was not successful completing the task “35 divide by …”, suggesting “35 divide by 5 equals 6 … no, 8”, appearing to recognise 35 as a multiple of 5, but lacking the insight that it is an odd multiple. After the teacher showed six 5-tiles, he corrected his answer.

Thus, over the course of the High 5s Sequence, Blair developed multiplicative strategies for odd multiples of 5 in a range of settings and orientations. Breakthroughs included overcoming confusions of units of 5 and 10; recognising the unevenness of odd multiples; structuring odd multiples in relation to neighbouring multiples; and structuring odd multiples as sums of partial products. However, he did not settle on a consistent strategy, and he could still miscue.

### 7.3.4 Developing strategies for division

Blair’s first insights into $N$-$F$-$T$ strategies for high 5s came in division orientation, in Lesson 10. He realised that for tasks $N \rightarrow F$—that is, in the division orientation—he could use a form of $2n \rightarrow F$ “You take the zero off the 30, then you just double the three”. The inverse orientation $F \rightarrow N$, which is ostensibly multiplication, was posed as the penultimate task in the lesson, and his apparent use of the inverse relationship $\frac{1}{2}F \rightarrow T$ appeared as a significant
insight. Furthermore, Blair could explain why his $2n \rightarrow F$ strategy for division worked: “You can fit two 5s into one 10. So it’s basically doubling that number”.

The same breakthroughs were recapitulated in each of Lessons 11 and 12. For tasks in division orientations, such as 80 find $T$ and $F$; and 60 find $T$ and $F$, he recalled his $2n \rightarrow F$ strategy from the outset, whereas for tasks in multiplication orientation such as six 5s, eight 5s, he at first used less sophisticated strategies, and only with instructional attention did he recall the $\frac{1}{2}F \rightarrow T$ strategy. Thus for Blair, in the context of the settings and tasks I was posing, an $N-F-T$ division strategy came first, while $N-F-T$ multiplication only followed as an inversion of that first strategy.

By Lessons 14 and 15, in the context of the parallel expressions notation, Blair was fluent, answering varied orientations tasks, and he appeared to treat both the multiplication and division orientations as just varying orientations of the same basic $N-F-T$ relationship. The parallel expressions notation appeared to hold the relationships clear for him now. Furthermore, in Segment 15.4, as I transitioned to using formal number sentence notation including using the division sign, he continued to use his $N-F-T$ strategy $2n \rightarrow F$. However, he could not quite explain why this strategy worked. His clearest effort was to explain why the inverse strategy worked: $\frac{1}{2}F \rightarrow n$. These explanations were developed in the 5-tile setting. So, in that setting at least, it appeared that Blair had become clearer on the multiplicative orientation than on the division orientation.

Another important development in handling division orientation tasks was to associate a division with a known multiplication. In Segment 12.4, Blair used for the first time a neighbouring multiple strategy for division of odd multiples: he solved 35 find $F$ via knowing 40 was eight 5s; and he solved 95 find $F$ via knowing 100 was twenty 5s. Such a strategy suggested he approached these division tasks as a search for multiples, or for multiplications. In Segment 16.4, for fixed table tasks with mixed high 3s, 4s, and 5s, he solved two divisions by associating them with known multiplications: $30 \div 5$ appeared to be solved via $6 \times 5$ being a known fact; and $45 \div 9$ was solved by recognising it as relating to the multiplication $9 \times 5$ in the
given table. This appeared to be an important breakthrough in Blair’s approach to division tasks in the high 5s. For the first time he did not try to complete a division calculation as such; rather, he sought a matching multiplication fact. In the next lesson, Lesson 17, this approach of matching a division task to a known multiplication worked for $40 \div $ ..., but not for $35 \div $ ..., which he at first failed to solve. For the task $27 \div $ he responded “Just wait. Let me figure this out. It can’t be $ \ldots \div $ 3 equals 9”. Thus, he recognised that 27 was not a 5s product, another aspect of knowledge for matching multiplication facts to division tasks.

I contend that these new strategies, associating division tasks with known multiplications, did not arise from learning something about division per se; rather, they arose from his increasing familiarity with multiples of 5, so he could now interpret a task in division orientation as just another task about a set of multiples he knew about. An implication of this argument, for instruction, is that there is an advantage in having a focus on a range of multiples, and developing division and multiplication together for that range. I will argue this in the Layer D analysis of the dimensions of instruction.

In summary, Blair developed in informal settings a multiplicative $N-F-T$ strategy for division orientation tasks, which became an invertible strategy between division and multiplication orientations. He sometimes continued to use this strategy for formal division tasks. He also began to solve some divisions by association with known multiplications.

### 7.3.5 Developing independence from the setting

In the earlier Layer C analysis of the Lesson 10 Sequence, the 5-tile setting was significant in Blair’s reasoning. He gave his answers in terms of the tiles, and sought to verify those answers by counting the visible tiles. When the tiles were visible, he developed a practice of counting them in pairs, whether he was counting dots, 5s, or 10s. Counting the visible tiles was a significant activity in working through his difficulties with distinguishing the three units of 1s, 5s, and 10s. When the tiles were screened, he appeared sometimes to be visualising the arrangement of tiles, as in 10.5.4, 40 and another 5, when he expected the arrangement “wouldn’t be even”; and in 10.6.5, 90 dots how many 10s, when he tried unsuccessfully to describe how the tiles might be
arranged. His first new insight, recognising a form of $2n \rightarrow F$, was expressed in terms of the tiles: “If you said there were 30 on 5-dotted cards, how many cards were there? Six cards”. While the final insight of Segment 10.7 did not refer to the 5-tiles, his language is suggestive of the reasoning he had been developing in the setting: “You can fit two 5s into one 10”. Through the last segment, 10.7, the analysis also suggests that Blair’s reasoning became increasingly independent of the tile setting. He stopped referring to the tiles, or verifying his answers by checking the tiles, becoming more focused on checking how his answers aligned in the new arrow sentence notation.

In Segments 11.3 and 11.4, Blair made a similar progression of increasing independence from the 5-tile setting. In Segment 11.3, the tasks were posed with visible tiles, then with screened tiles, and Blair checked answers with unscreened tiles. Then in Segment 11.4, as the new parallel expressions notation was introduced, he gave less attention to the screened tiles. On one task, five 10s find $F$, he had some confusion before answering ten 5s, and appeared to appreciate checking his answer by counting the unscreened tiles “One, two, three … ten. OK good”. For the other seven tasks in the segment, he discussed the notation, but stopped checking the unscreened tiles to verify his answers.

In subsequent segments, Blair seldom used the tiles. On occasions when the instruction brought the tiles to his attention again, he could still connect his reasoning to them. For example, in Segment 12.6, task $16 \times 5$ find $T$, he became confused about 5s and 10s:

Blair: Now, so 16 times 5. One times 10 …
T: Why is it one times 10?
Blair: Cos you’ve got the one there [pointing to the digit 1 in the written 16], you can only fit one 10 in that …

The teacher re-posed the task using 5-tiles:

T: All right. It’s not that we’ve got 16 dots. We’ve got a lot more than 16 dots. We’ve got 16 of these [holding up a 5-tile]. 16 5s.
Blair: Ahh.

With the task posed with 5-tiles, Blair was able to reason correctly about 16 5s:

Blair: Wait, there’d be 50, then that would be five 10s …
T:  There would be from the 50, but there’s also the 6 [pointing to the 6 of 16].
Blair:  There’s also the 6. Just wait, I’ve got to do some working … I’ll do it in my head. 5 times 6 – 5, 10, 15, 20, 25, 30, so 30. So 30, then 30 plus 50 equals 80, so then there’s eight 10s there.

While he reached a correct answer, he went back to skip-counting by 5s, and found the $T$ value of 8 via finding the $N$ value of 80. That is, the tiles appeared to help him make sense of the task, but with the tiles alone he did not use proportional reasoning from $F$ to $T$. Only after further instruction and a return to the parallel expressions notation did he recall the $\frac{1}{2}F\rightarrow T$ relation.

### 7.3.6 Relating to notation

Blair made significant use of the informal notations in developing his reasoning about $N-F$-$T$ relations. In later segments in the High 5s Sequence, he transitioned to using formal notation.

Blair paid attention to potential relationships in the notation. For example, in Segment 8.8 he readily recognised a pattern in the $T$ values in the table of number sentences, and used this pattern to solve a task: “It’s a pattern: 2, 4, 6, 8, 10”. Similarly, in Task 10.7.4 he noticed a sequence in the previous four $N$ values—80, 60, 40, 20—which was actually an accident of my task choices.

For a more constructive example, in Task 10.7.1, the second task with the new arrow sentence notation, he appeared to seek relations in the first arrow sentence to help answer the task, and by Task 10.7.3 was articulating the doubling relation he recognised between the $T$ and $F$ parts in the notation. In Segment 11.4, using this awareness of relationships in the notation, he began to organise his reasoning about a task using the parallel expressions notation: he would work through a task by working through the parts of the notation. For example, for the task 60 $N\rightarrow T$, he immediately wrote 60 into position “That there, 60 right there”, then answered both $T$ of 6 and $F$ of 12 to complete the notation. In a further indication of his attention to the relations in the notation, he took initiative to correct the alignment of the $F$ and $T$ values in the notation, though we had not yet discussed this alignment.

The following task was in inverted orientation: fourteen 5s $F\rightarrow T$ & $N$, the orientation for which he had not used an $N$-$F$-$T$ strategy earlier in the segment. This time, working through the parallel expressions notation,
he readily found \( T \) using the inverted relation \( \frac{1}{2} F \rightarrow T \): “If I halve fourteen, I’ve got seven, so then I’ve got 70 [writing 70 in place]. So that means, 7 times 10 [writing 10\times7 in place]”.

The climax episode of Segment 12.6 was an important example of his recognition of relationships in the context of the notation. In the lead-up in Segment 12.5, without the tile setting or the parallel expressions notation, he had been unable to find an \( N-F-T \) strategy for tasks in the orientation \( F \rightarrow N \).

Finally, in the task sixteen 5s in Segment 12.6, he initiated writing the task into the parallel expressions notation: “Just wait, let me do it this way. So … how’d you do it? …” I then helped him set up the notation, with the given 16 5s written in place, to which he remarked “OK. That’s better, that’s a bit easier now”. (Figure 7.5a shows the completed notation.) He did not yet recall an \( N-F-T \) strategy, eventually calculating \( N \) by skip-counting by 5s, and writing values for \( N \) and \( T \) into their places in the notation. It was then, while looking at the completed notation, that he suddenly recognised his earlier \( N-F-T \) strategies. The video still in Figure 7.5b shows his moment of recognition.

Blair: So then there’s 8 tens there. [writing 8 \times 10 in place] And then … I’ll just put 80 here [writing 80 in place] … Um … I get it now, I get it.

[Reading notation again] 16 5s … Oh, what?! Oh, why didn’t I figure that? Oh! Ah! I could have just taken that off, and halved that down.

Oh! I forgot about my own strategy!

Suddenly, the relation appeared obvious to him, and he proceeded to solve three more tasks fluently using this relation, the last not actually written in parallel expressions notation anymore. I argue that the completed notation was critical here, catalysing his recollection of the \( 2T \rightarrow F \) and \( \frac{1}{2}F \rightarrow T \) relations.

In subsequent segments, there was a sense of the parallel expressions notation holding the task for Blair: once he had written a known value in place in the notation, he could complete the unknown values fluently. A nice example was in Segment 15.3, with the task 60 find \( F \) and \( T \), when he talked out loud through completing the task, in parallel with completing the notation. “Just wait, let me write this down [writes 60 in place]. Then I need to go 6 dot 10 [writes 6\cdot10 in place]. Then add those up, I get 12 times 5 [writes 12\cdot5 in
I imagine “add those up” was about doubling $T$ of 6 to get $F$ of 12. On another task in the same segment, he skipped writing the notation, but apparently solved the task by visualising the notation “I didn’t even write that on a piece of paper”. In Segment 15.4, asked how he solved the task $40\div5$ posed with a conventional division sign, he pointed to some earlier parallel expressions notation from a different task and replied “I did that, I did, like, I took 40, times it by 2—8”. I interpret his expression “I did that” to mean, not writing or imagining the notation per se, but doing the associated $2n\rightarrow F$ strategy. For Blair, the notation could now stand for the $N\rightarrow F\rightarrow T$ strategies.

In Segment 15.4, Blair negotiated a transition from the informal parallel expressions notation to conventional number sentence notation. His handling of division revealed the work involved in the transition. At first, to notate a division task with a number sentence, he notated an unknown multiplier, as in $5\times_=80$, which imitated the approach we had used with the earlier informal notations. When I proposed using the division sign instead, the translation was not obvious for him.

T: Now. I could also write it this way. I could write 80 divided by five equals [writes $80 \div 5 =$].
Blair: What? Hmm.
T: How many 5s in 80. 80 divided by 5.
Blair: Okay.

In subsequent tasks he asked whether he was using the conventional notation correctly. By Segment 17.2 at the end of the High 5s Sequence, when a task was posed as I’m going to put some [5-tiles] down to make 45: how many do I need to put down? Blair responded “45 divided by 5 equals … 9” and wrote this as a number sentence. He had achieved a fluent translation of the context-based task into the formal notation of division.

7.3.7 Summary of progressions in Blair’s reasoning

In summary, Blair’s knowledge of multiplication and division of high 5s improved significantly over the High 5s Sequence. When he began Lesson 8, his main strategy for high 5s tasks was counting-on by 5s from 25. He made initial breakthroughs with multiplicative strategies in the contexts of the 5-tile setting, and the informal notations. By Lesson 17, he was achieving success and some fluency with efficient multiplicative strategies for solving high 5s tasks when posed amongst a mixed range of tasks in a formal bare number
setting. His progress involved several related threads. I have described six threads, which I list here again for later reference:

- distinguishing the three units 1s, 5s, and 10s;
- structuring $N$-$F$-$T$ relationships and related strategies for multiples of 5;
- developing strategies for odd multiples of 5;
- developing strategies for division with multiples of 5;
- developing independence from the setting;
- relating to informal and formal notation.

I finish this summary of the Layer C analysis by noting the texture of the progression in Blair’s reasoning overall. None of these threads were resolved in one lesson: instead, insights could be followed by omissions and confusions, and needed recapitulation in later lessons and different contexts. In turn, many tasks appeared significant for at least one thread of learning. Sometimes a single task brought an insight on one thread, simultaneously with a confusion on another thread.

Blair’s development in some aspects supported his understanding of other aspects. For example, his deeper insight into the $N$-$F$-$T$ relations supported his increased fluency with inverting tasks. His increased fluency with even multiples of 5 supported the development of better strategies with odd multiples, building from the even multiples. Furthermore, this support went in both directions. While his progress with even supported odd, also odd supported even: he first tried a partial products strategy with an odd multiple, and later applied this to an even multiple. Likewise, while knowledge of multiplication supported division calculations, it was in a division orientation that he first realised an $N$-$F$-$T$ strategy, and it was on division tasks that he first successfully used neighbouring multiples strategies.

Thus, the texture of Blair’s learning was convoluted, with multiple digressive interrelated threads. With this convoluted texture, a few example episodes are not sufficient to characterise the progression of his learning, and I have needed to trace his progress over the whole sequence of lesson segments. Such a texture has implications for an instruction seeking to be responsive to the learning. In effect, this convoluted texture of the learning progression is
suggestive of the texture of the instructional progression I am trying to characterise in this thesis:

- Instruction cannot progress in a straightforward way: it needs to accommodate setbacks, leaps forward, recursions, and recapitulations.
- Instruction needs a subtle flexibility to adjust to progress on several related threads.
- Instruction needs to bring several threads along together, to enable their mutual support.
- Instruction needs to support making links between threads: posing similar tasks from different orientations, in different settings, with different notations, and so on.

I turn now to investigate the instruction explicitly.
Chapter 8 – High 5s Sequence, Layers D, E, and F: Analysis of Instructional Progressions Through the Sequence

OK, that’s better, that’s a bit easier now. (Blair)

This is the second of the two chapters analysing the High 5s Sequence. The previous chapter has given a Layer A and B account of the sequence, and a Layer C analysis of key threads in the development of Blair’s reasoning. I turn now to the heart of the research: the Layers D, E, and F analyses of the mathematical dimensions in the instructional progression.

The instruction in these segments made progressions on all five dimensions. The 5-tiles, and various notations, were the basic tools used to draw attention to and reflect on the multiplicative $N$-$F$-$T$ relations, so progressions along the dimensions of setting (SET) and notation (NTN) were critical to the instruction. Also, adjustments in the range (RNG) and the orientation (ORN) of the tasks were used to create opportunities to generalise the structuring and strategies being developed. Each arithmetic task posed could be described as a subtle combination along these four dimensions: a certain range of numbers (RNG), a certain orientation of the knowns and unknowns (ORN), a certain distance of setting (SET), and a certain level of notation (NTN). Furthermore, enquiries and comments after arithmetic tasks—*How did you do that? Why does that trick work?*—gave attention to the fifth dimension STR: structuring multiplicative relations and related computational strategies.

I want to examine the role of each of these dimensions in the instruction. The analysis here seeks to extend the analysis already completed on the Lesson 10 Sequence. I am interested to track how the progressions and characteristics identified in that one lesson were continued and developed over the longer sequence of lesson segments. I am particularly interested in how instructional adjustments in the dimensions enabled a responsiveness to Blair’s progress in the different threads identified in Layer C.
The chapter is in three parts, one for each of the Layers D, E, and F. In the Layer D analysis, I will analyse each dimension separately, to investigate how it was used in instruction, and how it progressed. In the Layer E analysis, I will describe characteristic interactions between dimensions. In the Layer F analysis, I will develop a multidimensional account of the whole High 5s Sequence. Reflecting on the texture of this account, I will propose again that the instructional progression can be characterised as an interwoven calibration across all five dimensions.

8.1 **High 5s Sequence Layer D: Progressions in Each Dimension**

This section presents analysis Layer D, describing the activity in each of the five dimensions separately, through the High 5s Sequence. In Chapter 6, I wrote Layer D analyses for each dimension in the Lesson 10 Sequence. I developed there, for each dimension, an account of the use of the dimension, and of one or more progressions in the dimension. In the analysis in this chapter, I describe how those Lesson 10 uses and progressions continued through the High 5s Sequence, and in some cases, how new aspects of a dimension emerged in the later segments. Between the previous Lesson 10 Sequence account, and the present High 5s Sequence account, I seek to generate a viable account of potential progressions in each dimension. These accounts respond to the first and second research aims: to identify and to describe these key dimensions. My accounts of the progressions also enable me to track the adjustments in each dimension through the whole instructional sequence, which underpins the later Layer F construction of a multidimensional account of the sequence.

8.1.1 **RNG: Progressions in range dimension in the High 5s Sequence**

The basic use of range, dimension RNG, was to focus on high 5s separately from other multiplications, for these lesson segments dedicated to high 5s instruction. This adhered to the approach of the revised instructional design, developed during the teaching experiment, of using the revised scheme of ranges 1, 2, 3, and 4, as a basic organiser of the instruction (see Section 3.3.2). The first section below is an analysis of this basic technique of focusing on the high 5s range. There were two more local ways that range was manipulated within the instruction on high 5s. One was to adjust between restricting to even multiples of 5, and extending to a mixture of even and odd...
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Multiples. A second was to adjust between restricting the range to numbers up to ten 5s, and extending beyond ten 5s. I discuss each of these below.

8.1.1.1 **RNG: Focusing on the high 5s**
The basic organisation of instruction by ranges was part of the initial instructional framework. In Section 3.3.2 I described how, over the course of the teaching experiment, I revised the scheme of ranges, and clarified my use of ranges as the primary organiser of instruction. Focusing instructional attention on the high 5s was productive.

For the first lesson segments on high 5s, the instructional focus was on developing Blair’s structuring of the \(N-F-T\) relations. To this end, these lesson segments were devoted exclusively to multiples of 5, which allowed me to direct Blair’s attention to the recurring relations in those specific multiples. Some segments involved multiplication and division with 5-tiles, and were thus limited to multiples of 5. The segments in Lessons 8 and 10 included recording a table of number sentences all showing multiples of 5 relating to multiples of 10. Segments in Lessons 11 to 15 used parallel expressions notation, also relating multiples of 5 to multiples of 10. It was over the course of these lesson segments that Blair had a series of insights into the relations between multiples of 5 and multiples of 10, as described in the Layer C analysis of the development of his reasoning. These relations are only evident when the tasks are restricted to the multiples of 5, so this restriction to high 5s was a critical basic use of RNG.

In Lessons 16 and 17, I switched from segments exclusively on high 5s tasks, to segments mixing high 5s with high 3s and high 4s tasks. This extension in the main range challenged Blair to maintain his awareness of the \(N-F-T\) relations, amongst thoughts of other relations and tasks. For example, in 17.1, for the task “35 divided by ___”, he recognised the divisor could be 5, but his two proposed dividends were 6 and 8: that is, he did not maintain his finer awareness of the distinction between odd and even multiples of 5. His subsequent resolution of this task, after recognising three pairs of 5-tiles as 30, may have returned that finer awareness. In the task “27 divided by ___” he recognised the divisor couldn’t be 5—which shows a valuable broadening of his knowledge of multiples of 5s, to recognise what isn’t, as well as what is.
Thus, extending the range to mixed high 3s, 4s, and 5s continued to push Blair's cutting edge, and consolidate his knowledge of basic facts strategies and relations.

Note that the restrictions in RNG discussed above were in the lesson segments dedicated to high 5s. In other segments of the same lessons, I made opportunities to extend beyond the basic facts range. So, I was not narrowing his scope of mathematics overall. I was just being strategic about my learning aims for each lesson segment.

8.1.1.2 RNG: Adjusting between even and odd multiples

The even multiples of 5—2 5s, 4 5s, 6 5s, 8 5s, 10 5s—are the neat, familiar decuples. They are multiples of 10, and can be calculated using the $N-F-T$ relations. The odd multiples of 5—3 5s, 5 5s, 7 5s, 9 5s—are all odd numbers, ending in 5. Though they are not multiples of 10, they can be related to multiples of 10, as in knowing that 7 5s is half of 7 10s. Another strategy for the odd multiples is the neighbouring multiples strategy. Thus, in terms of multiplicative relations and strategies, the odd multiples of 5 are a distinct set of tasks from the even multiples of 5.

Through the early lesson segments, the odd multiples were included or excluded from task groups strategically, depending on the purpose of the task group. In Segment 8.8, the task group investigated relating $F$ to $T$, and so only considered even multiples of 5. Segment 10.5 was a task group of incrementing and decrementing by 5s. The purpose was to work with the additive relations between the multiples of 5, so odd multiples were as pertinent as even multiples. However, once Blair had his $2n \rightarrow F$ insight about even multiples, the subsequent segments 10.6 and 10.7 were limited to even multiples again to furnish him cases for his new strategy. Segments 11.4, 12.6, 13.1, and 14.3, continued to investigate the relation of $F$ to $T$, and so were limited to even multiples. Meanwhile, Segments 11.3, 12.4, and 12.5 posed tasks calculating various multiples of 5, including odd multiples, and in these segments Blair developed effective strategies for odd multiples, including neighbouring multiple strategies, and partial products strategies, as shown in Layer C. Segments 15.3, 15.4, and 15.5 were devoted to even multiples, while progressions were made in the notation. Segment 15.6
addressed strategies for a task with an odd multiple. From Segment 16.3 onward, with Blair succeeding with multiplicative strategies on even and odd multiples, the RNG involved both even and odd multiples.

At a finer scale of adjustment in range, within the lesson segments where odd multiples were included, the shift from even multiples to odd multiples could be an interesting point of instructional adjustment. Segment 11.3 posed tasks in orientation \( F \rightarrow N \) in the setting of screened pair-wise 5-tiles. Following introductory tasks of the low 5s in sequence, and the even multiples of 5s in sequence, I posed a series of tasks out of sequence: four 5s, five 5s, eight 5s, seven 5s, ten 5s, six 5s, nine 5s. In this series of tasks, I introduced each odd multiple following a neighbouring even multiple: five 5s following four 5s; seven 5s following eight 5s; and nine 5s two tasks after ten 5s, providing the opportunity for Blair to make a connection between the tasks. As it turned out, he probably solved five 5s as a known fact, but he did appear to derive seven 5s from eight 5s successfully, and sought to derive nine 5s from ten 5s, but confused his units, saying nine 5s would be “10 less than 50”.

In Segment 12.4, with tasks in orientation \( N \rightarrow F \& T \), the first three tasks gave even multiples of 5: 70, 40, 60. After these were answered successfully, I progressed to posing two less established odd multiples of 5: 35, 95. He solved both by relating to a neighbouring even multiple: for 35, he found \( F \) via the number of 5s in 40; for 95, he found \( F \) via the number of 5s in 100. In both these segments, the instructional adjustments between even and odd multiples appeared to challenge and support his developing strategy of relating odd multiples to even multiples.

### 8.1.1.3 RNG: Extending beyond ten 5s

Another form of adjustment in range was to extend higher than the high 5s, to multiples of 5 beyond 50. This extension of the range furnished more examples of even multiples, which appeared to support Blair’s investigation of the \( N-F-T \) relations. In Lesson 8, having succeeded with six 5s, eight 5s, and ten 5s, I extended to 16 5s, 22 5s, 26 5s, and finally 46 5s, seeking to extend beyond the reach of Blair’s skip-counting strategy and challenge him to find another strategy. He only partially met the challenge in this lesson. However, over Lessons 10 to 15 I continued to extend the range beyond ten
5s, up to eighteen 5s, to furnish sufficient examples of even multiples for Blair to test out and reflect on. These were the key lessons where Blair established insight into the multiplicative \( N-F-T \) relations with even multiples of 5. For example, Segment 11.4 posed tasks with even multiples of 5 which Blair could solve using his new insights about \( 2n \rightarrow F \) and \( \frac{1}{2}F \rightarrow T \) from Lesson 10. While working on the first tasks of six 5s, eight 5s, four 5s, and five 10s, Blair was still using some counting strategies. Continuing the task type with tasks in the range beyond 50—posing 80, 60, and fourteen 5s—Blair returned to using the new multiplicative strategies. He made a similar return to multiplicative strategies over the course of Segments 12.5 and 12.6, which again needed tasks in the range beyond 50 to furnish sufficient examples.

Thus, range was an important dimension that could be extended and restricted to adjust the instructional progression to the aims of each lesson segment, and to the cutting edge of Blair’s reasoning about high 5s.

8.1.2 ORN: Progressions in orientation dimension
In the High 5s Sequence, adjustments in ORN were significant in Blair’s learning of the \( N-F-T \) relations and strategies. I discuss three aspects of the progression in ORN:

- Progressions of frequent switching of orientation;
- Subtle adjustments in orientation;
- Not switching orientation.

8.1.2.1 ORN: Progressions of frequent switching of orientation
A main progression in ORN was to make several switches within a segment, in tasks involving \( N, F, \) and often \( T \). In the context of these segments, Blair made increasing links between the relations and strategies for these different orientations. The first segment with such switching ORN was 10.7. As discussed in the Lesson 10 analysis, Blair succeeded with tasks in three different orientations, and appeared to make a link between the orientations, using both \( 2n \rightarrow F \) and the inverse \( \frac{1}{2}F \rightarrow T \). He also articulated an awareness of the underlying relations of 5 and 10 units.
In Segment 11.4, the first three tasks had ORN of \( F \rightarrow T & N \), and Blair did not use the multiplicative relation of \( \frac{1}{2}F \rightarrow T \). I switched the ORN to \( T \rightarrow N & F \), and then \( N \rightarrow T & F \). For these tasks he did use the multiplicative relations \( 2T \rightarrow F \) and \( 2n \rightarrow F \). I then returned to the initial ORN of \( F \rightarrow T & N \). This time he did use the multiplicative relation \( \frac{1}{2}F \rightarrow T \). I argue that changing ORN to the orientations with \( F \) unknown, which he could solve more readily, supported him to recall the multiplicative relation. Then, returning ORN to the inverse orientation, he could construct the inverse multiplicative relation.

Segments 14.3 and 15.3 posed tasks with ORN switching between \( F \rightarrow T & N \) and inverse \( N \rightarrow T & F \). Blair managed them fluently, using the multiplicative strategies. The ORN switch served to consolidate the strategies developed in the previous lessons. Furthermore, switching orientation may have supported his sense that these were the same basic tasks, involving the same set of relations. As argued in the Layer C analysis of the High 5s Sequence, the changes in orientation appeared to be increasingly transparent for Blair in these segments, and he saw through to the linked \( N-F-T \) relations, at least in the context of the parallel expressions notation.

8.1.2.2 **ORN: Adjusting involvement of \( T \)**

Instruction could make subtler adjustments in ORN, moving away from involving \( T \) in tasks. While the instruction had developed the tasks involving \( N, F, \) and \( T \), the main instructional aim was for Blair to develop strategies for tasks involving just \( N \) and \( F \): high 5s multiplication and division. The \( T \) had been introduced to draw attention to the multiplicative relationships involved between 5s and 10s. In 10.7 and 12.6, Blair used \( \frac{1}{2}F \rightarrow T \) to find \( T \) from \( F \), but he did not answer tasks finding \( N \) directly from \( F \). Over the course of 14.3 and 15.3, as Blair appeared to develop fluency with the multiplicative strategies, the instruction adjusted ORN to involve \( T \) less, posing \( N \rightarrow T & F \), then \( N \rightarrow F \), and later \( F \rightarrow N \) tasks.

8.1.2.3 **ORN: Not switching orientation**

Another option for instructional progression was to maintain an orientation over a sequence of tasks, rather than switch it. As related above, in 11.4, when Blair didn’t use the \( \frac{1}{2}F \rightarrow T \) strategy for \( F \rightarrow T & N \) tasks, the instruction switched ORN, and later switched back. The switching ORN may have
supported Blair to recall an $N-F-T$ strategy, and invert it for the $F\rightarrow T&N$ orientation. In 12.5, Blair again didn’t use the $\frac{1}{2}F\rightarrow T$ strategy for $F\rightarrow T&N$ tasks. This time, the instruction *maintained* the challenging ORN. I pursued an extended sequence of tasks all in this orientation. Instead of the instruction changing the ORN for him, he was challenged to reason his way to the inversion himself. We ended up having an explicit conversation along these lines.

T: Last week, you saw something, and I want to help you use what you saw …

Blair: But to use the thing that I saw, I need to have like the answers first.

T: Mm.

Blair: Like 60 [pointing to earlier task in workbook], you double that for 12 times five.

T: Exactly, well I’m trying to now see if you can-

Blair: Switch.

T: -do it back the other way [gesturing turning hands past each other].

The instruction did not switch the ORN of the task, but it drew attention to the issue of switching orientations. Eventually, in the next task, supported by the parallel expressions notation, he successfully rediscovered the relation $\frac{1}{2}F\rightarrow T$. “Oh, why didn’t I figure that … Oh, I forgot about my own strategy!” He proceeded to use that strategy fluently in the three subsequent tasks.

I consider this a form of instructional progression from the previous lesson, involving withdrawing a level of instructional mediation about orientations. In Lesson 11, I switched ORN, which prompted Blair to recognise inverse relations. In Lesson 12, I did not switch ORN, and challenged Blair to find the inversion of relations within the tasks, which he finally did.

8.1.3 SET: Progressions in setting dimension

The setting of 5-tiles was a feature of the instructional progression in the High 5s Sequence. The tiles were most used in the Lesson 10 and 11 segments, and then the instruction made a general progression in distancing the setting, toward bare number tasks. However, the setting continued to have an interesting role in moments in the later lesson segments.
The analysis in the Lesson 10 Sequence reported on four aspects of the setting dimension:

- The setting as a context for reasoning about tasks;
- Structuring in the setting;
- Distancing the setting;
- Uses of screening and unscreening.

In this section, my analysis of the High 5s Sequence will elaborate on these four aspects. I will also address a fifth aspect:

- Returning to the tile setting from bare number work.

8.1.3.1 SET: Setting as a context for reasoning about tasks

A basic use of the 5-tiles was to provide a context for posing and reasoning about tasks. The Lesson 10 analysis observed that Segments 10.5 and 10.6 were posed as tasks about the tiles, asking: How many dots? and How many tiles? and checking answers against counting the tiles. This use of the tiles as the reference point was continued in Segments 11.3 and 11.4, posing tasks with screened tiles using phrases such as “I’ve got four 5s” and “I’ve got five 10s, how many 5s do I have?” The tiles were unscreened to check them after some answers, which Blair continued to find convincing, as argued in Layer C. For example, with the five 10s task, after some confusion in reaching an answer of ten 5s, the SET adjusted to unscreen the tiles, enabling Blair to count the tiles: “One, two, three … ten. OK good”.

8.1.3.2 SET: Structuring in the setting

Blair had difficulty distinguishing between the units of 1s, 5s, and 10s, as described in the Layer C analysis. The Lesson 10 analysis has already described how, in the Lesson 10 segments, the instruction offered the 5-tile setting as an important context for Blair structuring these units. This instructional use of the setting to promote Blair’s structuring of units continued through the High 5s Sequence.

For example, in Segment 11.3, for the nine 5s task posed with screened 5-tiles, Blair gave his confused answer of “10 less than 50” making 40. He appeared to have recognised that nine 5s is one unit less than ten 5s, but confused the missing unit as one 10, rather than one 5. The next instructional
adjustment was unscreening (SET), revealing four pairs of 5-tiles and one more 5-tile. Blair immediately recognised the answer was 45. Unscreening the 5-tiles helped him calculate the answer, and perhaps helped him clarify the distinction between the paired 5-tiles as 10s, and the unpaired 5-tile as 5. Six lessons later, in Segment 17.2, Blair correctly answered the related task: given \( N \) is 45 find \( F \), with screened 5-tiles. Again, I followed up with a SET adjustment of unscreening, revealing the same arrangement of nine paired 5-tiles, allowing him again to reflect on his structuring of 5s and 10s.

In Segment 11.4, I posed five 10s find \( F \) with screened tiles, Blair answered incorrectly 25. The next instructional adjustment was in SET dimension, showing not the whole arrangement of tiles, but just a brief flash of one pair, saying *Here’s one of my 10s there, I’ve got five of them. How many 5s?* Blair then corrected himself. As argued in the Layer C analysis, seeing the 10 as two tiles helped him organise his reasoning about 5s and 10s. In a similar fashion, in the later Segment 16.3, when Blair was posed the task eight 5s with screened tiles, the teacher’s prompt *I’ve got eight of these guys* while holding up one 5-tile, supported Blair to realise “oh, four 10s”.

8.1.3.3 SET: Screening and unscreening at the cutting edge

The Lesson 10 analysis suggested that the combination of screening and unscreening was frequently powerful in maintaining a subtle responsiveness to Blair’s activity. The examples described above from other lesson segments show this continuing feature in the instructional progression, the finely controlled shifts in screening and unscreening both challenging Blair, and supporting him to clarify his reasoning and further his structuring of multiplicative units.

The second task of Segment 11.3 offers a good example. The first task had been with SET visible, incrementing by 5-tiles up to six 5-tiles totalling 30 dots. The second task shifted SET to screened, and posed eight 5s, and Blair incorrectly answered 60. The instruction returned to the previous arrangement of six 5s, retreated SET to visible, then advanced SET to screened again, and reposed eight 5s by showing two 5-tiles before placing them behind the screen *Here’s another two 5s*. Blair answered correctly “… Oh, 40!” and finally the instruction retreated SET to visible again, to allow a check.
first screening of the tiles had proven too challenging for Blair; he had lost his bearings on the relationship of eight 5s to six 5s. Using subtle retreats and advances in SET, the instruction could repose the task with Blair seeing the additional 5-tiles, and suddenly the relationship seemed obvious to him, and he answered correctly. I see this instructional work with screening and unscreening as a strategic ratcheting back and forth along the SET dimension.

8.1.3.4 SET: Distancing the setting
Over the High 5s Sequence, there was an overall progression of distancing the tasks from the setting. Segments from Lessons 10 and 11 made much use of the 5-tile setting, whereas later segments became more independent of the tiles, and posed more tasks in bare numbers. This progression of starting from tiles, and then distancing toward bare numbers, was also enacted on a smaller scale within most segments. Such a progression is evident in Segment 10.7, where the Lesson 10 analysis described a subtle distancing in the relationship to the screened tiles over the six tasks, culminating in a bare number task. A similar progression occurred in Segment 11.4, where tasks were posed with screened tiles, but Blair paid decreasing attention to them, and the instruction stopped unscreening the tiles for verification of answers. Segment 12.4 began by invoking the tile setting used in the previous lessons, but the tiles were no longer actually laid out Now we also did these ... [pointing to the set of 5-tiles] If I made 70 with 5s, how many would I need? The four tasks of Segment 14.3 made a quick progression, from visible tiles, to screened tiles, to a bare number task; and Segment 15.3 made an even quicker progression, the screened tiles only appearing for the first task.

8.1.3.5 SET: Returning to 5-tiles from bare numbers
From Lesson 12 onward, most of the High 5s Sequence was presented with SET in bare numbers. In several episodes, the instruction returned SET briefly to the 5-tiles. These returns appeared constructive in supporting Blair to recall or reflect on the mathematics.

A first example of such a return was described in the analysis of Segment 10.7. The tasks in the segment had progressively distanced SET to bare number, then in Task 10.7.5, the SET returned to screened tiles, and to unscreened tiles for checking answers. Blair’s newly fluent counting of $N$, $F$, $$
and $T$ with the visible tiles appeared to be a significant consolidation of his developing facility.

The first task of Segment 12.5, $6 \times 5$, was posed with SET in bare numbers, and a different ORN to the previous tasks, and Blair couldn’t answer, appearing to lose his bearings on how to make meaning of the task. In response, I began to lay out 5-tiles. Blair said “No no no no no”, rejecting the return to the tiles, but he then counted by 5s and answered the task correctly. The instruction did not retreat SET properly to posing the task with the tiles; rather, it suggested the retreat, but this seemed sufficient for Blair to regather his reasoning about the task.

Segment 12.5 then involved a protracted search over several bare number tasks for Blair to recall an $N-F-T$ strategy, reaching a climax with task $16 \times 5 F \rightarrow T$, labelled as the first task of Segment 12.6. Blair notated the task in the parallel expressions notation, but still struggled to find $T$, getting confused about finding how many 10s in 16, rather than in $16 \times 5$. I then recalled the 5-tiles:

$$
\begin{align*}
T: & \quad \text{All right. It's not that we've got 16 dots. We've got a lot more than 16} \\
& \quad \text{dots. We've got 16 of these [holding up a 5-tile]. 16 5s.} \\
Blair: & \quad \text{Ahh.}
\end{align*}
$$

Blair was then able to reason correctly about 16 5s: the subtle return in SET appeared to support his reasoning.

Later, in Segment 12.6, after Blair finally rediscovered the $\frac{1}{2} F \rightarrow T$ strategy and successfully answered bare number tasks, I returned SET to the paired 5-tile setting to verify the answer to fourteen 5s:

$$
\begin{align*}
Blair: & \quad [\text{Writing answers}] 14 \text{ times 5. 7 times 10. That equals 70.} \\
T: & \quad \text{And do you believe that? Is fourteen 5s equal to seven 10s?} \\
Blair: & \quad \text{Seven 2s is 14.} \\
T: & \quad \text{But do you believe that if I had fourteen of these [indicating 5-tiles],} \\
& \quad \text{I'd have 70 dots altogether?} \\
Blair: & \quad \text{Yeah.} \\
T: & \quad \text{Why?} \\
Blair: & \quad \text{Because my maths told me.}
\end{align*}
$$

I then laid out 14 5-tiles in pairs. Blair counted the tiles to 14, and then counted the dots 10, 20, 30 … 70. I suggest that, while what his maths told
him was correct, the retreat in SET offered an important confirmation of his new success with multiplicative strategies in bare number tasks.

In Segment 17.1, a guess-the-divisor-and-the-dividend task was posed: 35 divided by __ in bare numbers. Blair proposed the correct divisor of 5, but the incorrect dividend, proposing 6, and after an exchange, changing to 8. The teacher then returned SET to visible 5-tiles, to show three pairs of 5-tiles, saying Here’s six 5s. Blair looked at the tiles, said “That’s 30”, then looked back at his written answer and said “35 … no it’s 7!” Again, the instructional adjustment did not re-pose the entire task with the tiles, but offered a return to visible tiles that supported Blair to clarify his structuring of the situation.

8.1.4 NTN: Progressions in notation dimension
Use of notation became a feature of the instruction over the High 5s Sequence. The informal parallel expressions notation was developed in Segment 11.4, and used in most segments until Segment 15.4, which transitioned to regular number sentence notation. I describe four aspects of the progression in the NTN dimension:

- using the notation to attend to multiplicative relationships;
- developing the role of informal notation;
- supporting the student’s increasing independence with notation;
- progressing to formal notation.

8.1.4.1 NTN: Attending to relationships in notation
The Layer D analysis of NTN in Segment 10.7 described the instruction using informal notation to attend to multiplicative relations, for example by attending to alignment of parts of the notation, and discussing patterns in the arrow sentences. In the next lesson, the parallel expressions notation was developed, and this became central to supporting Blair’s structuring of the N-F-T relations. I discuss three example episodes below.

In Segment 11.4, I introduced the new parallel expressions notation. Blair solved the early tasks, in orientation of $F\rightarrow N$ using skip-counting-based strategies. For the first task in the inverse orientation, given 80 find $F$, Blair recalled his $2n\rightarrow F$ reasoning: “Duplicate eight”, and the instruction then moved to record the answers in parallel expressions notation. Blair realised
which answers belonged in which parts of the notation. Furthermore, he corrected the writing to properly align $T$ and $F$, indicated a doubling relation $2T\to F$, and annotated this relation beside the original answers. Thus, the introduction of the notation appeared to help Blair clarify the different units as different parts in the notation, and helped him recognise the $2T\to F$ relation. One task later, fourteen 5s posed back in the $F\to N$ orientation, he did not go back to the skip-counting strategies, using instead the inverted multiplicative strategy $\frac{1}{2}F\to T$ “If I halve fourteen, I’ve got seven … 70”. As argued in the Layer C analysis, he appeared to reason this inversion in the context of writing the task in the parallel expressions notation. I contend that the shift in NTN of introducing the parallel expression notation in this segment was critical in supporting Blair to rediscover a multiplicative strategy, and its inversion.

The climax episode of the Lesson 12 segments was another significant example of notation supporting the structuring of relations. In Segment 12.5, I posed a series of formal, bare number $F\to T&N$ tasks, and persistently challenged Blair to find solutions without counting, without success. Prompts to recall past strategies, and recalling the 5-tile setting, did not seem to help. Finally, for task $16\times 5$, Segment 12.6, Blair asked to use the parallel expressions notation, and I supported him to set up the notation, placing the known number and pointing out the two unknowns: This belongs here. The question is that, and that. It was after he found $F$ and $T$ by other means, and wrote them into their places in the notation (see Figure 7.5a) that he suddenly realised the $\frac{1}{2}F\to T$ relation, “Oh, why didn’t I figure that … Oh! I forgot about my own strategy!” and he proceeded to solve three more tasks fluently using this relation. I argued in the Layer C analysis of this episode that having the completed notation brought the task into the realm of Blair’s multiplicative reasoning: he could see the more familiar direction of $2T\to F$, and then reason from this to the inverse relation, of $\frac{1}{2}F\to T$. The instructional adjustment in NTN to return to the informal notation enabled Blair’s reconstruction of his multiplicative strategy.
A third example to note was in Segment 15.5, at the end of our discussion seeking to explain why his $2n \rightarrow F$ strategy worked. In my explanation, I referred to a parallel expressions notation for $40 = 8 \times 5$ (Figure 7.7b):

T:  … Which is why we’ve been doing this here. 40 means four 10s [pointing to 4\(\cdot\)10], and four 10s means eight 5s [pointing to 8\(\cdot\)5].

Blair responded by annotating the parallel expressions with arrows:

Blair:  Mm. So it goes from up there to there, then from there to there. Is that what you’re saying?

After a recapitulation of these relationships, he exclaimed “Oh, it’s like a cycle”, and he created a new notation of this cycle from $N$ to $T$ to $F$ and back to $N$ (see Figure 7.7c). The structure of the notation, and the act of notating, appeared to be powerful contexts for attending to the multiplicative relations.

8.1.4.2 NTN: Developing the role of informal notation

Analysis of the use of informal notation through the High 5s Sequence reveals a development in the role of the informal notation.

An early role was simply to record results already found. When the arrow sentence was first introduced at the end of Task 10.6.5c, it had this role to record the values of $N$, $F$, and $T$. Once recorded, the notation also provided an opportunity to reflect on those values, as when the teacher asked “90 is nine 10s. Is that surprising?” After notating several tasks, the notation could be used to reflect on patterns in the values of those tasks, as at 10.7.3 and 10.7.6.

Following the success of the 10.7 instruction, I wanted to give the notation a clearer role in supporting Blair’s structuring of the $N$-$F$-$T$ relations. I planned to continue using notation in the next Lesson 11, but decided to try adjusting the notation to align $F$ and $T$, as I wanted to bring attention to this relationship. This planning also drew on the less successful experience of Segment 8.8, which used a table of number sentences in which Blair appeared to focus on patterns down the columns of $F$ values, and down the columns of $T$ values, rather than recognising a relationship between $F$ and $T$ values across a single row. So, in Segment 11.4, I developed the parallel expressions notation, which became important in Blair’s success with structuring $N$-$F$-$T$ relations and strategies over the subsequent lessons. The use of the notation to support Blair’s structuring was described in the previous section. Initial use of the notation in this role was to draw attention to parts of a particular task, but
by Segment 15.5, the instruction could use the notation to refer to the $N$-$F$-$T$ relations in general: “Which is why we’ve been doing this here [pointing to the notation]. 40 means four 10s [pointing to 4\times10], and four 10s means eight 5s [pointing to 8\times5].”

Related to the role of attending to $N$-$F$-$T$ relations, the notation also took on a role of supporting calculations. In the tasks of Segment 10.7 using the arrow sentences, and in 11.4 using the parallel expressions notation, Blair began to use the notation to help his reasoning toward the solutions. From the second task with the arrow sentences, 10.7.1, he wanted to look back at the previous sentence to think about which value was recorded where, “They were the 10s for the 90 weren’t they … ?” which required instructional support. At 10.7.3 he reviewed the previous sentences to check that his latest answer conformed to the pattern he had noticed in the notation. By the end of 11.4 he would organise his calculations by working through the three parts of the notation, as described in the Layer C analysis: “That there, 60 right there”. Increasingly, as further argued in the Layer C analysis, Blair recognised relationships between the values in the notation, until the notation appeared to hold the whole $N$-$F$-$T$ situation for him: he expected that, regardless of the orientation of a task, all values could be calculated and the relations would hold. Screening the previous lines of notation (10.7, 12.6) was an instructional adjustment in this role of the notation, challenging the student to organise his calculations without being able to see the notation.

When the informal notation was not being regularly used, the instruction could recall it in the role of reminder, supporting Blair to remember his earlier structuring and strategies. As described in the previous section, in the climax of Segment 12.6, Blair struggled to recall his multiplicative strategies when using number sentence notation, and the instruction supported a return to the parallel expressions notation, in the context of which Blair then recalled the strategy he needed. Once he had recalled this strategy, he completed subsequent tasks without using the informal notation. In Segment 14.3, Blair solved the first task using his old strategy of adding 5s from 25. The instruction recalled the parallel expressions notation for subsequent tasks, and Blair returned to using his multiplicative $N$-$F$-$T$-based strategies.
Thus, there is a progression in the role of the informal notation, through roles such as:

- recording results;
- distinguishing parts of tasks;
- reflecting on relationships in tasks;
- organising calculations; and
- reminding about the $N$-$F$-$T$ strategies.

8.1.4.3 NTN: Supporting the student’s increasing independence with notation

The Layer D analysis of Segment 10.7 revealed how the instruction made a progression increasing the student’s independence with using notation. Later segments in the High 5s Sequence made a similar progression.

In Segment 11.4, I began by demonstrating how to write the parallel expressions notation for one example task. I then handed the book and pen to Blair, saying *So, we’ll write a couple like that.* Blair then wrote the notations. In the third and fourth tasks, four 5s and ten 5s, he missed writing in the $N$ part of the notation, and I let this go at first. Then Blair realised what he’d missed, and went back to write in the missing numbers “Oh just wait. That’s 20 … that’s 50”. In the next task, given 80 dots, in a different orientation $N\rightarrow T&F$, Blair wrote the given number in the wrong place, and I moved to correct the writing *No, no ... Big number here.* In the next task, given 60 dots, I moved to refine the alignment of the notation, saying, *It’s good if the 10 is opposite the 5, so 5 times 12 above, 10 times 6 below.* The final task Blair wrote independently, without needing corrections. This is a typical progression in a segment, from teacher notating, through student notating with support, to student notating independently, navigated by the teacher’s strategic task-by-task shifts in the NTN dimension.

Later segments had moments of Blair showing independent initiative with using the parallel expressions notation. An important example was the episode in 12.6 discussed in the previous two sections when, struggling to recall a multiplicative strategy while working in formal notation, he initiated writing the task into parallel expressions notation, which proved helpful. In Segment 15.3, he initiated using the parallel expressions notation “Just wait,
Let me write this down”. In Segment 16.3, which Blair was recording in formal notation, he took initiative to also write in the parallel expressions notation relating $6 \times 5$ to $3 \times 10$, as a way to show me he could find $T$.

### 8.1.4.4 NTN: Progressing to formal notation

Over the High 5s Sequence, the instruction made a progression from informal to formal notation. Segments 10.4 and 10.5 were mostly verbal tasks, without notation; segments in Lessons 11 through 14 used mostly informal notations; and segments in Lessons 15, 16, and 17 used mostly standard number sentences.

The first main shift to formal number sentence notation was in Segment 12.5, but Blair stopped using his multiplicative strategies in these tasks, and the instruction to try to recall those strategies involved returning to informal notation for some of Segment 12.6.

The more successful transition to formal number sentence notation was managed as a fine-graded instructional progression in the NTN dimension through Segments 15.3 and 15.4. As described in the Layer C analysis, Blair appeared to depend on the parallel expressions notation for organising his $N\rightarrow F\rightarrow T$ strategies. The instructional shift to more formal notation was challenging his dependence on the informal notation. I progressed first from four tasks in parallel expressions notation, in varied orientations, to two multiplication tasks expressed as horizontal number sentences. Next, I asked Blair to translate a division task from parallel expressions notation into “normal” notation, and he proposed writing a missing multiplier sentence, shown in Figure 7.6a. After a second task in this form, I proposed expressing the division tasks using the division sign: *Now, I could also write it this way. I could write $80$ divided by $5$, shown in Figure 7.6b. The transition to the division sign expression was not immediate for Blair, but he did accept it, and succeeded on two more tasks written this way. Finally, I handed the pen to him to notate a task, $40$ divided by $5$, which he solved using his $2n\rightarrow F$. Thus, the instruction made a graded progression to the standard notation over the six tasks. Blair continued to solve the division tasks using his $2n\rightarrow F$ reasoning, and he could connect the notation and the reasoning back to the
parallel expressions notation. His reasoning had survived the transition to formally notated tasks.

For tasks in subsequent segments, while Blair had the responsibility to notate the task as a number sentence, I could continue to pay attention to his notation, as in this exchange in Segment 16.3:

T: I’ve got eight 5s [eight 5-tiles screened].
Blair: [Trying to write task] Eight times … ?
T: I’ve got eight of these [holding up a 5-tile].
Blair: So, 8 times 5 [writing $8 \cdot 5 =$].
T: Yes, that’s a good way to write it.

8.1.5 STR: Progressions in structuring and strategies dimension

Blair made circuitous but significant progress with multiplicative structuring and strategies, as the Layer C analysis revealed. The Lesson 10 analysis has already argued that, in the Lesson 10 segments, the instructional adjustments in STR responded closely to Blair’s learning of structuring and strategies. Here I argue that the responsiveness of the STR instruction continued through the High 5s Sequence.

In the Lesson 10 analysis, I characterised two aspects of the progression in STR: the regularity of instructional attention to STR, which supported the student’s attention; and the sense of progression through four types of STR instruction. Below I discuss further examples of these two aspects from the High 5s Sequence.

8.1.5.1 STR: Teacher’s attention supports student’s attention

In Lesson 10, I conjectured that the frequent instructional attention to STR supported Blair’s own attention to structuring and strategies. The analysis of the High 5s Sequence yields further examples for this conjecture. More than half of the tasks have a comment or sub-task labelled as STR. As the instructional progression continued to return to the STR dimension, attending to the different units, the $N$-$F$-$T$ relations, and the computation strategies, so Blair continued to bring his attention to the same aspects of the mathematics.

One example episode occurred in Segment 12.5. Blair solved the task $9 \times 5$ by counting by 5s from 5, and then, following a prompt, he offered an improved strategy of six 5s plus three 5s. I challenged *What about something even*
closer to 9 than 6? Do you know eight 5s or ten 5s? and he acknowledged he could use ten 5s is 50. Then for the next two tasks, he carried on using this suggestion of finding a closer multiple: he solved $12 \times 5$ as “ten 5s, 50, 60!” and he solved $15 \times 5$ using $12 \times 5 + 10 + 5$, explaining “I was going closer, using the closest.” A similar example occurred in Segment 15.6. After Blair solved $5 \times 9$ using $5 \times 5 + 20$, I suggested he could find a closer neighbour, and he proposed using $10 \times 5 - 5$. Then in Segment 16.3 the following day, he solved $8 \times 5$ as $10 \times 5 - 5 - 5$, apparently trying again to use the relationship of a close multiple.

For a more extended example, I can trace the layering of STR attention to a single aspect of the mathematics—the 5s and 10s parts of the informal notation. In 10.7, I clarified the 5s and 10s parts of the arrow sentence notation; in 11.4, I supported Blair to align the parts of the parallel expressions notation; in 12.6, I supported Blair in recalling the parts of the parallel expressions notation, which in turn helped him recall the inverse strategy $\frac{1}{2}F \rightarrow T$. Then in 15.5, following our discussion of the $2n \rightarrow F$ strategy, Blair initiated using the parts of the parallel expressions notation to notate the relations between $N$, $F$, and $T$ with a cycle of arrows. I contend that the teacher’s attention to the parts of notation in earlier segments laid the groundwork for the student’s initiative in using the notation in 15.5 to clarify the relationships between the parts. While not one of these STR moments is decisive, there is a layering up of attention to particular structuring that contributes to increasing awareness in Blair of this structuring.

8.1.5.2 STR: Progression in sophistication of structuring and strategies

In the analysis of the Lesson 10 Sequence, I identified four different types of STR instruction:

- enquiring about the student’s reasoning, and affirming his reasoning;
- prompting the student to recall or consolidate his earlier reasoning;
- challenging the student to improve his structuring or strategy; and
- challenging the student to explain a relationship or strategy.

Analysing the High 5s Sequence, I can continue to categorise the STR moments into these four types. I list examples below.
• Enquiring about the student’s reasoning, and affirming his reasoning
  o 12.4 40 find $F, T$: How did you figure that out?
  o 15.4 $14 \times 5$: Good, how’d you do that?
  o 15.6 $9 \times 5$: Good, good. I like that, write that down.
  o 16.3 $8 \times 5$: How did you know $9 \times 5$ is 45?
  o 17.2 \( \_ \times 5 = 45 \): How did you get that? ... That’s gold.

• Prompting the student to recall or consolidate his earlier reasoning
  o 11.3 Nine 5s: After Blair answers incorrectly as 40, using 10 less than 50: Sometimes we’ve got to think about 10s, sometimes think about 5s.
  o 12.5 $24 \times 5$: Last week, you saw something and I want to help you use what you saw. And it’s to do with what’s going on up here [pointing to top notations on page, showing numbers of 5s relating to number of 10s].

• Challenging the student to improve his structuring or strategy
  o 12.5 $24 \times 5$: After Blair gives incorrect answer of 48, teacher responds: I’ve got 48 10s? I’ve got $24 \times 5$, I’ve got twice as many 10s?
  o And later in the same task, discussing Blair’s earlier strategy: I’m trying to now see if you can do it back the other way [gesturing turning hands past each other].
  o 12.5 $9 \times 5$: After Blair counts by 5s from 5, teacher responds: Can we use what we’re learning so we don’t have to count all the 5s from one?
  o Then after Blair offers a strategy of $6 \times 5 + 3 \times 5$, the teacher continues: What about something even closer to nine than six? Do you know eight 5s or ten 5s?
  o 15.6 $5 \times 9$: But I reckon you know some 5 times that are closer to $5 \times 9$ than 5 times 5.
  o 16.3 $8 \times 5$: Now I’m wanting to see if you can think of another way. There’s eight 5s, how many 10s are there?
• Challenging the student to explain a relationship or strategy
  o  15.5  40 ÷ 5: Why does that trick work?...Why does it work to take that 4, and double it and make 8?

As suggested in the Lesson 10 analysis, there can be a sense of progression through these four different types of STR instruction. Over the High 5s Sequence, earlier segments in Lessons 10, 11, and 12 have more instances of the first two types—enquiring about and affirming the student’s reasoning, and prompting the student to recall his earlier reasoning—while later segments in Lessons 15, 16, and 17, have more instances of the last two types—challenging the student to improve his structuring or strategy, and challenging the student to explain a relationship or strategy.

In particular, tracing STR moments through the lesson segments, a coherent progression of attention to N-F-T relations can be discerned:

  10.5:  Affirming his counting 10s to find N
  10.5:  Affirming his new \(2n \rightarrow F\) insight
  10.6:  Prompting him to trial and recall \(2n \rightarrow F\) strategy
  10.6:  Challenging him to find \(T\) without counting
  10.7:  Challenging him to clarify counts of \(N, F,\) and \(T\)
  10.7:  Challenging him to explain \(2n \rightarrow F\) relation
  11.4:  Challenging him to clarify counts of \(N, F,\) and \(T\)
  12.6:  Prompting him to recall the inverse relation \(\frac{1}{2}F \rightarrow T\)
  15.5:  Challenging him to explain the \(2n \rightarrow F\) relation again, in a bare number context.
  15.6:  Challenging him to find a better strategy
  16.3:  Challenging him to find a different strategy or recall \(\frac{1}{2}F \rightarrow T\) strategy.
The earlier instruction introduced all three units, and challenged Blair simply on clarifying his counts of those units. It also affirmed his earlier insights, and asked him to recall his first multiplicative strategy. Later instruction challenged him to invert his strategy, to explain his strategy, to improve his strategy. As the Lesson 10 Sequence had a sense of culmination in a relationship-explaining task in 10.7, so the High 5s Sequence had a sense of culmination with a second relationship-explaining task in Segment 15.5. After this segment, I did not pursue this relation-explaining type of task again in the High 5s Sequence, but I brought attention to challenging Blair to improve his strategy, as in the examples listed above from 15.6 and 16.3. Thus, there is a progression in the level of sophistication being demanded, which emerges from the recurring attention to STR.

An STR episode in 16.3 suggests how the attention to strategy progressed over the High 5s Sequence. The task was eight 5s find $N$, and Blair used a strategy of 50 less two 5s. In the early lessons on high 5s, Blair was not connecting tasks to a near multiple, so such a strategy would have attracted an STR comment of affirmation, like the comments Blair’s insights attracted at the end of 10.5 and 10.7. But by Segment 16.3, I knew Blair could solve the task more directly, so the STR comment was to challenge Blair to improve his strategy. Thus, as the student’s cutting edge progressed, the type of STR comment could progress to match.

8.2 High 5s Sequence Layer E: Interactions Between Dimensions

The previous section addressed Layer D, analysing progressions in each of the five dimensions separately. This section addresses Layer E, seeking to identify and describe interactions between dimensions. This layer of analysis contributes to Research Aim 3, investigating interactions between multiple dimensions in the instructional progression. Below I introduce two interactions observed in the High 5s Sequence:

- varying ORN within a fixed RNG; and
- varying ORN to accompany advances in NTN.

I also describe three interactions identified in the Layer E analysis of the Lesson 10 Sequence (Section 6.3), elaborating how these interactions continued through the whole High 5s Sequence:
• coordinating progressions in SET and NTN;
• complementarity between ORN, SET, and NTN; and
• responding with STR to adjustments in other dimensions.

8.2.1 Dimension interactions: Varying ORN within a fixed RNG
One interaction between dimensions is to keep RNG steady, and vary ORN within that range. This interaction can support learning of a whole set of relations within the fixed range. In this case study, RNG was focused on high 5s for a sequence of lesson segments, while ORN was frequently varied. In the context of this instruction, Blair developed multiplication of multiples of 5 and division of multiples of 5 symbiotically. His first insight into an \( N\rightarrow F\rightarrow T \) strategy was in division orientation (10.5); he then realised a strategy for multiplication orientation as an inversion of the first strategy (10.7). His understanding of the two orientations reinforced each other as he established his sense that they were part of the same set of \( N\rightarrow F\rightarrow T \) relations in the context of the parallel expressions notation (Lessons 10 to 15). In developing division strategies for odd multiples, his familiarity with the multiples of 5 appeared to help him connect the division tasks to known multiplication facts (12.4, 15.6, 16.4, 17.2). Overall, by varying the ORN within a fixed RNG, the instruction focused on a set of multiplicative relations, not on multiplication or on division as such. In the teaching experiment, this approach of varying ORN within a fixed RNG was used in the other ranges as well as the high 5s, and one indicator of the success of this approach was in the improvement in Blair’s fluency in mental division.

8.2.2 Dimension interactions: Varying ORN to accompany advances in NTN
Each advance in NTN was accompanied by a cycle of shifts in ORN. These shifts may have supported Blair in understanding the new notations. After NTN first introduced arrow sentences, Task 10.7.1 was posed in ORN \( N\rightarrow T \) and Blair wanted to recognise where the 10s were in the notation—and was wrong at first. Similarly, when NTN introduced parallel expressions in 11.4, and ORN shifted for the first time, he struggled with the parts of the notation. Nevertheless, with continued shifts and attention in ORN, he developed fluency in both notations, and the parallel expressions notation came to hold the whole cycle of orientations in Segments 14.3 and 15.3. Then when NTN
introduced formal number sentences in 15.4, ORN was shifted slowly and systematically, from multiplication to a missing multiplier to a division with the ÷ sign. Also, with the NTN advanced to formal notation, ORN could be switched to investigate the ambiguity of the notation about the orientation of multiplier and multiplicand, as in 17.1 and 18.4. Thus, when NTN is advanced to a different notation, instruction can strategically adjust ORN, to challenge the student to clarify their understanding of the new notation.

8.2.3 Dimension interactions: Coordinating progressions in SET and NTN

As noted in the Lesson 10 analysis, the first advance in NTN to the ongoing informal arrow-sentences brought an associated advance in SET toward bare numbers. This association of SET and NTN advancing together was continued in the subsequent segments. Segment 11.3 returned SET to the screened tiles, and NTN was absent. In 11.4, NTN introduced an informal notation—the parallel expressions—in the context of the tiles setting. Then as NTN advanced the engagement with the notation, SET became increasingly independent of the tile setting. The segments in Lesson 12 continued with the notation, but with SET now in bare numbers. Both 14.3 and 15.3 began with SET in screened tiles, but these were quickly distanced to bare numbers, with the NTN of informal notation well established. In Segment 16.3, with the NTN advanced to conventional number sentences, SET returned to screened tiles, then 16.4 advanced to bare numbers again.

As has already been observed, setting and notation each serves as a context for the student’s sense-making. So, in particular, the instruction could draw on one to help make sense of the other. In the instructional progression overall, the advancement of both dimensions was closely coordinated.

8.2.4 Dimension interactions: Complementarity between ORN, SET and NTN

The Lesson 10 analysis suggested a complementarity between ORN and SET. While ORN shifts challenged Blair to solve a task in a particular orientation, retreats in SET to visible tiles provided a context for reflecting on the whole situation of the task. In subsequent segments, this role of the tile setting could also be played by the notation, especially the parallel expressions notation. Through the segments in Lessons 11, 12, 14, and 15, the instruction pursued
the NTN of parallel expressions, and Blair established a fluency with accommodating changes in the ORN. He appeared to begin to see the relations between elements of the notation as bi-directional. Whatever element was missing in the notation, he could recognise what relation the unknown needed to bear to the known, and so reason towards solving the task. For example, in 12.5 Blair was unable to recall the inverse of his $2n \rightarrow F$ strategy to find tasks in orientation $F \rightarrow N$. But in 12.6, after instruction returned to the parallel expression for a task, he could recall the $2n \rightarrow F$ strategy, and then he could reason using the inverse strategy $\frac{1}{2}F \rightarrow T$. In 15.5, when this system of $N$-$F$-$T$ relations was discussed, he notated the relationships between the different orientations by drawing a cycle of arrows on a set of parallel expressions. Thus, while ORN selected partial views of the $N$-$F$-$T$ system, instruction with the NTN of parallel expressions, and earlier with the SET of tiles, created a context for attending to the whole system of relations. I think the complementarity of these three dimensions in the instruction was significant in Blair’s progress in understanding the $N$-$F$-$T$ relations.

8.2.5 Dimension interactions: Responding with STR to adjustments in other dimensions

The STR dimension was often invoked to relate to an adjustment in one of the other four dimensions. I list some examples for each dimension. When RNG introduced an odd multiple, there could be an STR moment noticing a distinction between odd and even structures, as in Tasks 10.5.4 (40 and another 5), 11.3 (nine 5s), and 12.5 ($9\times5$). When ORN switched to finding a different unit count, there could be an STR moment seeking clarity about distinguishing the units, as in 10.6.5 (90, find $T$), 10.7.3 (four 10s), and 12.5. When SET was about to retreat from screened setting back to a visible setting, there could be an STR moment challenging the student to visualise the layout of the materials, as in 10.5.4 (40 and another 5), 10.6.4 (60 dots), and 17.1 ($35\div\_\_$). When NTN introduced new notation, there could be STR moments drawing attention to the parts of the notation, as in 10.7 (arrow sentences introduced), 11.4 (new parallel expressions notation introduced), and 15.4 (formal sentences introduced). When NTN finished a sequence of notated tasks, there could be an STR moment looking back over the notated
record to seek patterns or explanations, as in 8.8 (final task What’s the relationship?), 10.7 (Final task Why is it?), 12.6 (sixteen 5s, recall of relationships in parallel expressions notation), and 15.5 (Explain why). In each case, an adjustment in a dimension created an opportunity for noticing a distinction, for structuring relations. Instruction can attend to these moments by bringing STR to interact with the other dimension.

8.3 High 5s Sequence Layer F: Multidimensional Analysis of the Instructional Progression

Now I turn to the final Layer F of the analysis of the High 5s Sequence. This layer builds on all the prior layers, seeking to understand how the five dimensions are coordinated to create the instructional progression. As with the Layer F analysis of the Lesson 10 Sequence in Section 6.4, I develop what I call a multidimensional account of the instructional progression through the sequence. This is a concise account of the instruction, expressed as a series of adjustments along the five dimensions, indicating Blair’s responses to the instruction, and the instruction’s responses to Blair. Where the multidimensional account for the Lesson 10 Sequence described the adjustments task-by-task, this account for the High 5s Sequence describes the adjustments segment-by-segment. Following the multidimensional account, I describe the texture of the instructional progression, as viewed in the account. The analysis contributes to the final research aim: to characterise the instructional progression in terms of the multiple dimensions.

8.3.1 High 5s Sequence: Multidimensional account

The multidimensional account may be read in conjunction with the High 5s Analysis Chart (Figure 7.1) to recall the task types and tasks of each segment, and help track the progression of adjustments in the dimensions.

Segment 8.8 began the work on the high 5s topic, posing tasks seeking to relate $F$ with $T$. RNG was even multiples of 5, beginning in high 5s, and soon advancing well beyond 50. ORN was $F \rightarrow T$. The SET began briefly with paired 5-tiles, then advanced to bare numbers, with NTN recording results for $N$, $F$, and $T$ in a table of number sentences. STR tasks pressed for recognition of the multiplicative relationship between $F$ and $T$, but this was only slowly and unconvincingly reached by Blair.
Segment 10.5 began by retreating on all dimensions, then made more modest advances. RNG retreated to an introduction in low 5s, then focused on high 5s, including odd multiples; ORN retreated to incrementing by 5s, without involving T, then tracked both F and N; and SET retreated to visible 5-tiles, then advanced to screening, with frequent unscreening. NTN and STR were less involved. Blair responded with increased success, but continued to track F and N separately using skip-counting, until a breakthrough multiplicative insight of $2n \rightarrow F$ in the final Task 10.5.6: “You take the zero off the 30, then you just double the three”. NTN recording and STR affirmation responded to the insight.

Following up on this new insight, Segment 10.6 changed to a new task type with ORN shifted to $N \rightarrow F$, with RNG limited to even multiples and advancing from high 5s to beyond 50, and STR recalling Blair’s new strategy, while SET continued to use screening, and NTN remained uninvolved. Blair responded with new facility using his new multiplicative strategy and good structuring of the tiles over five tasks. The final task, Task 10.6.5 (90 dots) introduced a new challenge, *How many 10s?* with an ORN shift to find T, and an STR challenge, *Don’t count*. When Blair could not solve the task, SET retreated to visible tiles, ORN and STR adjusted to attend to unit counts, and NTN introduced an informal arrow notation. Blair resolved the task.

Segment 10.7 followed up with NTN advancing to use the new informal notation, and the other dimensions becoming more varied: RNG varied across low 5s, high 5s, and beyond 50; ORN varied across all $N-F-T$ orientations; SET advanced through screening to bare numbers; and STR advanced from enquiring about strategies to seeking explanations of strategies. Blair responded with increasing fluency in computation and unit coordination, including the first use of $\frac{1}{2}F \rightarrow T$, and an explanation of $2n \rightarrow F$: “You can fit two 5s into one 10. So it’s basically doubling that number”.

Segments 11.3 and 11.4 followed a progression similar to that of the Lesson 10 segments. In 11.3, RNG began in high 5s, including odd multiples, while ORN addressed only $F \rightarrow N$; then in 11.4 RNG focused on even multiples, and advanced to include beyond 50, while ORN advanced to vary between $N-F-T$ orientations. SET used screened 5-tiles throughout, but these became less
involved, while NTN introduced parallel expressions notation which increasingly held the tasks for Blair. Blair initially did not use $N\rightarrow F\rightarrow T$ strategies, but in the context of the parallel expressions notation, he progressed to recalling the $N\rightarrow F\rightarrow T$ strategies, both $2n\rightarrow F$ (“Duplicate eight”) and $\frac{1}{2}F\rightarrow T$ (“If I halve fourteen, I’ve got seven”).

Segments 12.4 and 12.5 continued to pursue two different ORN ($N\rightarrow F\&T$ and $F\rightarrow N$), while advancing SET to bare numbers, advancing RNG to include odd multiples, and advancing NTN to formal sentences. Blair developed new strategies of neighbouring multiple and partial products, and readily recalled $2n\rightarrow F$, however he could not recall his $N\rightarrow F\rightarrow T$ strategy $\frac{1}{2}F\rightarrow T$. In contrast to Segment 11.4, I did not switch ORN back to the successful $N\rightarrow F$; instead, I maintained ORN as $F\rightarrow N$, and persisted with an STR challenge to find a multiplicative strategy. In 12.6, a key episode in the High 5s Sequence, Blair succeeded in recalling $\frac{1}{2}F\rightarrow T$ (“Oh! I forgot about my own strategy!”) with the support of an NTN retreat to parallel expressions notation, alongside RNG restriction to even multiples, extensive STR attention, and brief SET recollections of visible 5-tiles.

Segments 14.3 and 15.3 continued with brief tasks like the final tasks of the Lesson 11 and 12 segments: RNG was focused on even multiples, in high 5s and beyond 50; ORN was varying through $N\rightarrow F\rightarrow T$ orientations; SET was recalling 5-tiles, but fading in significance and advancing to bare numbers; while NTN was using parallel expressions notation, which increasingly held the tasks for Blair. Blair responded with success and fluency of $N\rightarrow F\rightarrow T$ strategies (“For doing this type of thing, this strategy, you need an answer like 60, 40, or 30”). STR was quiet.

In Segment 15.4, another key episode, the instruction tried again to advance NTN to formal notation and SET to bare numbers, but more slowly than 12.5 had done. NTN linked deliberately from parallel expressions notation to formal number sentences ($How\ would\ we\ write\ that\ question\ normally?\ ...\ Now,\ I\ could\ also\ write\ it\ this\ way$), accompanied by careful STR attention to Blair’s use of $N\rightarrow F\rightarrow T$ strategies, and careful ORN attention shifting from multiplication to division notation. Blair succeeded, continuing his multiplicative strategies in the formal bare number context. Following Blair’s
success, Segment 15.5 advanced STR to seek an explanation for the \(N\)-\(F\)-\(T\) strategies (Why does that trick work?). To support the ensuing discussion, SET recalled the paired 5-tiles, which appeared not to help Blair, then NTN recalled and elaborated the parallel expressions notation, which appeared to help, leading to Blair annotating a cycle of arrows (“So it goes from up there to there, then from there to there”). Segment 15.6 then sought to continue the bare number SET and formal NTN, while advancing RNG to an odd multiple task, and bringing new STR emphasis to refining strategies. Blair achieved an improved strategy for odd multiples.

Segment 16.3 continued the new NTN of formal number sentences, with RNG in the target range of even and odd high 5s, while focusing ORN on the multiplication orientation \(F\rightarrow N\), for which Blair had not yet consolidated his \(N\)-\(F\)-\(T\) strategies. STR and a return in SET to screened tiles (I’ve got eight of these guys) supported Blair’s attention to his strategies, which he succeeded in consolidating. Segments 16.4, 17.1, and 17.2 then continued NTN in formal number sentences, and advanced ORN to mixed multiplication and division orientations, no longer explicitly involving \(T\). The major advancement was in RNG, expanding to mix high 3s and high 4s with high 5s. Two different task types were used to support this advance in RNG: the type “varied orientations with varied n-tiles” (17.2), with SET advanced to a mixture of n-tiles; and the type “varied orientations with a fixed table” (16.4 and 17.1), with SET advanced to bare numbers. Thus, at this point, the instruction was close to the instructional aims of mixed conventional multiplication and division tasks in formal bare number. Satisfied with Blair’s increasing fluency and attention to strategies, the instruction used little STR.

This ends the High 5s Sequence. The shape of this whole instructional progression through the High 5s Sequence can be summarised using the dimensions as follows. The first segment advanced too rapidly in all dimensions, and Blair struggled to progress. Subsequent segments brought a focus to structuring \(N\)-\(F\)-\(T\) relations with more modest shifts in the dimensions: restricting RNG to even multiples; varying ORN carefully between all \(N\)-\(F\)-\(T\) orientations; using frequent screening and unscreening in SET; and advancing to bare number in coordination with advancing NTN to informal strategies. Attention to STR was frequent, and advanced in a few
key episodes. Blair had several important insights, but also retreats and confusions, and in response the dimensions had small advancements and recursions. As Blair consolidated his $N$-$F$-$T$ strategies, each of these dimensions was advanced, in a staggered interwoven way: RNG to include odd multiples, and then finally to mix high 3s and high 4s with high 5s; ORN to settle on mixed conventional multiplication and division; SET to bare numbers; and NTN to formal number sentences. STR brought attention to each of these advancements, then receded in the final segments, as Blair achieved some fluency. By the end, the instructional aims of multiplicative structuring and strategies for bare, formal high 5s tasks were largely achieved.

8.3.2 Characterising the multidimensional instructional progression

I will consider the characterisation of the instructional progression just described, and briefly highlight some features evident in the multidimensional account. I will discuss further these features, and the proposed characterisation overall, in the next chapter.

Following the task-by-task multidimensional account for the Lesson 10 Sequence (Section 6.4.2), I proposed that the instructional progression could be characterised as:

\begin{quote}
\textit{a strategic interwoven calibration of the five dimensions, toward the instructional aims of keeping at the learner’s cutting edge and attending to the mathematics of interest.}
\end{quote}

I contend that the segment-by-segment multidimensional account of the High 5s Sequence just presented can be characterised in the same way. The dimensions progressed further than in Lesson 10 alone, and they have moved through a greater variety of interwoven combinations. Through this extension and variety, the proposed characterisation remains apt, and I find the characterisation more compelling. The instruction in each segment of the whole High 5s sequence can be coherently understood as a subtle adjustment of calibrations across the five dimensions, responding to Blair’s insights and confusions, and working towards the instructional aims of multiplicative strategies for formal high 5s.

As noted for Lesson 10, most tasks involved adjustments in one or two dimensions, while each segment involved adjustments of most dimensions,
but in different combinations. Also, again, the adjustments tended to be small. From segment to segment, there was typically significant advancement in only one or two dimensions. For example, from Segment 11.3 to Segment 11.4, the main advancement was in variety of ORN; from Segment 15.3 through Segment 15.4 the main advancement was toward formal NTN.

As noted for Lesson 10, the multidimensional account reveals the responsiveness of the calibrations: dimensions are advanced or retreated in response to Blair’s apparent success and understanding. Blair was consistently thinking hard on the mathematics of interest, and generally succeeding.

Other features of this characterisation emerge from charting the longer High 5s Sequence. One feature to note is how each of the five dimensions both advanced and retreated multiple times over the course of the whole sequence. The instruction did not progress consistently forwards on any single front. This is visible in the back-and-forth movement down each column of the analysis chart (Figure 7.1) For example, while SET was largely advanced to bare numbers by the Lesson 12 segments, there were still constructive returns to the visible and screened 5-tile settings. After the RNG advanced to odd multiples in Segments 12.4 and 12.5, it retreated to even multiples for several segments, before returning to odd multiples again in 15.6. This recursion in each dimension is, I think, a distinctive characteristic in the instruction.

A second feature evident in the multidimensional account is how the adjustments in each segment were experimental. Segment 11.3 tried to introduce a new informal NTN, the parallel expressions notation, and Blair responded well, so the NTN carried on. Segment 12.5 tried to advance NTN to formal number sentences, and Blair lost his sense of the $\frac{1}{2}F \rightarrow T$ strategy, so dimension adjustments were made. The outcome of the adjustments was never known in advance; instead, Blair’s activity was observed and responded to.

A third feature, arising from this sense of experimentation, is how each segment was distinctive. Across the 18 segments of the High 5s Sequence, there is little that appears as routine in the progression. The segments display a rich range of textures in the dimensions and dimension adjustments.
8.4 Chapter Summary

In this chapter analysing the High 5s Sequence, I have given:

• a Layer D account of the progression in each of the five dimensions;

• a Layer E account of five interactions between dimensions; and

• a Layer F account of the multidimensional instructional progression.

These accounts layer up rich insights into the highly responsive instruction throughout the sequence, and culminate in the Layer F view of the instruction as an interwoven calibration of the five dimensions. The accounts have reinforced and enriched the findings from the corresponding analyses of the Lesson 10 Sequence.

There is ample material now to draw on, in responding to the three research aims of identifying the key dimensions, describing the progressions in the dimensions, and characterising the instructional progression overall. Chapter 9 will systematically address these research aims, drawing on these last four chapters of analysis.
Chapter 9 – Discussion

So basically, that’s what I wanted this book for. That is really sick.
I look all mathy. (Blair)

In the four previous chapters, I analysed instructional progressions within lesson segments, within a lesson, and across a whole sequence of lessons on the topic of high 5s. In each chapter, I have analysed the instruction in terms of five mathematical dimensions of instructional progression. I have sought to build up layers of analysis from the video data, to illuminate significant task-by-task adjustments along these dimensions of instruction.

I have described how adjustments in the instruction were able to respond to the student’s activity, engage the student at the cutting edge of his learning, and support advancements in the student’s learning. Overall, the instruction was highly responsive, and the advancement in learning was significant.

Having worked through this layered analysis, I want now to gather together coherent responses to the three research aims:
1. Identify key mathematical dimensions of instructional progression.
2. Describe the progressions in each dimension.
3. Characterise the instructional progression, using the dimensions.

The analytic chapters sought to describe the dimensions and overall progressions within specific segments and sequences in the case study. This discussion chapter will propose more generic characterisations of viable progressions in instruction in multiplication and division. I will discuss each of the three aims in turn, drawing on the examples developed in the preceding case study. The Lesson 10 Analysis Chart (Figure 5.1) and the High 5s Analysis Chart (Figure 7.1) can serve as references for the examples throughout this discussion.
9.1 **Aim 1: Identifying Key Mathematical Dimensions of Instructional Progression**

The first aim of this study is to identify key mathematical dimensions of instructional progression for the instruction developing multiplicative strategies for basic facts. The five dimensions identified are:

- range (RNG);
- orientation (ORN);
- setting (SET);
- notation (NTN); and
- attention to structuring and strategies (STR).

9.1.1 **These five dimensions sufficient to characterise the instruction**

Together, this set of five dimensions has proved sufficient to characterise much of the instructional progression. The multidimensional account of the Lesson 10 Sequence, in Section 6.4, and the multidimensional account of the High 5s Sequence, in Section 8.3, described the instructional progressions entirely in terms of adjustments along these five dimensions. Both accounts emerge as coherent, substantial descriptions of how the instruction progressed. Both accounts are sufficient to reveal how the instruction responded to the student’s thinking, how the instruction brought challenge and support along the student’s cutting edge, and how the instruction directed the student towards the instructional aims.

As well as being sufficient to characterise the instruction overall, each of these five dimensions has been necessary to capture significant aspects of the instruction. Each dimension has been involved in almost every lesson segment studied. Also, each dimension has featured in significant moments in the instruction, as in the examples listed below.

**RNG**

10.6 and 10.7: RNG restriction to even 5s supported development of $2T \rightarrow F$ insight. “You can fit two 5s into one 10”.

16.4: RNG expansion to include high 3s and 4s supported new fluency with basic facts.

**ORN**

12.6: Persistence with ORN of $F \rightarrow T \& N$ supported breakthrough reconstruction of $\frac{1}{2}F \rightarrow T$ strategy. “Oh! I forgot about my own strategy!”
15.4: Attentive shift of ORN from multiplication to division supported transition to using ÷ sign.

SET

10.5: SET of screened 5-tiles basis for initial $2T \rightarrow F$ insight “Take the zero off the 30, then just double the three”.

10.7: SET returns to unscreened 5-tiles supported new fluency in coordinating $N$-$F$-$T$ units.

NTN

12.6: NTN return to parallel expressions notation supported breakthrough reconstruction of $\frac{1}{2}F \rightarrow T$ strategy.

15.4: Attentive NTN progression linking three notations supported transition to formal number sentences.

STR

10.7: STR attention and final task *Why is it?* supporting $2T \rightarrow F$ insight.

15.5: STR attention with task *Why does that trick work?* supporting clarity of $N$-$F$-$T$ relations “So it goes from up there to there, then from there to there”.

I propose these five dimensions are the key dimensions in the instructional progression.

### 9.1.2 The dimensions are all mathematically significant

Each of these five dimensions is mathematically significant. They are not simply dimensions of, for example, perception, or task organisation. Rather, each of them is a dimension that matters mathematically. Regarding the dimension of range, the identification of different ranges of numbers is important in many areas of mathematics. In the domain of multiplication and division, significant arithmetic differences emerge in different ranges of factors: for example, between the even and odd multiples of 5s, and also between multiples of 5 and multiples of 4. Orientation is important in many areas of mathematics, particularly the inversion of mathematical operations and relationships. In the domain of multiplication and division, the shift of orientation between multiplication and its inverse division is, of course, fundamental. The setting dimension relates to the fundamental mathematical activity of abstraction from contexts to more general relations and properties (e.g. Gray & Tall, 2007; Mitchelmore & White, 2004; Skemp, 1986). Notating and symbolising have been central aspects of mathematical practice through history (e.g. Gray & Tall, 1994; Sfard, 2000; Vasco, 2007), and induction into conventional notation is a standard part of the mathematics.
curriculum. Structuring is central to mathematics (e.g. Freudenthal, 1983; Mulligan & Mitchelmore, 2009; Pirie & Kieren, 1994) and the development of computation strategies is central to arithmetic (e.g. Anghileri, 2001; Treffers, 1991).

The mathematical significance of the dimensions is a strength. Using significant mathematical dimensions to describe mathematics instruction accords with the RME principles of instruction discussed in Section 2.2.2, such as progressing through levels of increasing mathematical sophistication (Freudenthal, 1991). These five significant mathematical dimensions are a sound basis for understanding the instructional progression, and in turn for describing an instructional design.

9.1.3 **The dimensions are different but the set is coherent**

Each of these five dimensions is a very different aspect of arithmetic instruction. I will consider in turn, the role of each dimension in general arithmetic instruction.

The range of numbers is commonly invoked in circumscribing curriculum topics and expectations. The domain of instruction under consideration here, the multiplicative basic facts, is defined in part by the range of factors, which is customarily from 1 to 10. Instructional plans in this domain would typically identify a range of factors or products as an instructional focus—such as a lesson on the 5 times tables. However, planning task-by-task adjustments in range, as I have observed here, is less typical.

Orientation, in the guise of the formal distinction between multiplication and division, is also commonly invoked in circumscribing curriculum topics: some curricula teach division separately from multiplication. However, while many ranges can be distinguished, there are only a few orientations to distinguish: switching from multiplication to division, and switching multiplier and multiplicand, which is equivalent to switching partition and quotition. So, range is used more than orientation to organise curricula. At the same time, learning the relationships *between* different orientations—for example, between the facts $5 \times 7 = 35$ and $35 \div 7 = 5$—is a more explicit aim of instruction than learning the relationships between ranges, or between
notations. So orientation is distinctive in this way, and has a close association with the instructional aim of structuring these multiplicative relations.

Settings, by contrast, are not typically considered part of the curriculum as such. Rather, a setting can be considered as an instructional device, used to support students’ reasoning about the topic. Students do not need to learn about 5-tiles—I just use 5-tiles to support students to think about multiples of 5. We know informal settings are important in mathematics instruction, and the abstraction or horizontal mathematisation from visible material settings to bare numbers is a fundamental aspect of mathematics learning (Gravemeijer, 1991; Gravemeijer & Stephan, 2002; Pirie & Kieren, 1994).

Notation has some similarities to setting in its role in instruction. Some notation can be considered an instructional device to support students’ thinking about the mathematics, as is the case, for example, with the parallel expressions notation in the High 5s Sequence. A notation can be a context in which to pose tasks, as a setting can be a context for tasks. Also, the progression from informal to formal notation has a quality of abstraction similar to the progression from visible settings to bare numbers. Indeed, I have observed that in this domain, these two progressions can be closely interwoven in instruction. However, formal notation is also an important part of the curriculum to be learned, in a way that settings are not. Students do need to learn to use the × and ÷ symbols.

The dimension of structuring and strategies is different again. The student’s structuring of number relationships, and their development of sophisticated computation strategies, may not be identified as objectives in a bare mathematics curriculum. However, we can propose these are major instructional aims, underpinning the standard goal of “learning the multiplication basic facts”. On the other hand, the teacher cannot directly adjust a student’s structuring or strategies. In this respect, the fifth dimension is distinct from the first four: the teacher can directly adjust the range, the orientation, the setting, and the notation used in an arithmetic task, but they can only adjust how they direct attention to the structuring and strategies used. This dimension is often involved in reflections and follow-up tasks after a basic arithmetic task, rather than in the original task.
Given these major differences, gathering these aspects of instruction under one rubric of “the five dimensions” could be unwieldy. So, I emphasise that I do not label them all dimensions to pretend they are the same kinds of activity or material. I label them dimensions of instructional progression because each of them affords significant instructional adjustments. Each, in different ways and for different reasons, is a significant dial for calibrating instruction, for fine-tuning instruction to a student’s cutting edge. Furthermore, I have found that there can be significant coordination between these dimensions, different though they are, in the calibration of instructional progressions. That is, when trying to advance one of these dimensions, other dimensions may be strategically adjusted or maintained to support that advancement. So the five dimensions work together. Using these five dimensions, I find I can characterise a great variety of instructional progressions that arise in this domain. This is what I seek: a small set of dimensions for characterising responsive instructional progression.

9.1.4 Comparison with dimensions developed for earlier domains
The Mathematics Intervention Specialist Project (MISP) has developed instructional designs for earlier domains of arithmetic instruction, using multiple dimensions. As described in the literature review, the aims of the present study arose in part from seeking to emulate the success of those instructional designs for the domain of Multiplicative Basic Facts. There is also an interest in elaborating an understanding of how to identify dimensions, and how to use them in instructional design. So, it is appropriate to compare the dimensions developed for this domain with those developed for the earlier domains in MISP.

9.1.4.1 Conceptual Place Value
The domain of Conceptual Place Value (CPV) involves flexibly incrementing and decrementing numbers by 1s, 10s, and 100s. Learning CPV serves as a foundation for the development of mental strategies for multi-digit addition and subtraction. The instructional design for CPV is organised around three dimensions of progression: (a) extending the range of numbers; (b) making the increments and decrements more complex; and (c) distancing the setting (Wright et al., 2012, pp. 80-81).
The setting dimension (c) is similar to the SET dimension in the present multiplicative framework, and indeed my use of the setting in the teaching experiment evolved in part from my experience with the use of settings in CPV instruction.

Both domains have a dimension of range. The progression in range (a) in CPV is a simple extension of number size: the range to 100, to 1000, and beyond 1000. For the RNG dimension in the present multiplicative framework, the scheme of ranges is more refined, and the potential adjustments are more intricate, to attend to the richness in multiplicative structure of different factors and products.

The CPV dimension of increasing complexity (c) does not have an equivalent in the present framework. However, some complexity adjustments in CPV are similar to ORN adjustments in the multiplicative framework. For example, the CPV complexity dimension includes the switch from incrementing to decrementing, which could be considered an adjustment of additive orientation. Likewise, the CPV complexity dimension includes the shift to posing unknown increments—*There is 87: how many more to make 100?*—which is a shift in orientation regarding which elements in the task are known and unknown.

CPV does not have a separate notation dimension—notation in the CPV instruction is described within the progression of the setting dimension. Finally, CPV does not have an STR dimension.

### 9.1.4.2 Structuring Numbers to 20

The domain of Structuring Numbers to 20 involves addition and subtraction in the range 1 to 20. The aim is for students to structure numbers as additive combinations and partitions, and to develop facility with related mental computation strategies that do not involve counting by ones. The Structuring Numbers domain and the multiplicative domain share important parallels. Structuring Numbers aims to cultivate the structuring of additive relations, and the development of advanced additive strategies; the multiplicative domain aims to cultivate the structuring of multiplicative relations, and the development of advanced multiplicative strategies. Structuring Numbers
ultimately addresses facility with the additive basic facts; the multiplicative domain with the multiplicative basic facts.

The instructional design for Structuring Numbers to 20 is described using four dimensions of progression: (a) from key combinations to all combinations; (b) from lower range (1–5, 1–10) to higher range (1–20); (c) from additive to subtractive tasks; and (d) distancing the setting from visible to bare number (Wright & Ellemor-Collins, 2018). Both (a) and (b) are progressions in range. The key additive combinations in (a), such as doubles and partitions of 10, have some parallel in the multiplicative domain. For example, it is recommended that the multiples of 2 and of 10 be learned early, as other multiples are often derived from them. However, such key multiplicative products are less clearly identified than the key additive combinations. Likewise, the progression in additive range (b) is more clearly from lower numbers to higher numbers, than the progression in range in multiplication, where, for example, the multiples of 6 may well be addressed later than the multiples of 9 or 10. Further, the RNG dimension has more ranges to progress through. Instead of the two clear progressions in the addition instruction, the single RNG dimension developed here for multiplication is a richer notion, incorporating various options for adjusting the multiplicand, the multiplier, odds and evens, and so on.

The progression (c) from additive to subtractive tasks corresponds to the ORN dimension. ORN has more options: switching between multiplier and multiplicand, and between related units such as number of tens $T$ and number of fives $F$. The progression in setting (d) corresponds to the SET dimension. Again, there is more potential complexity in the adjustments for multiplication, as the settings can be structured multiplicatively, such as organising the 5-tiles into pairs to make 10s.

As with CPV, Structuring Numbers 1 to 20 has not been described with a separate dimension for notation. There is less emphasis on notation in this domain, and the use of notation can be sufficiently explained within the setting dimension. Likewise, there is not a dimension for structuring and strategies. The domain certainly involves attention to structuring and strategies, but that hasn’t been described as a dimension of progression, more
as part of the teaching procedures, or part of the general approach to one-to-one teaching.

In summary, there are five dimensions for organising the multiplicative domain, compared with three dimensions for CPV and four for Structuring Numbers to 20. Comparing the proposed five dimensions with the earlier dimensions:

- RNG has correspondences, with more levels and subtleties.
- ORN has some correspondences, but more complexity.
- SET is similar, with more potential complexity.
- NTN is determined to be its own dimension.
- STR is newly identified as a dimension.

This comparison makes sense. The multiplicative domain is more complicated mathematically, so having more complicated range and orientation is appropriate. The domain is also more mathematically advanced, so makes more use of notation. Finally, I have argued for including attention to structuring and strategies as a dimension now, to more clearly characterise the instructional progressions.

9.2 Aim 2: Describing Progressions in Each Dimension

Having identified five key dimensions, the second aim of the study is to describe potential progressions in each dimension. This aim draws on the Layer D analysis in the nested case studies. The Layer D analyses clarified the main ways the instruction progressed in each dimension, and also offered insight into subtle aspects of these progressions. Below, I draw together a coherent description of instructional progression in each dimension. Developing these rich descriptions of each dimension makes a significant contribution to the detail of the instructional design.

9.2.1 RNG: Progressions in the range dimension

The main progression in RNG is the progression through the scheme of Ranges 1 to 4, from lower, simpler factors through to higher, more complex factors. For the High 5s Sequence studied, the RNG was kept steady on multiples of 5, while the term of instruction overall made the progression from Ranges 1 to 4. Two finer progressions used within a single basic range
were a progression in parity—in this case, adjusting between even and odd multiples of 5—and a progression in size—in this case, adjusting between low 5s, high 5s, and multiples beyond 50.

9.2.1.1 RNG: Progression through the main scheme of ranges

RNG can progress from Range 1 to Range 4 in the revised scheme of ranges (see Table 3.6). This is a broad, slow progression which I used as the main organiser for the instruction in the multiplication and division domain overall. Other instructional plans for multiplication and division also organise instruction around fact families in various ways (e.g. Anghileri, 2006; Van de Walle, 2012). The case analysed here involved a sequence of lesson segments which remained steady on this dimension, devoted to instruction in one range of products within Range 3, the high 5s. In the last lesson segments of the sequence, the RNG advanced to include all Range 3, high 3s and 4s as well as 5s.

In the analysis of the High 5s Sequence, I argued that this focus on a single range of products was effective. The $N$-$F$-$T$ relations and $N$-$F$-$T$-based strategies are unique to the multiples of 5. For example, the $N$-$F$-$T$ strategies tend to treat 5 as a multiplicand, and use relations of 5 to 10; by contrast, Blair’s strategies for high 4s treated 4 as a multiplier, and used relations to known doubles. With the RNG focus on high 5s, the instruction could support Blair’s structuring of $N$-$F$-$T$ relations, and $N$-$F$-$T$-based computation strategies. With the later advancement of RNG to include high 3s and 4s, Blair refined his knowledge of high 5s, such as by recognising products that could not be multiples of 5, and consolidated his knowledge of high 5s within a larger set of known products.

Note that outside the case study of high 5s segments, the lessons these segments are drawn from each had other segments addressing other ranges (see chart of Blair’s lessons, Figure 4.1). So, while this sequence maintained a focus on a single range, the instruction overall forged a progression through the ranges. Roughly, that progression across the term can be described thus: Consistently attending to several ranges within each lesson; attending to each range over an extended sequence of lessons—as with the high 5s case; progressing within each range from developing structuring and strategies
towards automatising basic facts; and progressing across the ranges from a focus on lower ranges to higher ranges and mixed ranges. This broad progression in RNG deserves further study for proper description. For the purposes of this thesis, it is important simply to put this short sequence of high 5s segments into the context of the broader progression in RNG.

9.2.1.2 **RNG: Progressions within one range**

Within the sequence focused on high 5s, an adjustment in RNG between even and odd multiples proved effective. The even and odd multiples can have distinct computation strategies. The analysis of Lesson 10 noted that the instruction established a focus on even multiples of 5, which continued in some segments of Lessons 11 to 15, pursuing the sustained attention on the $N$-$F$-$T$ relations of the even multiples. In other segments such as 11.3, 12.4, and 15.6, advancing RNG from even to odd multiples challenged the cutting edge of Blair’s developing facility, and led to the development of new strategies such as neighbouring multiple strategy.

Another progression within the range of high 5s was in size, adjusting RNG between low 5s, high 5s, and beyond 50. Anghileri (2006) recommends extending instruction beyond the basic facts range while developing multiplicative strategies. The Lesson 10 analysis suggested that the extension in RNG beyond 50 initially challenged Blair’s cutting edge with his new $2n \rightarrow F$ strategy, but ultimately supported the consolidation of this strategy. The High 5s Sequence analysis showed that the extension of RNG beyond 50 continued in later segments to serve in similar fashion, furnishing sufficient examples of even multiples for Blair to establish his new multiplicative strategies. At the other end of the dimension, retreats to low 5s could serve as introductory tasks, or recovery tasks after an impasse.

9.2.2 **ORN: Progressions in the orientation dimension**

The ORN dimension incorporated various adjustments to which values were known and unknown in a task. A multiplicative situation typically involves three unit counts: a multiplier, a multiplicand, and a product. The instruction can choose which unit counts are given in a task, leading to three main orientations: giving multiplier and multiplicand—*I have eight 5s, how many altogether?*; giving product and multiplicand—*I have 40 altogether, how
many 5s?; or giving product and multiplier—*I have 40 altogether made with eight tiles, which tiles am I using?*. Related units can also be counted, such as counting the number of 10s and the number of 5s in the same situation. The instruction can also adjust which unit count is sought, as in the early task type of incrementing and decrementing by 5-tiles, switching between asking “how many dots?” and “how many 5s?”, or later tasks giving *N* and switching between “how many 5s?” and “how many 10s?”. Awareness can also be given to the switch between the roles of multiplier and multiplicand. The instructional progression in high 5s maintained 5 as the multiplicand, until later bare number tasks allowed the interpretation of 5 as a multiplier. Below I discuss further the issues of switching orientations, intertwining multiplication and division, and advancement in the ORN dimension.

9.2.2.1 ORN: Switching between orientations
Instruction can choose one orientation to support learning about that orientation. However, there was also evidence in the study that tasks in one orientation can support learning about other orientations. For example, in Segment 10.5, the tasks were posed with an orientation $F \rightarrow N$, but they led to Blair’s $2n \rightarrow F$ insight for a different orientation $N \rightarrow F$. Likewise, Blair’s experience using the $2n \rightarrow F$ insight to solve $N \rightarrow F$ tasks in Segments 10.6 and 10.7 appeared to inform his reasoning for solving Task 10.7.5 in the inverse orientation $F \rightarrow N$.

Furthermore, instruction can switch ORN back and forth between orientations to promote learning about the whole multiplicative relationship of unit counts. I argued that, in the context of the switching of ORN in Segment 10.7, Blair began to work with the $N$-$F$-$T$ relations as a whole. More broadly, the instructional progression through the high 5s segments in Lessons 10 through 15, which involved considerable switching of ORN, appeared to support Blair’s clarification of the $N$-$F$-$T$ relations overall.

9.2.2.2 ORN: Intertwining multiplication and division orientations
In particular, instruction can adjust the ORN dimension to develop learning of division strategies alongside multiplication strategies within a given range. Blair’s learning of division and multiplication of high 5s was closely intertwined. His first multiplicative strategy, the $2n \rightarrow F$ insight arising at the
end of 10.5, was in division orientation, which went on to support the development of strategies in multiplication orientation. In turn, Blair’s knowledge of high 5s multiplication products supported his thinking about bare number division tasks. The instruction made choices in ORN, switching from multiplication to division, and from division back to multiplication, to facilitate the intertwining of this learning. These switches were frequently sources of valuable challenge and problem-solving for Blair. These observations are in accord with Downton’s (2008) conclusions that instruction can bring division strategies in close connection with multiplication strategies, and with Fosnot and Dolk’s (2001a) recommendation that “using multiplication and division interchangeably is powerful as long as children are clear what (the whole or which part) they are trying to determine” (p. 57).

Such an intertwining of division with multiplication along the ORN dimension is a distinctive approach to the instructional progression. Some instructional approaches organise division and multiplication more separately. As a typical example, the ORIGO Education materials (Burnett, Irons, & Turton, 2007) have separate teaching guides and separate scope and sequence charts for multiplication number facts and division number facts.

**9.2.2.3 ORN: Advancement in orientation**

Unlike some of the dimensions, ORN does not have a standard direction of advancement. It may be expected that the multiplication orientation should come before the division orientation, but as described above, Blair ended up finding strategies for division tasks first. Likewise, there are different potential orders for learning strategies for a number as multiplier or multiplicand. In the range of high 5s, 5 was first treated as a multiplicand, whereas in the range of high 4s, Blair’s first strategies involved interpreting 4 as a multiplier. However, while I do not propose a standard direction of advancement for ORN; I have established that in a given task or lesson segment, one ORN may be more familiar or make more sense for the student than another. With the teacher’s awareness of this context-specific difference in orientations, shifts in ORN can advance or retreat the level of challenge for the student.
Furthermore, handling only one orientation is generally easier than switching or combining orientations. At Task 10.6.5, the introduction of finding $T$ as well as $F$ became suddenly more challenging, and subsequent switches between finding $N$, $F$, and $T$ challenged Blair to clarify the distinction between the three units, ultimately leading to deeper insight about the $F$ and $T$ relations through Segment 10.7. The sequence of lesson segments overall progressed from segments addressing a single main orientation, to segments switching orientations, in which Blair increasingly related to the whole set of $N$-$F$-$T$ relations as a single entity. So, I propose the sense of advancement for the ORN dimensions overall, is from single orientations to switching and coordinating orientations, seeking to comprehend the whole set of relations.

9.2.3 SET: Progressions in the setting dimension

The study confirmed that a setting was valuable in the instruction, and that the distancing of the setting toward bare number tasks was an important progression in the instruction. The Layer D analyses of the Lesson 10 Sequence, and of the whole High 5s Sequence, reported on five aspects of SET:

- the setting as a context for reasoning about tasks;
- the setting supporting multiplicative structuring;
- the distancing progression in the SET dimension;
- uses of screening and unscreening; and
- the setting as a context to return to from bare number work.

In this section, I discuss the first three aspects. The analysis of the last two I incorporate into a fourth aspect:

- the recursivity of the progression in SET.

These four aspects help me articulate the texture of this dimension of instructional progression.

9.2.3.1 SET: Setting as a context for reasoning about tasks

The setting had a role in the instruction as a context for making sense of the tasks and the calculations. Throughout Segments 10.5 and 10.6, and again in 11.3, the tasks were posed in terms of the tile setting, and there is evidence that Blair understood the tasks as being about the tiles: he gave answers in
terms of the tiles and dots, and treated counts of the actual tiles and dots as verifying the answers. With this role of establishing the sense of the tasks, the instruction spent time with the setting in the early segments, and in early tasks within each segment. I understand this use of the setting to be in accord with the context principle of RME instruction, that learning needs to be anchored in a context that students can imagine and make sense of, as reviewed in Section 2.2.2.

The progression also involved recalling the tile setting in later tasks, when Blair appeared to have lost his sense of the tasks. In these episodes, the recollection of the tiles sometimes appeared to support Blair to regather his sense of the tasks, and to use reasoning to find an answer. These returns sometimes came after a short time within a segment, for example, in 10.6.5 with 90 dots, after his major block with finding $T$, when it was only when the instruction returned to visible tiles that Blair was able to resolve the task. The returns could also come after a longer time in later segments when the tile setting had become less prominent. For example, in Segment 16.3 finding $T$ for eight 5s (*I've got eight of these guys*), and in Segment 17.1 solving $35 \div 5$, an instructional gesture to the tiles appeared to initiate a breakthrough for Blair.

Gravemeijer (1997, 1999), in his theory of emergent modelling, describes how a student’s thinking might return to an earlier less abstract image or context, as part of reasoning with a more abstract model of a task: “Discussions on the level of general activity and of formal mathematical reasoning frequently fold back to referential activity or even to activity in the setting” (1997, p. 163). McClain and Cobb (1998) and Pirie and Kieren (1994) make similar observations of students’ folding back to support their reasoning. I think in these episodes of returning to the tile setting, Blair’s success can be explained as drawing on his earlier modelling of the tiles. In which case the instruction prompting the connection to the tiles is supporting his reasoning, and supporting his attempts to develop from his model of the tiles, a model for more abstract reasoning about the numbers.

While the setting could support the student’s sense-making, it could also hinder it. For example, in Segment 15.5, with Blair trying to explain why his
2n → F strategy works, his use of the tiles appeared to confuse him. In a sense, this incident highlights some of the sensitivity of settings in teaching mathematics. Though it was in the paired 5-tile setting that Blair first had his 2n → F insight (10.5), trying to use the setting to justify the strategy demands a subtle structuring of the tiles, and the strategy may be easier to actually explain without using the tiles. In general, our early sense-making of a phenomenon is likely to be context-dependent, but our clearest sense of a structure or pattern may come precisely as we develop a more abstract, context-independent understanding (Sfard, 1991). Instructional progressions do well to be sensitive to both sides of the setting coin.

9.2.3.2 SET: Structuring in the setting
The analysis suggests that the instruction could work with the tile setting to support the student’s structuring of the multiplicative units and multiplicative relations. In Segments 10.5, 10.6, 10.7, 11.3 and 11.4, the instruction used adjustments to unscreen tiles to challenge Blair to distinguish the different units of 1s, 5s, and 10s, and to organise how to count N alongside counting F and T. In some episodes, the SET adjustment involved explicitly showing just a single 5-tile and saying, That’s one 5 there, or showing a pair of tiles and saying, Here’s one of my 10s, and Blair could then organise his thinking about 5s and 10s. Challenges to visualise the arrangement of tiles while keeping SET screened were also productive, as when Blair structured 60 as a double row of three pairs of 5-tiles. Overall, the strategic ratcheting back and forth along the SET dimension by screening and un screening the setting, had Blair at his cutting edge, thinking hard about the structuring of 1s, 5s, and 10s.

The setting also enabled structuring the distinction between multiplicand and multiplier. The early segments on high 5s, in the setting of 5-tiles, all treated 5 as the multiplicand. When the setting shifted to bare number, this role could be renegotiated. For example, in Segment 15.6, for the bare number task of 5 times 9, Blair asked “You want me to do 9 times 5 or 5 times 9?” I replied “5 times 9”. But he still solved using knowledge of multiples of 5: five 5s plus four 5s. Of interest in this regard was an episode in Lesson 18, after the High 5s Sequence, when Blair appeared to interpret some division tasks as quotition and others as partition, to suit relating the task to known multiples.
of 5: for example, 30÷6 was solved as six 5s (partition), while 35÷5 was solved as seven 5s (quotition). Gray and Tall (1994) emphasise that this ability to treat the meaning of a sign—in this case the verbal term “divide by”—as ambiguous, and to interpret the meaning to suit one’s purposes, is critical to mathematical activity.

9.2.3.3 SET: Distancing the setting

The basic progression in setting advances by distancing the student from the materials: progressing from visible materials, to screened materials, to bare number tasks. This progression in distancing occurred in the case study over the High 5s Sequence. Lessons 8, 10, and 11 involved the 5-tile setting, visible at first but increasingly screened. Later segments were largely in bare numbers. The distancing progression was fundamental to the success of the instructional progression overall. In the setting of the visible and screened tiles, Blair had several breakthroughs in his structuring of $N\times F\times T$ relations and his development of multiplicative strategies. Yet the instructional aim was for facility in a bare number context, so the tiles needed to be removed somehow. As several moments in the analysis revealed, the transition of his new insights from a setting of tiles to bare number tasks was not immediate for Blair. Instead, the instruction made a careful progression along this SET dimension, supporting his eventual success in the bare number context.

The progression of distancing the setting can involve subtle adjustments. The close analysis of the Lesson 10 Sequence revealed a range of subtle adjustments: screening; partial screening; flashing the whole tile arrangement; flashing a single tile; adjusting the screening when posing tasks, and while working on tasks; using the setting in checking answers by unscreening; counting tiles; re-arranging tiles; challenges to imagine or describe the setting; and shifts to less setting-dependent terms, such as shifting from “cards” to “5s”. This gradual instructional progression in SET is in accord with the findings of Sullivan, Clarke, Cheeseman, and Mulligan (2001) “that students will develop more robust conceptualisations of multiplication and division if teachers pose problems that gradually but explicitly remove physical prompts or supports, and encourage students to form mental images, in multiplicative situations” (pp. 233–234).
9.2.3.4 SET: A recursive progression

The progression in SET did not happen once only. In the Lesson 10 Sequence, the SET dimension was adjusted within most tasks, from screened back to visible and forward to screened again. On the scale of lessons, the instruction in high 5s made some form of progression from tiles to bare numbers in each of Lessons 8, 10, 11, 14, 15, and 16. On a longer scale again, the whole High 5s Sequence made a progression from tiles to bare numbers. Then, when the RNG shifted to mix high 5s with high 3s and 4s, the instruction returned to the tile setting again in Segment 17.2. Furthermore, over the scale of the whole term of teaching, the instruction in ranges other than the high 5s also made progressions in SET. In general, the tile setting is used to establish early reasoning and structuring, but is also returned to repeatedly to refine, consolidate, and extend that reasoning and structuring. So there is a progression in SET from tiles to bare numbers, but that progression recurs multiple times over multiple scales in the instructional progression. I describe the progression in SET as \textit{recursive}. The recursivity is evident in the zig-zagging movement down the SET column on the analysis charts.

This recursivity of SET is a significant feature in the texture of the instructional progression. By contrast, for example, the MR learning framework (Wright, Martland, & Stafford, 2006) proposes a single hierarchy of levels for the whole multiplicative domain, advancing from counting strategies in a setting through to facility in bare number tasks (see Section 2.4.3). The evidence of the case study here suggests that instruction may follow a different design, making this progression from visible setting to bare numbers many times within and across different ranges of multiples.

9.2.4 NTN: Progressions in the notation dimension

Notating and notations can be powerful contexts for attending to multiplicative relations, as argued in the analysis in Section 8.1.4. A basic progression in the NTN dimension is from no notation, to informal notations, to formal notation. This NTN progression was evident in the case study over the sequence of high 5s lesson segments, and it made a significant contribution to the instructional progression overall. The analysis has revealed several potential progressions within this dimension.
9.2.4.1 NTN: Developing informal notations

Informal notation can be introduced in an unplanned, experimental way, responding to what has occurred in a task. It may then simply remain as a one-off instance of notating, which can be constructive, as in the recording of Blair’s insight about array structure after Task 10.5.1, or of his new $2n \rightarrow F$ insight at Task 10.5.6. Alternatively, the instruction can pursue using the new notation over subsequent tasks, as happened with the development of a sequence of arrow sentences in 10.7. Building on the success of the arrow sentences, in 11.4 I adapted the arrow sentence notation to develop the parallel expressions notation with the relations of $F$ and $T$ more clearly aligned, and this led to the notation-based tasks of the next few lesson segments.

Thus, the progression in NTN may involve introducing, refining or changing a notation to suit the instructional aims. Also, informal notation may be introduced, not as a way station toward formal notation, but to support a progression in structuring, or another aspect of learning. Nevertheless, having introduced an informal notation, a transition to conventional notation may be necessary, as discussed below.

9.2.4.2 NTN: Progressions in the role of notation

Another progression in the NTN dimension can involve shifting the role of the notation in the tasks. In the case study, the role of the informal arrow sentence and parallel expressions notations evolved subtly over the lesson segments, as analysed in Section 8.1.4.2.

Notation can be used simply to record results already found; later, this record provides an opportunity to reflect on a result (*90 is nine 10s. Is that surprising?*), or on a pattern through a sequence of results, as happened at the end of 10.7 (*Why is it?*).

Notation can have a role in supporting the calculating of answers, as Blair used the arrow sentences in 10.7, and the parallel expressions notation in 11.4. Screening previous lines of notation to challenge the student’s reasoning (10.7, 12.6) was an instructional adjustment in this role of the notation. Also, when a notation is not in current use, instruction can recall it temporarily to help support the student’s reasoning, as when the parallel expressions...
notation was recalled to help resolve Blair’s impasse in Segment 12.6 (“Oh! I forgot about my own strategy!”).

Notation can come to stand for the task, as when Blair developed a practice of completing a parallel expressions notation when given the value of any part. Notation can also come to stand for the strategies developed in the notation, as when Blair pointed to some parallel expression notation saying “I did that”, to mean he’d used his \( 2n \rightarrow F \) strategy. Furthermore, notation can be used to support more general discussions about strategies and structure, as in the 15.5 discussion when Blair ended up annotating parallel expressions with a cycle of arrows (see Figure 7.7b).

9.2.4.3 **NTN: Supporting the student's increasing independence with notation**  
Another progression in the NTN dimension can involve increasing the student’s independence with notation. The analysis of Segment 10.7, for example, revealed a subtle transition from teacher writing to student writing including:

- teacher writing whole notation sentence;
- teacher indicating where to write each part of the sentence, and student writing;
- teacher writing one part, leaving student to complete the sentence;
- teacher asking student to write the whole sentence;
- teacher asking student to write the whole sentence, while screening previous lines of notation so the student cannot see earlier examples; and
- student taking own initiative to write the sentence.

Several segments involving notation involved some form of this progression, with the teacher initiating the use of the notation, then increasingly handing over to the student. Reaching a point where the student can take initiative with notation may support the learning from the notation, as when Blair took initiative to recall the parallel expressions notation in Segments 12.6 and 15.5, helping his progress in those tasks.

9.2.4.4 **NTN: Progressing to formal notation**  
The progression to conventional number sentence notation can be negotiated strategically. Segments 15.3 and 15.4 involved a sequence of tasks
progressing from the parallel expressions notation to using number sentences. The first tasks recalled the informal notation for tasks in different orientations. Next, multiplication tasks were introduced using number sentence notation. Then, a task in division orientation was posed, and the student was asked to express the task with a number sentence, which he did as a missing multiplier sentence. This form of notation was repeated. Finally, the division sign was introduced as an alternative for expressing the same task. Blair’s hesitations in accepting the division sign here are evidence that this translation cannot be taken for granted. Significantly, he managed to carry on using the division symbol correctly for subsequent tasks, while continuing to use the multiplicative strategies he had developed in the context of the earlier informal notations. This transition of his strategies into the formal context was essential to the success of the instruction overall in achieving the aim of Blair using multiplicative strategies for formal bare number tasks. The attentive progression from informal to formal notation supported that transition.

The development of notating through the High 5s Sequence overall was reflexively related to the development of Blair’s reasoning: clarifying units led to the arrow sentence notation, which led to insights into N-F-T relations, which supported the parallel expressions notation, with which new tasks and strategies were established, which in turn provided a context for the introduction of conventional notation. Such reflexive developments have been analysed using the theory of emergent modelling by Gravemeijer and Stephan (2002), and in related studies of chains of signification, such as by Cobb, Gravemeijer and colleagues (1997). While I have not made an analysis in terms of emergent models, I find the narrative of the NTN progression accords with their conclusions, that “the use of symbols as tools supports the development of meaning, while at the same time this new meaning lays the basis for the introduction of new symbolisations in a meaningful way” (Gravemeijer & Stephan, 2002, p. 163).

9.2.5 STR: Progressions in the dimension of structuring and strategies

The STR dimension is frequently involved in the instruction. In the study, more than half of the tasks have a comment or sub-task labelled as STR. However, the progressions in the STR dimension are less clear than in the
other dimensions. As explained in the section above on identifying the five dimensions, STR is a distinctive dimension, in that structuring and strategies are not amenable to a teacher’s direct adjustment: what the teacher can adjust is actions and words to direct the attention to structuring and strategies. Because of this, the sequence of STR instruction becomes more elusive. For example, the teacher might not need to ask about the student’s strategy in a task if the student reveals his strategy in his speech and actions, and the teacher might not need to challenge the student to generalise, if the student takes initiative in generalising unprompted. Also, adjustments in STR are often associated with adjustments in other dimensions, a characteristic identified in the Layer E analyses of interactions between dimensions. For these reasons, I leave the STR dimension in a less defined form than the other four. Nevertheless, I find it helpful to recognise a sense of progression in STR, and use it as part of the richer description of the instructional progression overall.

A basic characteristic of the instructional progression has been the repeated attention to STR. I relate this frequent attention to the teaching role described by Davis, Sumara, and Luce-Kapler (2008): “The teacher acts as the consciousness of the collective—selecting from and orienting towards the interpretive possibilities that are presented” (p. 203). The analysis described examples of how the repeated attention to structuring and strategies—in this case, to the units of 5 and 10, to the relations between $N$, $F$, and $T$, and to multiplicative strategies for high 5s—supported Blair’s attention to the same aspects of the mathematics. Examples included episodes where the STR instruction challenged Blair to find a closer multiple for a neighbouring multiple strategy, and then in later tasks he continued to look for closer multiples. Other examples suggested a layering of STR instruction as a background to Blair taking initiative with noticing a pattern in the notation. Blair was learning that these questions were valued in the class: how a result was calculated, how the tiles were visualised, how to distinguish the 5s and 10s, what patterns appeared in the notation, and so on, so he sought such questions himself. The review of literature on teacher attention in Section 2.2.5—which examined the work of Wheeler, Mason, Jacobs and others—flagged this potential effect. As Erickson (2011b) emphasised, building on
Dewey’s insights before him, “What teachers notice … ends up affecting what students notice … in an ecosystem of mutual influence” (p. 22). The instructional practice of recurring attention to STR can cultivate a shared practice of attention to STR.

Furthermore, a progression is discernible through different types of STR instruction, along the following lines:

- enquiring about the student’s thinking and affirming his thinking;
- prompting the student to recall or consolidate his earlier thinking;
- challenging the student to improve his structuring or strategy; and
- challenging the student to explain a relationship or strategy.

A progression like this was traced through the STR example in Lesson 10, and a similar progression was traced through the whole High 5s Sequence. These progressions in the STR dimension reflect the sense of the instructional progression overall, which moved from observing and affirming Blair’s new multiplicative strategies, through supporting him to consolidate and clarify his reasoning, to challenging him to see new relationships and develop better strategies. I see these STR interactions as being closely linked with the aspiration to highly responsive instruction, as described in Section 2.2: teaching with attention, teaching at the cutting edge, and teaching towards mathematisation.

### 9.2.6 Relative variability in the progressions in each dimension

I have described potential progressions in each of the five dimensions. In the course of a sequence of segments, the instruction makes adjustments along these potential progressions, in all five dimensions. Some dimensions tend to get adjusted more often than others. Below, I reflect on the relative variability across the five dimensions. These reflections serve as a summary of the progressions in the five dimensions.

RNG is the steadiest dimension. There is typically a focus on one basic range within a lesson segment, and over a sequence of lesson segments. However, there are also subtler adjustments within RNG, such as revisiting lower or more basic ranges, and making forays into higher or more complex ranges, and these can be varied within a single segment.
ORN is a flexible dimension. ORN can be held steady over a sequence of tasks. This happened early in the learning of a topic, as in Segment 10.6, and late in a topic, as in rehearsing division basic facts together. But ORN was not held steady over more than one lesson within a range: alternative orientations were soon posed. At times, shifts in orientation can become a focus of instruction, and the ORN is adjusted at almost every task, or even within tasks. Connecting division to multiplication and back happened repeatedly in the lesson sequence.

SET can be the dimension that is adjusted most often. When there is a material setting involved, the basic technique of adjusting from screened to unscreened tiles can be effectively used for almost every task in a segment. On the other hand, when segments are posed in bare numbers, the SET dimension may remain steady for a whole segment. Still, the material setting may be invoked occasionally.

NTN is the second-steadiest dimension. In one sequence of segments, NTN can make a broad progression from absent, to informal, to formal notations. Within a single segment, NTN is likely to be relatively steady, or perhaps work through one main shift. However, within that broad formalising progression, there are subtler adjustments possible, as there are with RNG.

STR is a distinctive dimension, for which I have offered a sense of adjustment that is less defined than for the other four dimensions. The STR dimension was involved in more than half the tasks, and adjustments in the type of STR instruction occurred in most lesson segments. So, overall, STR was active and varied, but not as variable as SET or NTN.

In summary, I have developed a set of five dimensions, a sense of the potential progressions in each dimension, and a sense of the relative variability across the five dimensions. This is already a useful account of the instructional progression, which can serve instructional design. The next section discusses how the instructional progression can be understood as a coordination of all five dimensions.
9.3 Aim 3: Characterising The Instructional Progression Using the Dimensions

The third aim of the study is to characterise the instructional progression as a whole, using the dimensions. So far in this discussion, I have described progressions in each dimension separately. I turn now to describing how the dimensions are coordinated and interwoven to enact the instructional progression as a whole. This aim draws on analysis Layer E, which described interactions between particular dimensions observed in the case studies, and especially on analysis Layer F, which developed the multidimensional accounts of the case studies.

Highly responsive inquiry-based instruction can be convoluted and idiosyncratic. The aim here has been to find ways, despite the convolution and idiosyncrasy, to characterise how this instruction can progress from task to task, and lesson segment to lesson segment. The approach is to express the instructional progression in terms of adjustments in the five dimensions, drawing on the Layer F multidimensional accounts. This discussion is the culmination of the thesis, trying to find a simple, coherent characterisation of the instructional progression.

9.3.1 Instructional progression as an interwoven calibration across the five dimensions

I propose a basic characterisation of instructional progression as:

*a strategic, interwoven calibration across the five dimensions.*

That is: the progression of instruction tends to involve subtle, fine-grained calibrations in all five dimensions; the calibrations are interwoven and coordinated across the five dimensions; and the calibrating is strategic in seeking the cutting edge of the student’s thinking, and directing attention toward the instructional aims. I will develop this characterisation over the following sub-headings.

9.3.1.1 Instructional progression involves advancing and retreating on the dimensions

Each dimension can offer directions to advance or retreat. These directions may not be consistent across all segments. In ORN for example, in one segment shifting to the $N \rightarrow F$ orientation may increase the challenge, but in another segment shifting to $N \rightarrow F$ may decrease the challenge. The sense of
advance and retreat is relative to the current tasks and the student’s current thinking. Nevertheless, for a particular student and a particular task, I have described potential directions for progression in each dimension. Throughout the case study analysis, each step in the instruction could be interpreted as responding to the student’s latest activity, by advancing or retreating the level of challenge for the next task or sub-task. These adjustments in the instruction were created, task by task, by strategic calibrations of the five dimensions.

9.3.1.2 Instructional progression adjusts one or two dimensions at a time
The instruction tends to progress by adjusting only one or two dimensions at a time. This characteristic was evident in the analysis of the multidimensional accounts, especially the task-by-task account of the Lesson 10 Sequence. When I made an adjustment along one dimension, I tended to hold most of the other dimensions steady. When I did change another dimension, it was often in the next sub-task—perhaps unscreening the tiles, or posing an STR follow-up reflection.

The alternative, advancing multiple dimensions at once, can run into trouble. An example is Segment 8.8. After an initial task with visible SET, I shifted to tasks with SET advanced to bare numbers, NTN advanced to a table of formal number sentences, and RNG rapidly advanced through high 5s to beyond 50. When I sought STR discussion of relations, with switching ORN, Blair appeared to lose his bearings, struggling to recognise the relationship between $F$ and $T$ values. One way to characterise the segment is that I advanced on too many dimensions too quickly.

Another example was Segment 12.5. Following Blair’s success in 12.4, I switched the ORN, advanced the NTN from informal notation to number sentences, kept the SET advanced at bare numbers, kept the RNG advanced at beyond 50, and included odd multiples as well as even ones. Blair succeeded with the tasks, but he did not continue to use the neat $N$-$F$-$T$-based strategy he had been developing the previous segments. When I pressed him to find that strategy (STR), it was our retreats in NTN and RNG in Segment 12.6, that supported his success. So, 12.5 could be characterised as an experiment with advancing hard on at least two dimensions at once, which proved a stretch for Blair.
9.3.1.3 Instructional progression involves multiple dimensions
While the instruction tends to only adjust one or two dimensions at a time, it rarely dwells on progress in only a single dimension for long. Instead, after some success in an advancement in one dimension, the instruction can switch to advancing a different dimension, or it can switch back and forth between two dimensions, and meanwhile, another dimension might be retreated to support the adjustments. Thus, the instructional progression overall involves adjustments across multiple dimensions. This characteristic is apparent in the analysis charts, when viewing down each coloured dimension column. No dimension calibration stayed consistent for long. Each lesson segment involved adjustments in most dimensions. If a dimension was relatively inactive for a segment, it typically became active in the next segment.

Furthermore, there is no clear standard order for switching between the dimensions being advanced. I cannot describe the instructional progression in the high 5s segments as advancing first in SET, then in NTN, then in ORN, and finally in RNG, for example. Instead, there was significant adjustment, back and forth, on each of these dimensions, over the sequence of segments. The instructional progression overall appears as an interweaving of adjustments across multiple dimensions.

9.3.1.4 Progressions in the dimensions are interdependent
The dimensions do not work independently. Rather, they are interdependent. When instruction adjusts in one dimension, the significance of this adjustment depends on the calibration of the other dimensions. This interdependence is revealed especially in the Layer E analyses of characteristic interactions between dimensions. Following are some examples. The effect of repeatedly switching the ORN in some segments depended on holding the RNG steady in the range of multiples of 5, so the different orientations were working on the same set of \(N-F-T\) relations. Advances in NTN to new notations were supported by switches in ORN, challenging the student to clarify his understanding of the new notation. Advances of SET to bare number made sense for the student in the context of coordinated advances of NTN with informal notation. STR questions seeking to clarify units required the context of the SET retreating to visible tiles. Other STR questions challenging the student to generalize a pattern required pointing to written records generated
by NTN persisting with a notation over a sequence of tasks. These examples reveal how the dimensions are interdependent aspects of a whole instructional progression.

### 9.3.1.5 Instructional adjustments can be subtle

Each adjustment on a dimension tends to be relatively small. The potential progressions described in each dimension in Section 9.2 often revealed subtle gradations: for example, in the shifts from even to odd multiples in RNG, in the distancing of SET, or the transition to formal NTN. Yet these adjustments, though small, still appear to challenge the student. Indeed, the accounts of the progression of the student’s learning revealed regular challenges and difficulties, moments of significant insight, and important progress in several aspects of knowledge.

I suggest that the adjustments needed to be small, to keep within the student’s cutting edge. In light of the difficulties he experienced, for example in recognising a relationship between $F$ and $T$ in Segment 8.8, in finding $T$ in Task 10.6.5, in transferring his $N-F-T$ strategies to a bare number context in 12.5, and in transitioning to number sentences in 15.4, I think he needed the small steps of this instructional progression. Small adjustments can actually support significant progress.

While each adjustment can be small, the interweaving of the adjustments across the five dimensions can be dense. The two multidimensional accounts of instructional progression developed in Sections 6.4 and 8.3 are rich tapestries of interwoven adjustments. Consider the multidimensional account of the Lesson 10 Sequence (see Section 6.4.2). Segment 10.5 involved mostly a steady RNG and ORN as the warp, with regular screening and unscreening in SET the interweaving weft. Following a significant STR response, Segment 10.6 shifted the ORN, and the RNG advanced soon after as well, while the SET frequently retreated to unscreened tiles, and the STR attention increased. Then a new ORN (find 7) prompted an engagement with different orientations, the introduction of informal NTN, and a temporary advancement of SET to bare numbers. This was followed by a flurry of adjustments in Segment 10.7. ORN shifted in every second task, and the new engagement with a steady NTN brought swinging advances and retreats in SET, crossing...
with retreats and advances in RNG. The final major STR task had all
dimensions reaching a relatively advanced calibration, and Blair reaching a
relatively advanced insight into $N$-$F$-$T$ relations: “You can fit two 5s into one
10…so it’s basically doubling that number”.

To capture all that is happening in these progressions demands a fine-grained
analysis. These progressions could not be planned in detail. Instead, they
appear as responsive, strategic, continually fine-tuned instructional
experiments, and the affordances for tuning are in the five dimensions.
Hence, the instructional progression is characterised as a strategic, interwoven
calibration across the dimensions.

**9.3.1.6 Interwoven calibration of dimensions is a compelling characterisation**
The instructional progression could be described in other ways. For example,
the range is determined by the syllabus, the orientation is determined by the
task type, the settings and informal notations could be considered
instructional aids, and the computation strategies are attended to by the
instructional style. However, I think that observing all of these as different
dimensions of progression helps reveal a significant characteristic of the
instruction: the interwoven calibration across all the dimensions. When any
one of these dimensions is adjusted, the calibration of the other four is
significant, and any of the other four may also be adjusted to support the
current instructional purpose.

Also, we could disregard the small adjustments in the dimensions, and try to
characterise the instructional progression in three or four phases: a setting
phase, an informal-notation phase, a formal notation phase, and a rehearsal
phase, for example. There could be value in identifying phases. However, the
analysis demonstrates that such phase descriptions would not capture the
dense, subtle, experimental progressions involved. The little retreats and
advances, the blurring of simple phase descriptions, were not just messy
edges; they were central to the success of the instruction. I find the
interwoven calibration of dimensions to be a compelling characterisation of
the instructional progression.
9.3.2 Features of the instructional progression
Beyond the basic characterisation of the progressions as interwoven calibrations of the dimensions, I have identified three features that help to further characterise the progressions: they are recursive, they are experimental, and they develop distinctive textures.

9.3.2.1 Instructional progressions involve recursive elaboration
The progressions in each of the dimensions are recursive. That is, a dimension does not just progress once and for all, but often progresses in little repeated cycles over a single topic, even within a single lesson segment. For example, in the SET dimension, the 5-tile setting was invoked and then distanced to bare numbers at some episode in each of the lessons in the case study. Likewise, the other dimensions are recursive. The RNG extends from low 5s to high 5s repeatedly. ORN connects from multiplication to division repeatedly. NTN shifts between informal and formal notations in several episodes. STR repeatedly moves from simply clarifying units to generalising relationships.

At the same time, the return in one dimension does not imply that the instruction is simply repeating, that the student is simply rehearsing known material. On the contrary, each time a dimension returns to an earlier calibration, the other dimensions can be at a different point, the task as a whole can be different, and the student can be bringing a different understanding to the situation. In this way, returning to an earlier calibration in a dimension can support the student in developing their understanding.

For example, the 5-tiles were first used when Blair was first structuring a multiplicative relationship between $F$ and $T$ (10.5). The tiles continued to be both screened and unscreened, as Blair consolidated his new strategy (10.6). The instruction returned to the visible tiles when Blair foundered on the new ORN, to find $T$ (10.6.5), then progressed to bare numbers as he established new fluency (10.7). The instruction returned to the tiles as Blair developed a new notation NTN (11.3–11.4). The tiles were briefly invoked then distanced as the notation and his strategies were consolidated (12.4, 14.3). The instruction returned to the tiles when Blair was challenged to explain how his strategies worked (15.5). Later, the tiles were invoked when RNG extended to
mixing high 5s with high 3s and 4s (17.1 and 17.2). Each return in the SET dimension was supporting new instruction, new learning.

A similar story could be told for each of the dimensions. The ORN switched to $N\rightarrow F$ in many episodes, with many different instructional purposes: to trial a new strategy, later to connect to finding new unit count $T$, later to develop a new notation, later to recall a strategy in the new SET of bare numbers, later to transition to formal NTN, and so on.

Pirie and Kieren (1994) identified folding back as a key characteristic of the progression of mathematical understanding. Referring to their model of expanding levels of understanding, they propose that:

- when faced with a problem or question at any level, which is not immediately solvable, one needs to fold back to an inner level in order to extend ones current, inadequate understanding. This returned-to, inner level activity, however, is not identical to the original inner level actions; it is now informed and shaped by outer level interests and understandings … This inner level action is part of a recursive reconstruction of knowledge, necessary to further build outer level knowing. (p. 69)

I propose that the responsive recursion in dimensions of instruction can support students’ folding back activity. Davis, Sumara, and Luce-Kapler (2008) recommend this kind of instruction as recursive elaboration: returning to earlier learning and elaborating it, then doing this again, and again. The instructional progression in the case study features such recursive elaboration. The recursion arises when the instruction retreats on dimensions by returning to familiar ranges, orientations, settings, or notations. The elaboration arises when the instruction advances on dimensions by extending the range, introducing a new orientation, distancing the setting, formalising some notation, refining a strategy, or explaining a relationship. Since there are multiple dimensions, an instructional progression can both retreat and elaborate. Thus, the account of the instruction in terms of dimensions helps reveal this feature of recursive elaboration.

### 9.3.2.2 Instructional progression can be experimental

A plan was prepared for each lesson in the case study. Several significant developments in the instruction arose from planning, including: returning to the 5-tiles at the beginning of the Lesson 10 Sequence; developing parallel
expressions notation in Lesson 11; shifting to conventional number sentences in Lesson 15; and extending the range to include high 3s, 4s, and 5s in Lesson 16. At the same time, the adjustments made in progressing from task to task are often spontaneous and experimental. The teacher has not prepared them, and does not know in advance how the student will respond. They are an attempt to respond in the moment to something the teacher notices in the student. Even the planned instruction is experimental in this sense. Yet such experimental adjustments can prove productive, and may lead to whole new productive episodes of instruction. So the instructional progression has in part the opportunistic character of experiment and improvisation. Following are some examples to illustrate.

At Task 10.6.5a, introducing the ORN sub-task of finding $T$ was a new experiment. Blair had shown increasing fluency over the previous four tasks, so seemed due for an extra challenge. I was interested in him linking $F$ and $T$, in what I have come to call shifts in the ORN dimension, so introducing the new sub-task was strategic. But I did not know it would prove so difficult for him. Nor did I anticipate the subsequent sequence of tasks I came to call Segment 10.7, centred on varying the ORN between $N$, $F$, and $T$. Yet this task type of varying ORN became a blueprint for productive tasks over several subsequent lesson segments.

Likewise, in 10.6.5c the NTN introduction of arrow sentence notation was a one-off experiment for the purpose of consolidating the hard-won results of that task. I was aware that notation can be significant in mathematics learning, and I thought it was worth trying here. Yet, it was only after the notation was completed that I considered continuing the notation in the subsequent task. Then, as Blair gave increasing attention to the notation, it grew in importance in the instruction. In each case, the initiative arose from an experiment on a dimension, leading to a whole new progression of instruction. Furthermore, I pursued these experiments because of my awareness of those potential dimensions of instruction.

Some roads are not taken, too. In Task 10.5.4, with nine 5-tiles, Blair indicated that he visualised the odd multiple of 5 “wouldn’t be even”. This was a promising visual structuring which I might have followed up soon after.
However, two tasks later at 10.5.6, Blair had his $2n \rightarrow F$ insight about even multiples, which I decided to pursue instead, with shifts in STR and ORN for Segment 10.6. This led to the entire productive remainder of the lesson, which was all about even multiples, while odd multiples were left to be returned to in later lessons.

So, the instructional progression cannot be characterised as a *routine* calibration of the dimensions. Rather, the instruction made *experimental* calibrations, through awareness of the potential adjustments in the dimensions.

In describing the aspiration to highly responsive instruction in Section 2.2, I drew on Davis’s (1997) notion of *hermeneutic* listening in teaching, which engages in “the messy process of negotiation of meaning and understanding” (Crespo, 2000, p. 156). Davis, Sumara, and Luce-Kapler (2008) identify that a key role for the teacher within this messy negotiation is to be “attentive to emergent possibility” (p. 203). I associate this role with the notion of recognising significant *pedagogical opportunities* (Leatham, Peterson, Stockero, & Van Zoest, 2015). I understand the experimental quality of the instructional adjustments in dimensions to arise from this role of attending to and responding to emergent possibility.

### 9.3.2.3 Distinctive textures in instructional progression

Lesson segments can have distinctive textures of instructional progression, which can be described in terms of the dimensions. A sense of these distinctive textures is visible in the Lesson 10 Analysis Chart (Figure 5.1). Segment 10.5, after the introductory first task, comprised a sequence of five tasks, each of which maintained the same RNG (high 5s), alternated the same two ORN (find $N$, find $F$), and alternated the SET from screened to unscreened tiles, with no use of NTN. The final task was distinctive, introducing incidental NTN and STR. In Segment 10.6, by contrast, the RNG began to vary between high 5s and beyond 50, the ORN shifted to $N \rightarrow F$, and STR was involved in each task. SET continued to alternate from screened to unscreened, and NTN remained unused. Again, the final task of the segment was distinctive. In Segment 10.7, NTN introduced an ongoing notation, RNG, ORN, and SET became more variable and varied over more calibrations,
while STR eased in frequency. Each of these three segments had its own instructional purpose, which was served by a different texture in the progression of the dimensions. The analysis of dimensions helps to reveal and articulate these different textures.

In summary, using the analysis of the dimensions, the instructional progressions can be characterised as a strategic, interwoven calibration of dimensions. Cycles of retreating and advancing on the dimensions support a rich recursive elaboration in the instructional progression. The progression of adjustments in dimensions is experimental, and leads to distinctive textures in different segments of instruction.

9.3.3 Reflections on the instructional progression
To finish this discussion chapter, I offer three reflections on this characterisation of the instructional progression in terms of dimensions.

9.3.3.1 Instruction toward opening, not narrowing
I have developed an image of a teacher stepping back and forth along multiple dimensions to guide a student toward mathematical progress. This image could begin to look like the teacher narrowing the student’s view towards the single aim of formal, bare, sophisticated multiplicative thinking. We might imagine the student’s thinking like a mouse, scuttling around the floor, as we try to herd it into a box in the corner: we step back to the left to cut off this escape, then leap to the right to block that one, and press forward along all dimensions until the mouse is forced into the box.

This is not how I see the instruction. If anything, I think the instruction is opening to a more expansive view for the student. The student’s thinking is initially limited. He does not see many potential relationships or strategies. He has few options for approaching the tasks. When a student who knows only how to count by 5s faces a task like $7 \times 5$, it is not an expansive prospect. Then the instruction seeks, at each task, to support the student to see more clearly the number relationships involved. Turn to the tiles, then try to visualise the tiles while screening them. Turn to the notation, then reflect on patterns in the notation. Try with lower ranges, now can that thinking be extended to higher ranges? Try in one orientation, now can that relationship be seen from a different orientation? Now, can you connect that new
orientation back to the tiles, and to the notation? These instructional movements along the dimensions are not blocking off different escapes of a scurrying mind. Rather, they are holding out different aspects of the task for a clearer look.

Drawing on Battista’s (2011, p. 514) metaphor for a learning trajectory, we could have the student climbing up an unfamiliar hillside, struggling to see anywhere to step. The teacher is pointing out places to look for footholds and handholds. With attention, the student gets better at seeing the rocks and roots, and increasingly can see a potential path ahead. By the end of the week, the student is up and down the slope like a spider.

As the student is supported and stretched along these dimensions, they construct a clearer and clearer view of the numbers: a clearer, and more coherent, and more extensive view. When a student arrives at an insight, they do not experience it like arriving in a box. Rather, it is like the camera coming into focus, or indeed, like the light being switched on in a dim room. Even when the learning is not such a singular moment of insight, it is clearly the case that the student can see more and more, not less and less. Instruction responding with calibrations of dimensions can be opening, not narrowing.

9.3.3.2 Characterising instruction with dimensions rather than phases
A common approach to research into a progression in learning, or a progression in instruction, is to identify phases in the progression. The progression will be messy, but with attentive study of the data, distinctive phases can be crystallised out of the data. What such research offers to the community is insight into how to recognise these phases, and what instruction might be effective in each phase. For example, Wright and colleagues have used such an approach in describing instruction in multi-digit addition and subtraction, listing three phases: (1) developing foundational knowledge, (2) consolidating early strategies, and (3) refining strategies and extending tasks (Wright et al., 2012, p. 110).

My experience of my one-to-one teaching was that looking out for phases, and changing the instruction to the phases, was not the most helpful guide for how to proceed. While at times I did refer to phases, I think it was hard to notice a phase at the time of teaching. What seemed to be a more useful guide
in the case of one-to-one responsive teaching was trying to stay at the student’s current cutting edge by constantly tuning in to the responses of the student, and tuning the pitch of the instruction to match and stretch the student. What I found most helpful, and what I find striking in the data, is a view of the dimensions for progression at any given point. So what I want to offer is not an account of major phases, and the instruction that suits them. Rather, I offer a map of dimensions of progression, each of which affords options for tuning the instruction at any point.

To highlight what is different about the structure I am offering, here is a metaphor. To describe how to travel from town A to town B, instead of characterising the three villages you pass though along some main road, I’m characterising northerliness, and westerliness, and uphillness, and suggesting if you keep working in those directions of NOR and WST and UP, you’ll end up getting to town B. Instead of identifying lateral bands that get travelled across, I am identifying longitudinal dimensions that get travelled along.

When researchers and teachers identify phases in instruction, we readily acknowledge that these phases are not clear-cut, that there is overlap and transition between them. Nevertheless, we are deciding that there is something helpful about distinguishing the phases. I would agree that distinguishing phases often does help; however, something can also be lost by distinguishing phases. An account of instruction by phases typically draws our attention to the clear examples of each phase, but ignores the overlap and transition times. I am interested, instead, in drawing attention to the overlap and transition times. I think these may be some of the most important times in teaching; indeed, it may be that most times are transition times.

Consider an example in the case study. The first few lessons in high 5s involved the 5-tile setting. The later lessons did not. We could reasonably suggest two phases here: Phase A with the tiles but without notation, and Phase B without tiles and with notation. Furthermore, we could then identify Segments 10.7 and 11.4 as the transition times which involved introducing notation and letting go of the setting. So, we could recommend that in their instructional design, the teacher should look out for when Phase A is ending, and shift to Phase B. But that was not my experience of those transitional
lesson segments, and I doubt such a recommendation would serve a teacher so well. In Segment 10.7, notation was introduced, not because of some increasing accomplishment, but because a confusion arose for the student about finding $T$, and the notation was an attempt to consolidate Blair’s understanding of the mathematics at that point, and so find again the cutting edge of his understanding. Then the student’s attention shifted towards using and understanding the notation. In pursuing the student’s cutting edge with his thinking around the notation, the setting was sometimes involved, but sometimes it was not. Over the next two lessons, in pursuing the cutting edge of the students’ thinking about multiples of 5s, the tile setting became less relevant. I continued to have the tiles to hand for several lessons after that, but found few occasions to bring them into the tasks. So, I would describe the organisation of instruction through this time, not as an attempt to transition between two phases, rather as a consistent attempt to keep Blair at his cutting edge, strategically using two dimensions of progression to do so: the setting and the notation. I find most illuminating the interaction between these two dimensions. As described in the Layer E analyses, the interaction included: the way the notation emerged from confusions with the setting, the way reflections on the setting clarified Blair’s learning of the notation, and the way the learning of the notation supported Blair to become increasingly independent of the tiles. Segments 10.7 and 11.4, supposedly transitional between Phases A and B, emerge as significant lesson segments for me, where we can learn much about the nature of the instructional progression.

So, I have offered a characterisation of the instructional progression in terms of dimensions, rather than phases. I am interested in whether this account proves helpful for instructional design and teaching.

9.3.3.3 Dimensions create a space for instructional progression

The analysis of Blair’s activity in the case study confirms the expectation that the path of learning is circuitous and idiosyncratic—a labyrinth, as Treffers (1978/1987, p. 249) suggested in the epigraph to Chapter 1. In turn, responsive instruction is also circuitous, and does not follow the same path twice. Therefore, instructional design for a domain needs some way to hold multiple, circuitous paths.
This study does not propose that the five dimensions have clear, routine roles in the instruction. The calibrating could not be planned in detail before the teaching. Neither do I think the calibrating would suit some decision flow-chart: “If student succeeds on Task 1, advance on dimension A, otherwise, retreat on B; after the next two successes, advance on dimension C”, and so on. There are too many dimensions and factors involved for such an approach. Instead, the teacher can use the potential adjustments in each dimension to respond to the different instructional needs in different moments. The dimensions are potential directions or affordances to be aware of as a teacher, and to experiment with. In turn, the dimensions are about the raising of the awareness of the student, of each of these aspects of the mathematics.

This study seeks to inform an instructional design for highly responsive instruction. In Section 2.2, drawing on the research literature reviewed, I suggest that such an instructional design would need to:

- offer ways to adjust tasks in small jumps, so the teacher can pitch tasks just beyond a cutting edge, and can keep adjusting tasks along a changing cutting edge;

- offer potential directions for mathematisation in finer meshes;

- create space for the teacher to respond to the dialogue with a student by going sideways and backwards.

I suggest that an instructional design based on dimensions of progression can meet these needs. The dimensions can construct an instructional framework as an instructional space. Laying out the dimensions creates the directions, calibrations, and space for the responsive, attentive instruction sought. They offer an instructional design that can hold the multiple, circuitous paths of highly responsive instruction.
Chapter 10 – Conclusion

One cannot prescribe practices, but one can guide practice by means of explanatory frameworks accompanied by data, evidence, and argument. (Confrey, 2006, p. 139)

I’ve mastered it! Well, not really mastered. (Blair)

The previous chapter has worked through the three research aims, discussing the findings arising from the analysis of the two nested cases of the Lesson 10 Sequence, and the High 5s Sequence. This final chapter briefly reviews the whole study and summarises those research findings. Following the summary, I explain limitations of the study, and implications of the findings, particularly for my larger aims of instructional design. I also suggest directions for future research arising from the study.

10.1 Summary of Research Findings

This study set out to track, in the words of Treffers (1978/1987, pp. 248–249), the “finer meshes and stepwise structure of mathematising at the micro-level”, and how “the teacher, knowing the right path, can help the pupils to find their way up to the top” in the labyrinthine passage from one mental level to the next. The study arose in the context of developing instructional design for highly responsive intervention instruction, in the domain of multiplication and division. I sought to illuminate the instructional progressions involved in supporting students’ transitions from counting-based strategies to multiplicative strategies, in terms of mathematical dimensions of instructional progression. The study has addressed three research aims:

1. Identify key mathematical dimensions of instructional progression,

2. Describe the progressions in each dimension,

3. Characterise the instructional progression, using the dimensions.

The intention is that such findings can inform the refinement of an instructional framework for multiplication and division.
To pursue these aims, I have undertaken a close analysis of two nested sequences of instruction with an intervention student, Blair. The sequences are drawn from an experimental intervention program, in which Blair made significant progress in developing from counting-based to multiplicative strategies, particularly with multiples of 5. The analysis of the sequences was organised in layers. Building on observations and local interpretations of the activity in each task, I developed accounts of progressions in the student’s activity, and of progressions in each dimension of instruction. Drawing on these accounts, I developed multidimensional accounts of the progression in each sequence. I have identified interactions between the dimensions, and characterised the multidimensional texture of the instructional progression overall.

Over the course of the layered data analysis, I developed responses to the three research aims, which are discussed extensively in Chapter 9. The research findings are summarised in Table 10.1, on pages 314–315.

10.1.1 Aim 1: Identifying key mathematical dimensions of instructional progression

Instruction to support the transition from counting-based to multiplicative strategies can progress along five principal dimensions of instructional progression:

- range;
- orientation;
- setting;
- notation; and
- attention to structuring and strategies.

In the instruction, each of these dimensions is significant. Almost all instructional adjustments can be expressed in terms of these five dimensions. Together, they make a coherent set of dimensions for describing the instruction.
### 10.1.2 Aim 2: Describing progressions in each dimension

The main progressions, and some finer progressions, have been described for each dimension. I have also described how progressions in one dimension often interact with other dimensions. The study has furnished rich illustrations of these progressions and interactions. They are summarised in Table 10.1.

### 10.1.3 Aim 3: Characterising the instructional progression using the dimensions

The principal achievement of this thesis has been to distil a simple, coherent characterisation of the progression of highly responsive instruction. I have argued that the instructional progression can be characterised as:

> *a strategic, interwoven calibration across the five dimensions.*

I explain the characterisation in the following terms.

The instruction involves *calibration* in the sense of making subtle, fine-graded adjustments, tuning the pitch of advance or retreat in each dimension. The calibrations are *interwoven* in the sense that each instructional task can be determined by the pitch of each of the five dimensions. For example, a task may be calibrated with RAN high 5s, ORN multiplication, SET screened 5-tiles, NTN arrow sentences, and STR enquiring about strategy. Furthermore, the calibrations in the dimensions are interdependent, so the significance of an adjustment in one dimension depends on the calibrations of the other dimensions. For example, screening a setting alongside a notated task, can have a different significance than screening a setting while not using notation. The calibrations are typically coordinated so while one or two dimensions are adjusted, the other dimensions are held steady. Nevertheless, over any sequence of tasks, progressions along several dimensions are interwoven.

The calibrations are *strategic*, in that they aim to meet the student’s cutting edge, and guide toward the mathematical aims. Strategic adjustments may be only subtle or small: small adjustments can still be densely interwoven, and can support significant progress. Also, strategic adjustments may not involve a neat or standard order of dimensions. Instead, over time, there is an interweaving of adjustments across all five dimensions.
<table>
<thead>
<tr>
<th>Dimension</th>
<th>Progressions</th>
<th>Interactions</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Range (RNG)</strong></td>
<td>Main progression is through scheme of ranges:</td>
<td>Within a sequence of lesson segments, RNG keeps main range stable while other dimensions are varied, developing knowledge of the structuring and strategies peculiar to that range.</td>
</tr>
<tr>
<td>Steadiest dimension: mainly steady through sequence of segments.</td>
<td>• lower simpler factors → higher more complex factors.</td>
<td></td>
</tr>
<tr>
<td>Two finer progressions within a single basic range:</td>
<td>• Progression in size, e.g. low 5s → high 5s → beyond 10 5s.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Progression in parity, e.g. even → odd multiples of 5.</td>
<td></td>
</tr>
<tr>
<td><strong>Orientation (ORN)</strong></td>
<td>Varying which values are known and unknown in tasks:</td>
<td>Within a fixed RNG, ORN can vary, to intertwine the learning of multiplication and division relationships within the range.</td>
</tr>
<tr>
<td>Flexible dimension: sometimes steady, sometimes varied.</td>
<td>• the product, the multiplier, the multiplicand, and other related units.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Different orientations are more challenging in different contexts. Instruction can:</td>
<td></td>
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<tr>
<td></td>
<td>• focus on one orientation, or</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• switch back and forth between orientations, to promote learning about the whole multiplicative relationship.</td>
<td></td>
</tr>
<tr>
<td><strong>Setting (SET)</strong></td>
<td>Main progression involves distancing the setting:</td>
<td>SET and ORN can be complementary:</td>
</tr>
<tr>
<td>Recursive progression, can be most frequently adjusted dimension.</td>
<td>• visible setting → screened setting → bare number tasks.</td>
<td>• Shifts in ORN select partial views, challenging clarification of different parts of the setting, while</td>
</tr>
<tr>
<td></td>
<td>Subtle adjustments include:</td>
<td>• Instruction with SET can illuminate whole system of relations between all orientations.</td>
</tr>
<tr>
<td></td>
<td>• counting tiles, rearranging tiles, visualizing tiles;</td>
<td>Can coordinate advances in SET and NTN:</td>
</tr>
<tr>
<td></td>
<td>• partial screening, flashing whole setting, flashing a single tile;</td>
<td>• Reference to setting can support understanding advances in NTN, while</td>
</tr>
<tr>
<td></td>
<td>• negotiating terms related to the setting, e.g. “tiles” → “5s”</td>
<td>• Developing notation can support advances in SET towards bare numbers.</td>
</tr>
<tr>
<td>Distancing progression recurs multiple times over multiple scales:</td>
<td>• within tasks, within segments, over sequences, over whole domain.</td>
<td></td>
</tr>
<tr>
<td></td>
<td><em>Early:</em> establish setting as context for making sense of tasks.</td>
<td></td>
</tr>
<tr>
<td></td>
<td><em>Middle:</em> frequent screening and unscreening, to challenge structuring of units.</td>
<td></td>
</tr>
<tr>
<td></td>
<td><em>Later:</em> during bare number work, can return to setting to recall reasoning.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Activity in material setting can distinguish multiplicand and multiplier, while progression to bare numbers develops important ambiguity of factors.</td>
<td></td>
</tr>
</tbody>
</table>

Table 10.1 Summary of research findings: Progressions and Interactions in five dimensions, and a characterisation of instructional progression.
### Table 10.1 continued

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Progressions</th>
<th>Interactions</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Notation (NTN)</strong></td>
<td>Main progression involves formalising the notation of tasks:</td>
<td>NTN and ORN can be complementary (like SET and ORN):</td>
</tr>
<tr>
<td></td>
<td>- No notation → informal notation → formal notation.</td>
<td>- Shifts in ORN can clarify different parts of the notation, while</td>
</tr>
<tr>
<td></td>
<td>A finer progression involves developing different roles for notation:</td>
<td>- Development of NTN can illuminate whole system of relations between all</td>
</tr>
<tr>
<td></td>
<td>- Recording results, reflecting on a result, reflecting on multiple results,</td>
<td>orientations.</td>
</tr>
<tr>
<td></td>
<td>supporting calculations, explaining general relationships.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>A second finer progression involves developing student’s independence:</td>
<td></td>
</tr>
<tr>
<td></td>
<td>- Teacher notating → teacher directing shared notating →</td>
<td></td>
</tr>
<tr>
<td></td>
<td>teacher initiating shared notating → student initiating notating</td>
<td></td>
</tr>
<tr>
<td></td>
<td>The transition to formal notation may involve attentive negotiation.</td>
<td></td>
</tr>
<tr>
<td><strong>Attention to Structuring and Strategies (STR)</strong></td>
<td>Progression less marked, can involve changing types of STR instruction:</td>
<td>STR often invoked when an adjustment in another dimension can illuminate a</td>
</tr>
<tr>
<td>Active, varied, distinctive dimension.</td>
<td>- Enquiring about and affirming the student’s thinking</td>
<td>distinction or relationship of interest, e.g.:</td>
</tr>
<tr>
<td></td>
<td>- Prompting student to recall earlier thinking</td>
<td>- RNG switching from even to odd;</td>
</tr>
<tr>
<td></td>
<td>- Challenging student to improve structuring or strategy</td>
<td>- ORN switching orientation;</td>
</tr>
<tr>
<td></td>
<td>- Challenging student to explain a relationship or strategy</td>
<td>- SET unscreening a setting;</td>
</tr>
<tr>
<td></td>
<td>Involves directing attention, rather than adjusting arithmetic tasks per se.</td>
<td>- NTN completing a table of results.</td>
</tr>
<tr>
<td></td>
<td>Instructional practice of recurring attention to structuring and strategies</td>
<td></td>
</tr>
<tr>
<td></td>
<td>can cultivate student’s attention to structuring and strategies.</td>
<td></td>
</tr>
<tr>
<td><strong>Characterising the Instructional progression as a whole</strong></td>
<td>Instructional progression is a strategic, interwoven calibration across the five dimensions.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>- Each task involves subtle calibrations of all five dimensions.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>- The significance of the calibration of one dimension depends on the calibrations of other dimensions.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>- When one or two dimensions are adjusted, other dimensions are typically held steady.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>- Multiple dimensions are involved over any sequence of tasks.</td>
<td></td>
</tr>
<tr>
<td></td>
<td><strong>Recursive elaboration:</strong> Interwoven cycles of retreating and advancing across the dimensions.</td>
<td></td>
</tr>
<tr>
<td></td>
<td><strong>Experimental progressions:</strong> Each new calibration is an experiment, testing a possibility, observing the student’s responses.</td>
<td></td>
</tr>
<tr>
<td></td>
<td><strong>Distinctive textures:</strong> Lesson segments develop distinctive textures from the calibration of each dimension, how each dimension is varying or holding steady, and the combination of dimensions involved.</td>
<td></td>
</tr>
</tbody>
</table>
Overall, the instructional progressions appear as responsive, strategic, continually fine-tuned instructional experiments, where the affordances for tuning are in the five dimensions. The dimensions are the metaphorical threads through the labyrinth of the highly responsive instruction. I regard this characterisation as the major contribution of the thesis.

10.1.4 Three features of the instructional progression

Drawing on this characterisation of the instruction in terms of calibrating dimensions, I have articulated three further features of the instructional progression.

Firstly, the calibration involves recursive elaboration (Davis et al., 2008). Each dimension is recursive, progressing from less advanced to more advanced calibrations in repeated cycles over the course of an instructional sequence, and sometimes even within a single lesson segment. At the same time, while one dimension folds back to a less advanced calibration, the instruction elaborates and extends by advancing other dimensions. By calibrating across the dimensions, the instructional progression becomes a rich interweaving of folding back and elaboration.

Secondly, the instruction progresses as a continuously experimental calibration of dimensions, testing out how the student will respond to each new adjustment (Simon, 1995). For example, following up on a student’s insightful answer, the teacher introduces an informal notation, aware that notating can help clarify an insight. Observing the student’s engagement with the notation, the teacher continues the notation in the next task, and tries screening the setting, which may help him to assess the student’s grasp of the notation. Observing the student’s facility when the setting is screened, the teacher continues with the notation and the screened setting, and challenges the student with a new orientation. In this approach, each new calibration is a closely observed experiment.

Thirdly, the experimental calibrations lead to distinctive textures in the dimensions of each lesson segment. For example, one segment might keep RNG and ORN steady, have recurring screening and unscreening of SET, and use no NTN, while a second segment has distinctly varying RNG and ORN,
and a third segment is marked by the development of NTN, while SET is distanced. Each segment can weave a different texture in the dimensions.

The characterisation and three features of the instructional progression are summarised at the end of Table 10.1.

10.2 Limitations of the Research
The research findings need to be considered in the context of the limitations of the research methodology. The account of instruction developed pertains to the observations made in this particular teaching experiment, with my teaching and my student, in the complexity of one local school. The account does not seek to compare or evaluate instructional design, and the findings cannot be generalised in a hypothetical sense. Rather, the findings can illuminate a viable approach to intervention instruction, and inform instructional design (P. Cobb et al., 2003; Steffe & Thompson, 2000). The account contributes to local instructional theory by describing how dimensions progress and how the adjustments in dimensions can support learning. These descriptions can inform other teachers and researchers when drawing on this research in their own contexts (Gravemeijer & Cobb, 2006). As Confrey (2006) asserts, “one cannot prescribe practices, but one can guide practice by means of explanatory frameworks accompanied by data, evidence, and argument” (p. 139).

In Chapter 3, I remarked on the breadth and depth I offer in this study. Any one of the five dimensions analysed here can be studied in greater depth, individually. Likewise, students’ activity in developing multiplicative strategies can be the focus of deeper and broader study. What this study offers is an investigation of the interweaving of these dimensions, and how they respond to a student’s activity, over a sustained sequence of instruction. By limiting the investigation of the individual dimensions, and by limiting the data to one main sequence with one student, the study can offer depth in understanding a responsive instructional progression as a whole.

10.3 Implications of the Research
Below I discuss the implications of the study for

a) revising the instructional framework for intervention in multiplication and division;
b) informing classroom instruction in multiplication and division; and

c) describing the instructional approach for intervention.

10.3.1 Revising the instructional framework for intervention

This study was conducted to inform the revision of an instructional design. As explained in Chapter 1, within MISP we wanted to refine an instructional framework for intervention in the domain of multiplication and division, to guide instruction toward grounded habituation of basic facts. I have now developed such a revised instructional framework (Wright & Ellemor-Collins, 2018, p. 131), drawing partly on this study, and partly on other work in the teaching experiment. I have not presented the revised framework as an outcome of this study; rather, I note here how the implications of the study have been able to inform the revised framework.

Figure 10.1 is a schematic chart for the revised framework. I will briefly explain the chart. The chart shows how the revised framework is organised as the span of the five key dimensions: RNG, ORN, SET, NTN, and STR; with the five dimensions compressed into a two-dimensional chart. Each inner white cell in the chart represents a calibration of instruction. The corresponding row header indicates the calibration of RNG, and the topic progressing from STR through to rehearsal mode. The column header indicates the main calibrations of SET and NTN. For example, the top left white cell represents instruction in Range 1, attending to structuring and strategies, with a setting of visible n-tiles, and no notation. ORN can be varied within each cell, indicated by the orange cyclic arrows.

The study contributed to this revised framework firstly by identifying the five key dimensions, which became the basis of the framework. Furthermore, the study described the progressions in each dimension, and characterised instruction as an interweaving of these progressions. In turn, the framework has been designed to indicate the interweaving. The broad progression in the range dimension is indicated vertically down the chart; the structuring and strategies dimension is highlighted in rows within each range; the progressions in the setting and notation dimensions are indicated alongside each other horizontally across the rows; while the orientation dimension is indicated to vary within each cell.
Figure 10.1  Overview of a revised instructional framework for multiplication and division, indicating progressions in five dimensions

The matrix of white cells represent calibrations of instruction, across the dimensions of range, structuring and strategies, setting, and notation. The dimension of orientation varies within each cell.

<table>
<thead>
<tr>
<th>Range: 1: 2s, 5s, 10s</th>
<th>Structuring &amp; strategies</th>
<th>Rehearsal</th>
<th>Structuring &amp; strategies</th>
<th>Rehearsal</th>
<th>Structuring &amp; strategies</th>
<th>Rehearsal</th>
<th>Structuring &amp; strategies</th>
<th>Rehearsal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low × low</td>
<td></td>
<td></td>
<td>Low × low</td>
<td></td>
<td>Low × high</td>
<td></td>
<td>Low × high</td>
<td></td>
</tr>
<tr>
<td>High 3s</td>
<td></td>
<td></td>
<td>High 4s</td>
<td></td>
<td>High 5s</td>
<td></td>
<td>High 6s</td>
<td></td>
</tr>
<tr>
<td>High 7s</td>
<td></td>
<td></td>
<td>High 8s</td>
<td></td>
<td>High 9s</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Legend:
- SET  
- NTN
- ORN
- RNG
The study affirmed the practice of each lesson segment focusing on a steady main range, while varying the other dimensions. So, the framework is organised broadly into ranges, with the other dimensions varying within each range. Instruction can focus on the multiplicative structuring and strategies peculiar to each range of multiples, such as the high 5s in the case study. The intention is that, within a single lesson, different lesson segments address different ranges, with instruction in each range at a different point of progression, as in the study.

Within a single range, the study found effective a sustained sequence of instruction focused on the transition from counting-based to multiplicative strategies, before ongoing rehearsal and habituation of those strategies. The framework includes, within each range, a phase on structuring and strategies, followed by a phase on habituation. The former phase indicates a focus on the dimension of attention to structuring and strategies.

The study found a close interaction between progressions in setting and progressions in notation. The revised framework indicates these two dimensions progressing alongside each other, across the rows in each range.

Finally, the study found that varying the orientation at points throughout the instructional sequences was constructive. The framework indicates variation in orientation is available within each cell of the chart. Minor variations in range, such as from even to odd multiples, are also available in each cell.

The analysis and discussion offered in the study, as summarised in Table 10.1 above, contribute local instructional theory to accompany this revised instructional framework, detailing how the progressions in each dimension are enacted in instruction, and how the dimensions can be interwoven to support student learning.

10.3.2 Recommendations for classroom instruction in multiplication and division

The study has sought the fine-grained detail helpful in designing intervention instruction. Nevertheless, broader findings from the study can be applied to general classroom instruction in multiplication and division, where supporting the transition from counting-based to multiplicative strategies is still a
significant issue (as reviewed in Section 2.4). I make four recommendations below.

Classroom instruction could focus on developing multiplicative strategies within each range or family of multiples, and recognise that the transition to multiplicative strategies may need support within each range, rather than happening once and for all. In particular, instruction could work on multiplication and division together within each range (Downton, 2013), rather than regarding division as a separate, later topic. Regarding the high 5s range, multiples of 5 are often assumed to be relatively straightforward, but the case study suggests this may not be the case for all students: while skip-counting by 5s may be straightforward, developing multiplicative strategies for multiples of 5 is another matter.

Classroom instruction could incorporate screening of settings, recognising the significance of visualising and checking structured settings in students’ development of multiplicative reasoning (Anghileri, 2006). Instruction could also incorporate occasional folding back to settings when working in bare numbers.

Classroom instruction could incorporate the development of informal notations (Gravemeijer et al., 2000) as a significant phase in the development of multiplicative strategies. The study also alerts teachers to the care needed in negotiating the use of formal notation, especially the use of the division sign.

Finally, classroom instruction in the domain overall could be framed in terms of coordinating progressions on mathematical dimensions, just as intervention instruction is framed. A lesson plan could propose the specific range, orientations, setting, notation, and attention to structuring and strategies to be worked on. Teachers could anticipate what adjustments or extensions might arise in each dimension for responsive interaction with students. Indeed, an instructional framework for the classroom could align with the intervention framework, but simply have less detail.
10.3.3 Articulating the instructional approach for intervention

The study aimed to inform instructional design by identifying key dimensions and progressions in those dimensions. Yet because it investigates the micro-scale of task-by-task progressions, and characterises the whole texture of progression, the study has implications for the instructional approach as well. Our approach of highly responsive instruction for one-to-one intervention has not been articulated in terms of calibrations of dimensions before. I think this closely examined study suggests this would be a helpful way to articulate the instructional approach in the future.

Intervention teachers could see their role as making fine adjustments within a space spanned by several mathematical dimensions of progression (Section 9.3.3.3). They could become explicitly aware of:

- what dimensions they are working with in a domain (Section 9.1); and
- what progressions they make on those dimensions (Section 9.2).

They could practice how adjustments in those dimensions are enacted in teaching procedures, for example:

- how to pose a task in three different orientations;
- how to partially screen a setting;
- how to prompt a student to complete an informal notation; and
- how to word a question about a pattern in a sequence of notations.

They could also become aware of how they coordinate the dimensions:

- how they hold some steady while they adjust others, and what happens when several are adjusted at once (Section 9.3.1.2);
- how calibrations in one dimension affect the significance of calibrations in others (Section 9.3.1.4);
- how instruction can develop recursive elaboration through cycles of folding back and advancing in each dimension (Section 9.3.2.1); and
- how distinctive textures can develop in different segments of their instruction (Section 9.3.3.3).

I recommend intervention teachers think of their work, at the task-by-task scale, as comprising experimental, strategic, responsive calibrations.
interwoven across several mathematical dimensions of instructional progression.

10.4 Directions for Further Research

This study aims to inform the refinement of an instructional framework for intervention in multiplication and division. That refinement deserves further study. Since this has been an investigation of a small case study, a study of other relevant cases is warranted, including other cases within this same teaching experiment. Also, this study has analysed dimensions on a micro-scale: instructional progression task-by-task. Further studies could broaden the scale of analysis to a meso-scale—instructional progression through different topics or ranges—and a macro-scale—instructional progression over the whole domain of multiplication and division.

Other issues emerged in the teaching experiment which are relevant to refining the instructional framework, and which intersected with the analysis for the present study, but which could not be addressed within the scope of the present study. I mention three which would complement the present findings:

a) investigating the scheme of ranges used to broadly organise the instruction in multiplication and division;

b) investigating how instruction in multiplication and division can focus more on multiplicative structuring, rather than on specific multiplicative strategies; and

c) investigating how to coordinate instruction in multiplicative knowledge with instruction in additive knowledge.

The approach of this study also suggests directions for further research beyond the refinement of the instructional framework for multiplication and division.

For this study, I developed a distinctive approach to the analysis of instructional progression. The approach has involved analysing long sequences of instruction, tracking several simultaneous dimensions, using several layers of analysis, and compiling multidimensional charts. I am
satisfied with this as a first attempt, but the approach could be improved. In particular, I would like to analyse more periods of instruction, but present the analysis with fewer words. How can a study of interweaving dimensions of instructional progression report less data and analysis, while remaining rigorous and informative?

Finally, the form of instructional framework used here, involving mathematical dimensions of instructional progression, warrants further research. In Section 2.5, I reviewed research literature recommending the potential of learning trajectories and frameworks, and seeking to develop these forms of instructional design. The framework of dimensions is a promising form of design, and potentially makes a distinctive contribution to our developing notions of instructional framework and learning trajectory. In Section 9.3.3 I distinguished a framework of dimensions from a framework of phases, and suggested the potential contribution of the former in illuminating responsive instruction. I also reflected on how a framework of dimensions can create a space that supports responsive instruction. I propose further investigating the framework of dimensions as a form of instructional design.

I conclude this thesis with a lithograph by Dutch artist M. C. Escher (see Figure 10.2). A young mathematician runs joyfully from the dim steps and passageways of a labyrinth, out into the light. Working down to the foundations, they see that the labyrinth is constructed from an interwoven tessellation of many actions of student and teacher. Yet, lifting out of the complexity of interactions, the multidimensional form of the interactions is revealed, and a clearer view emerges. The mathematician can reflect on their own place in the interactions, and return inside, to continue the cycle of teaching and learning.
Figure 10.2  M. C. Escher: Cycle
Reference list

Amit, M. (2010). Commentary 1 on "Re-conceptualizing mathematics education as a design science". In B. Sriraman & L. D. English (Eds.), *Theories of mathematics education: Seeking new frontiers* (pp. 147-149). Heidelberg, Germany: Springer.


Gravemeijer, K., Bowers, J., & Stephan, M. (2003). Continuing the design research cycle: A revised measurement and arithmetic sequence. In M. Stephan, J. Bowers, P. Cobb & K. Gravemeijer (Eds.), *Supporting students' development of measuring conceptions: Analyzing students' learning in social context (Journal for research in mathematics education, Monograph number 12)* (pp. 103-122). Reston, VA: NCTM.


Mathematics Education Research Group of Australasia (pp. 496–503). Canberra, ACT: MERGA.


Tran, L. T. (2016). Targeted, one-to-one instruction in whole-number arithmetic: A framework of key elements. (Doctoral dissertation), Southern Cross University, Lismore, NSW.


Treffers, A. (2001a). Grade 1 (and 2) - Calculation up to 20. In M. van den Heuvel-Panhuizen (Ed.), Children learn mathematics (pp. 43-60). Utrecht, The Netherlands: Freudenthal Institute, Utrecht University/SLO.


Appendix 1 - Research Ethics Approval

Please note that the Chair of the Higher Degrees Research Committee, under delegated authority, has approved a change from the proposed title shown in the ethics approval document, to the current research title:

*Threads through a labyrinth: Characterising intervention instruction for multiplicative strategies as an interweaving of five dimensions of progression.*
Appendices

Appendix 2 - SERAP Approval

This is the approval given to the study by the State Education Research Approval Process (SERAP), within the New South Wales Department of Education and Communities.

Mr David Ellemor-Collins  
509 Wallace Road  
THE CHANNON NSW 2480

Dear Mr Ellemor-Collins

I refer to your application to conduct a research project in NSW government schools entitled *Designing arithmetic instruction for intervention with low-attaining students*. I am pleased to inform you that your application has been approved. You may contact principals of the nominated schools to seek their participation. You should include a copy of this letter with the documents you send to schools.

This approval will remain valid until 22 April 2014.

The following researchers or research assistants have fulfilled the Working with Children screening requirements to interact with or observe children for the purposes of this research for the period indicated:

<table>
<thead>
<tr>
<th>Name</th>
<th>Approval expires</th>
</tr>
</thead>
<tbody>
<tr>
<td>David Leslie Ellemor-Collins</td>
<td>14/10/2014</td>
</tr>
</tbody>
</table>

I draw your attention to the following requirements for all researchers in NSW government schools:

- School principals have the right to withdraw the school from the study at any time. The approval of the principal for the specific method of gathering information must also be sought.
- The privacy of the school and the students is to be protected.
- The participation of teachers and students must be voluntary and must be at the school’s convenience.
- Any proposal to publish the outcomes of the study should be discussed with the research approvals officer before publication proceeds.

When your study is completed please forward your report to: Manager, Quality Assurance/Research, Department of Education and Communities, Locked Bag 53, Darlinghurst, NSW 1300.

You may also be asked to present on the findings of your research.

I wish you every success with your research.

Yours sincerely

Dr Susan Harriman  
Leader, Quality Assurance  
/\ December 2013

Policy, Planning and Reporting Directorate  
NSW Department of Education and Communities  
Level 1, 1 Oxford Street, Darlinghurst NSW 2010 – Locked Bag 53, Darlinghurst NSW 1300  
Telephone: 02 9244 5060 – Email: sabrad@det.nsw.edu.au
Appendix 3 - MISP Assessment Schedules 3B, 3C, 3D, 3E, 3F

The assessment schedules 3B, 3C, 3D, 3E, and 3F were used in the teaching experiment, for the initial assessments and the three subsequent assessments, as described in Sections 3.2.4 and 3.2.5.
## 3B. Structuring Numbers 1 to 20

### Assessment Schedule version 2012

**MATERIALS: YELLOW CARDS**

<table>
<thead>
<tr>
<th>Student Name:</th>
<th>Interview Date:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>DoB:</th>
<th>Age: (yrs) (mths)</th>
<th>Interviewer:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Structuring Numbers 1 to 20 – framework levels

0. Emergent spatial patterns and finger patterns.
1. Spatial patterns 1–6 and finger patterns 1–5.
2. Small doubles and small partitions of 10.
3. 5-plus and partitions of 5.
4. Facile structuring numbers 1 to 10.
5. Formal addition (parts ≤ 10).
6. Formal addition & subtraction (parts ≤ 10).
7. Formal addition & subtraction (whole ≤ 20).

Note: For levels 1-7, student has to use facile strategies, that is, not counting by ones.

In each task group, if student uses counting, ask *Can you do it without counting?*
If student can only solve by counting, no need to pose remainder of task group.

1. **Making finger patterns (6-10)** (>Level 1)
   
   *Show me a number on your fingers, as quickly as you can. Show me ____.*

<table>
<thead>
<tr>
<th>10</th>
<th>8</th>
<th>7</th>
<th>9</th>
<th>6</th>
<th>6 another way?</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

   If not facile, pose tasks 6 and 7 from Schedule 2 (fingers 1-5, configurations 1-6).

2. **Small doubles** (Level 2)

<table>
<thead>
<tr>
<th>2 &amp; 2</th>
<th>5 &amp; 5</th>
<th>3 &amp; 3</th>
<th>4 &amp; 4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3. **Partitions of 10**

   *I’ll say a number, and you say how many more to make 10.*

   *For example, I say “5”, you say … (Can prompt “5”).*

   3.1) **Small partitions** (Level 2)

<table>
<thead>
<tr>
<th>9</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

   3.2) **Big partitions** (Level 4)

<table>
<thead>
<tr>
<th>2</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4. **Partitions of 5** (Level 3)

   *I’ll say a number, and you say how many more to make 5.*

   *For example, I say “4”, you say … (Can prompt “1”).*

<table>
<thead>
<tr>
<th>2</th>
<th>1</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5. **5-plus facts** (Level 3)

<table>
<thead>
<tr>
<th>5 plus 2</th>
<th>5 plus 4</th>
<th>5 plus 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 plus 3</td>
<td>1 plus 5</td>
<td>3 plus 5</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

6.1 **Partitions in range 1 to 9** (Level 4)

   *I am going to give two examples. 3&1 add up to 4, yes? Also, 2&2 add up to 4.*

   *Now, can you tell me two numbers that add up to ____?*

   *Can you tell me another two numbers that add up to ____?*

   | 6 | 9 |
6.2 **Partitions in range 11 to 20** (Level 7)
*Can you tell me two numbers that add up to ___?...Another two?

| 12 | 19 |

7.1 **10-plus facts** (Level 5)

| 10 plus 4 | 10 plus 8 | 10 plus 2 |
| 7 plus 10 | 5 plus 10 | 1 plus 10 |

7.2 **Teen plus 10**

| 16 plus 10 | 12 plus 10 |

8. **Big doubles** (Level 5)

| 10 & 10 | 6 & 6 | 8 & 8 | 9 & 9 | 7 & 7 |

If student only uses count-by-1s in task groups 1 to 8, END ASSESSMENT HERE.

**Written Addition & Subtraction 1-20**

[CARDS—NUMBER SENTENCES]

Show the card: *Read this please...What is the answer?*

Generally need to enquire about strategy used.

If the student counts, ask: *Can you do it without counting?*

9. **Formal addition (Whole ≤ 10)** (Level 4)

| 4 + 3 | 3 + 6 | 2 + 7 |

10. **Formal addition (Parts ≤ 10)** (Level 5)

| 6 + 5 | 9 + 6 | 8 + 7 |

11. **Formal subtraction (Parts ≤ 10)** (Level 6)

| 8 – 5 | 14 – 7 | 11 – 4 | 17 – 9 |

12. **Formal addition and subtraction (Whole ≤ 20)** (Level 7)

| 13 + 3 | 11 + 8 | 17 - 15 | 19 – 13 |
3C. Conceptual Place Value
Assessment Schedule  version 2012

MATERIALS: ORANGE CARDS, 15 x 10-STICK BUNDLES, 13 x 100-DOT SQUARES, COVER

Student Name: .............................................................. Interview Date: .........................
DoB: ...................................... Age: .................(yrs) ...............(mts) Interviewer: .........................

Conceptual Place Value – framework levels
0. Emergent inc/decrementing by ten
1. Inc/decrementing by 10 off the decuple with materials
2. Inc/decrementing by 10 formal to 100
3. Inc/decrementing by 10 formal to 1000

1. **Incrementing by tens ON the decuple** [BUNDLING STICKS]
   - Establish there are ten sticks in each bundle.
   - Place one bundle under cover. *How many sticks are under there?* (10)
   - Continue placing bundles under cover. *Now how many sticks are under there?*
     20 30 40 50 60 70
   - At 70 ask *How many sticks? How many bundles?*
   - Continue incrementing, asking *How many sticks?*
     80 90 100 110 120

2. **Incrementing by tens OFF the decuple** [BUNDLING STICKS]
   - Place out four sticks. *How many sticks?* (4)
   After answer, place sticks under the cover.
   - Place a bundle beside the cover.
   *If I add this bundle under the cover, how many sticks will that be?* (14)
   After answer, place bundle under the cover.
   - Continue adding bundles. *Now how many sticks are under there?*
     24 34 44 54 64 74 84

IF STUDENT IS UNSUCCESSFUL ON TASK 2, SKIP TASK 3.

3. **Decrementing by tens OFF the decuple** [BUNDLING STICKS]
   - Place out 147 sticks: a big bundle of 10 bundles, 4 more bundles, and 7 sticks.
   - Explain that the big bundle has 100 sticks.
   - Ask *How many sticks are there altogether?*
     If the student doesn’t establish how many, just tell him/her there are 147.
   - Cover the sticks. *Okay, there are 147 sticks under there.*
   Remove a bundle. *Now how many sticks are there?* (137)
   - Continue removing bundles. *Now how many are there?*
     127 117 107 97 87 77
4. **Jump of ten—more** [NUMERAL CARDS]

Show number on card. *Which number is ten more than this?*

| 40 | 90 | 79 |

If jumping without counting-by-1s OK, try tasks in 3-digit range:

| 356 | 306 | 195 |

If 3-digit range OK, try a task across 1 000:

| 999 |

5. **Jump of ten—less** [NUMERAL CARDS]

Show number on card. *Which number is ten less than this?*

| 40 | 79 |

If jumping without counting-by-1s OK, try tasks in 3-digit range:

| 356 | 306 |

If 3-digit range OK, try a task across 1 000:

| 1 005 |

6. **Jump of one hundred** [NUMERAL CARDS]

Show number on card. *Which number is...?*

| one hundred more than 50? | one hundred more than 306? |

| one hundred more than 973? | one hundred LESS than 108? |

7. **Incrementing by 100s on the hundred** [100-DOT SQUARES]

- Establish there are 100 dots on each 100-square.
- Place one square under cover. *How many dots are under there?* (100)
- Continue placing squares under cover. *Now how many dots are under there?*
  
  | 200 | 300 | 400 | 500 | 600 |

- At 600 ask *How many dots? How many squares?*
- Continue incrementing, asking *How many dots?*
  
  | 700 | 800 | 900 | 1 000 | 1 100 | 1 200 | 1 300 |
3D. Addition and Subtraction to 100

Assessment Schedule version 2012

MATERIALS: RED CARDS

Student Name: ___________________________ Interview Date: ______________
DoB: ___________________________ Age: __________ (yrs) __________ (mths) Interviewer: ___________________________

Addition and Subtraction to 100 – framework levels
0. Emergent addition and subtraction to 100
1. Add up from/subtract down to decuple
2. Add up to/subtract down from decuple – small
3. Add up to/subtract down from decuple – large
4. Add/subtract across a decuple
5. 2-digit addition with regrouping
6. 2-digit addition and subtraction with regrouping

Note: For levels 1-6, student has to use facile strategies, that is, not counting by ones.

Written addition and subtraction

Present tasks on CARDS. Read this please... What is the answer?
Generally need to enquire about strategy used.

1. 2-digit (and 3-digit) addition/subtraction with regrouping

1.1) 58 + 24

1.2) 43 + 19

1.3) 51 – 25

1.4) 82 – 39

1.5) 386 + 240

1.6) 705 – 698

IF UNSUCCESSFUL ON 1.1-1.4 → ALSO POSE 2 and 3.

2. Add/subtract a decuple

2.1) 61 + 30

2.2) 87 – 50

3. 2-digit addition/subtraction without regrouping

3.1) 36 + 22

3.2) 65 – 13
Adding and subtracting to and from a decuple

Preliminaries
Show the DECUPLE LINE. What do you call these numbers?
(Tens, whole tens, round numbers, tens, decades…?)
I’m going to turn this card over, and then ask you some questions about decuples.
(Use the student’s name for the decuples.)

Remove the decuple line. Show the NUMERAL CARD for each task.

4. Decuple after/before a number

<table>
<thead>
<tr>
<th>40</th>
<th>What’s that number?</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>What is the decuple after 40?</td>
</tr>
<tr>
<td></td>
<td>What is the decuple before 40?</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>58</th>
<th>What’s that number?</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>What is the decuple after 58?</td>
</tr>
<tr>
<td></td>
<td>What is the decuple before 58?</td>
</tr>
</tbody>
</table>

5. Add-up-from/subtract-down-to a decuple

<table>
<thead>
<tr>
<th>20</th>
<th>What’s that number?</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>What’s 3 more than 20?</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>36</th>
<th>What’s that number?</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>What is the decuple before 36? [30]</td>
</tr>
<tr>
<td></td>
<td>How far is it back to 30? [6]</td>
</tr>
</tbody>
</table>

6. Add-up-to/subtract-down-from a decuple (small 1-5)

<table>
<thead>
<tr>
<th>56</th>
<th>What’s that number?</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>What is the decuple after 56?[60]</td>
</tr>
<tr>
<td></td>
<td>How far is it up to 60? [4]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>90</th>
<th>What’s that number?</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>What is 2 less than 90?</td>
</tr>
</tbody>
</table>

7. Add-up-to/subtract-down-from a decuple (large 6-9)

<table>
<thead>
<tr>
<th>43</th>
<th>What’s that number?</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>What is the decuple after 43? [50]</td>
</tr>
<tr>
<td></td>
<td>How far is it up to 50? [7]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>80</th>
<th>What’s that number?</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>What is 8 less than 80?</td>
</tr>
</tbody>
</table>

8. Add/subtract across a decuple

<table>
<thead>
<tr>
<th>37</th>
<th>What’s that number?</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>What is 6 more than 37?</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>63</th>
<th>What’s that number?</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>What is 8 less than 63?</td>
</tr>
</tbody>
</table>
3E. Multiplicative Strategies
Assessment Schedule version 2012

MATERIALS: BLUE CARDS, COUNTERS, SCREEN

<table>
<thead>
<tr>
<th>Student Name:</th>
<th>Interview Date:</th>
</tr>
</thead>
<tbody>
<tr>
<td>DoB:</td>
<td>Age: (yrs) (mths)</td>
</tr>
<tr>
<td>Interviewer:</td>
<td></td>
</tr>
</tbody>
</table>

Multiplicative strategies – framework levels
0. Emergent grouping
1. Perceptual items, counted by ones
2. Perceptual items, counted in multiples
3. Figurative items, counted in multiples
4. Abstract groups, items counted in multiples
5. Abstract groups, facile addition and subtraction
6. Multiplication and division as operations

Tasks with Equal Groups

1. Forming equal groups: \(4 \times 5\) [COUNTERS]

   1.1) Present a pile of more than 30 counters. Using these counters, make groups of 5. Stop the student at four groups of 5.
   1.2) Move the excess counters aside. How many groups of five have you made? How many counters did you use altogether?

2. Multiplication with equal groups [3-DOT CARDS]

   2.1) \(4 \times 3\)
   
   Show a 3-dot card. Each card has three dots. Place four 3-dot cards face down and screen the four cards. There are four cards under here. How many dots altogether?
   
   If unsuccessful → unscreen the four cards. If still unsuccessful → flash one card. If still unsuccessful → turn up all cards.

   If facile with 2.1, pose:

   2.2) Extension: \(8 \times 3\)
   Can you use that to work out eight 3s?

3. Quotition division with equal groups: \(? \times 4 = 20\) [COUNTERS]

   Present a pile of 20 counters. There are 20 here. Screen all the counters. I am putting them into groups of four (briefly show one group of four). How many groups of four can I make?
Tasks with Arrays

4. **Introductory array task** [ARRAYS]

   | Present arrays. Have you seen these before in class work? What do you call them? We can call it an array. |
   | Present the 6x4 array. Place a ruler under the first row. How many dots in the top row? Move ruler down a row. How many in the next row? Move ruler down a row. How many in the next row? (If hasn’t shown awareness that rows are equal: What do you notice?) Finally, ask How many rows? |

5. **Multiplication with an array**

   5.1) **5 × 3** [5X3 ARRAY]
   Briefly show then screen the 5x3 array. Five rows of 3. How many dots altogether?
   If unsuccessful → unscreen top row. If still unsuccessful → unscreen whole array. Observe closely how student counts.
   If facile with 5.1 (i.e. uses a known fact), pose:

   5.2) **Extension: 5 × 6** [5X6 ARRAY]
   What about five rows of 6? Briefly show then screen 5x6 array. How many dots altogether?

6. **Quotition division with an array**

   Briefly display array, then screen all but the top row.
   There are 12 spots altogether. Here is one row. How many rows are there altogether?

   6.1) **12 ÷ 2** [6X2 ARRAY]  6.2) **12 ÷ 4** [3X4 ARRAY]

Skip Counting

7. **Skip counting**

   Count by ____. I’ll tell you when to stop.

   7.1) by 2s (to 30)  7.2) by 5s (to 50)  7.3) by 3s (to 36)

If student is level 3 or lower (E.g. task 6 unsuccessful) → END ASSESSMENT HERE
## Advanced Multiplication and Division Tasks

### 8. Tasks beyond the basic facts
Present on card. Read this please. Can you work this out?

<table>
<thead>
<tr>
<th>8.1) 3 × 12</th>
<th>8.2) 15 × 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.3) 24 ÷ 3</td>
<td>8.4) 65 ÷ 5</td>
</tr>
</tbody>
</table>

### 9. Using commutativity, inverse, and distributivity properties
Present complete number sentence(s) on card. Read this please. Present incomplete number sentence below. Can you use that, to help you do this?

<table>
<thead>
<tr>
<th>9.1) Commutativity</th>
<th>9.2) Inverse</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 × 24 = 72</td>
<td>12 × 9 = 108</td>
</tr>
<tr>
<td>24 × 3 = □</td>
<td>108 ÷ 9 = □</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>9.3) Distributivity/associativity</th>
<th>9.4) Distributivity</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 × 13 = 52</td>
<td>10 × 15 = 150</td>
</tr>
<tr>
<td>8 × 13 = □</td>
<td>9 × 15 = □</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>9.5) Distributivity</th>
<th>9.6) Distributivity</th>
</tr>
</thead>
<tbody>
<tr>
<td>6 × 10 = 60</td>
<td>5 × 14 = 70</td>
</tr>
<tr>
<td>6 × 3 = 18</td>
<td>2 × 14 = 28</td>
</tr>
<tr>
<td>6 × 13 = □</td>
<td>7 × 14 = □</td>
</tr>
</tbody>
</table>

### 10. A formal use of distributive property

(7×12) + (3×12) = □
Read this. Can you think of a short-cut way to work this out?
3F. Multiplication Basic Facts
Assessment Schedule version 2012

MATERIALS: PURPLE CARDS

<table>
<thead>
<tr>
<th>Student Name:</th>
<th>Interview Date:</th>
</tr>
</thead>
<tbody>
<tr>
<td>DoB:</td>
<td>Age: (yrs) (mths)</td>
</tr>
<tr>
<td>Interviewer:</td>
<td></td>
</tr>
</tbody>
</table>

### Multiplication basic facts – framework levels

0. Emergent basic facts
1. Both factors up to 5
2. One factor is: 1, 2, 5, or 10
3. One factor is: 0, 1, 2, 3, 4, 5, or 10
4. Both factors up to 10

**Basic plan.** Pose tasks in order. The student is considered facile with a task when he/she answers very quickly, without counting or skip-counting. While student shows such facility with the tasks, keep asking. When student stops showing facility, pose the remainder of the row, then end the assessment.

Observe how the student handles the last row he/she is asked. Derives unknown facts from known facts? If so, in what ways? If not, skip-counts? Or what?

Present on card. *Read this please. Can you work this out?*

1. **Both factors up to 5**
   
<table>
<thead>
<tr>
<th>3 × 2</th>
<th>2 × 4</th>
<th>3 × 3</th>
<th>4 × 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 × 4</td>
<td>5 × 3</td>
<td>4 × 4</td>
<td>5 × 5</td>
</tr>
</tbody>
</table>

2. **One factor is: 1, 2, 5, or 10**
   
   *new facts*: 1s, 10s, high 2s, high 5s (and turns)
   
<table>
<thead>
<tr>
<th>1 × 3</th>
<th>7 × 1</th>
<th>6 × 2</th>
<th>8 × 10</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 × 9</td>
<td>10 × 7</td>
<td>8 × 5</td>
<td>5 × 9</td>
</tr>
</tbody>
</table>

3. **One factor is: 0, 1, 2, 3, 4, 5, or 10**
   
   *new facts*: 0s, high 3s, high 4s (and turns)
   
<table>
<thead>
<tr>
<th>0 × 7</th>
<th>6 × 3</th>
<th>3 × 9</th>
<th>7 × 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 × 0</td>
<td>8 × 4</td>
<td>4 × 7</td>
<td>4 × 9</td>
</tr>
</tbody>
</table>

4. **Both factors up to 10**
   
   *new facts*: 6–9 × 6–9
   
<table>
<thead>
<tr>
<th>6 × 6</th>
<th>8 × 8</th>
<th>7 × 6</th>
<th>9 × 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>6 × 8</td>
<td>9 × 7</td>
<td>6 × 9</td>
<td>8 × 7</td>
</tr>
</tbody>
</table>
Appendices

Appendix 4  -  MISP Teaching Charts 3E and 3F (A3 size)

The teaching charts 3E and 3F were used in the initial instructional design for multiplication and division in the teaching experiment, as described in Section 3.3.1.
<table>
<thead>
<tr>
<th>MIDDLE NUMBER BAND</th>
<th>Domain 3E: Multiplicative Strategies</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Topic</strong></td>
<td><strong>Teaching Procedures</strong></td>
</tr>
<tr>
<td><strong>PHASE I: Multiplicative tasks with small factors</strong> <em>(Red Book chapter 7, “instruction phase 1)</em></td>
<td></td>
</tr>
</tbody>
</table>
| **3E1 Multiplication w/ equal groups**  
Determine total number | **3EX.1**  
*N-groups out to manipulate/ to see only*  
Given number in each group and number of groups, determine total. | **3EX.2**  
*N-groups face down*  
------------------------> | **3EX.3**  
*N-groups screened*  
------------------------> |
| **3E2 Quotitive division w/ equal groups**  
Determine number of groups | **3EX.4**  
*Counters out to manipulate/ to see only*  
Given total and number in each group, determine number of groups. | | | *(N-groups, with written task beside)*  
------------------------> |
| **3E3 Partitive division w/ equal groups**  
Determine number in each group | **3EX.5**  
*Counters out to manipulate/ to see only*  
Given total and number of groups, determine number in each group. | **3EX.6**  
*N-groups face down*  
------------------------> | **3EX.7**  
*N-groups screened*  
------------------------> |
| **3E4 Introducing arrays**  
| **3E5 Multiplication w/ arrays**  
Array shown  
Describe array in terms of rows, columns. | **3EX.8**  
*N-tiles out to manipulate*  
Build arrays, then describe them. | | |
| **3E6 Division w/ arrays**  
Array shown  
Given number in each row and number of rows, determine total number. | **3EX.9**  
*Array partially screened*  
------------------------> | **3EX.10**  
*Array screened*  
------------------------> | **3EX.11**  
*(Array, with written task alongside)*  
------------------------> |
| **PHASE II: Incrementing and decrementing by 2s, 5s, 4s, and 3s** *(Red Book chapter 7, instruction phase 3)* | | |
| **3E7 Inc- & decrementing by 2s in the range 2 to 20**  
N-tiles shown  
After each inc-/decrement, say number of tiles and number of dots. (Extend to multiple unit jumps.) | **3EX.12**  
*N-tiles face-down*  
------------------------> | **3EX.13**  
*N-tiles screened*  
------------------------> | **3EX.14**  
Written  
Record sequence of products systematically.  
------------------------> |
| **3E8 Inc- & decrementing by 5s in the range 5 to 50** | | | | |
| **3E9 Inc- & decrementing by 4s in the range 4 to 40** | | | | |
| **3E10 Inc- & decrementing by 3s in the range 3 to 30** | | | | |

Table continued overleaf…
### Teaching table 3E: Multiplicative Strategies

#### PHASE III: Developing multiplicative strategies: two 1-digit factors (Red Book chapter 7, instruction phase 4)

<table>
<thead>
<tr>
<th>Topic</th>
<th>3E11 Multiplication involving a given factor: 2; 10; 5</th>
<th>3E12 Multiplication involving a given factor: 3; 4</th>
<th>3E13 Multiplication involving a given factor: 9; 6; 8; 7</th>
<th>3E14 Multiplication involving any two factors 0 to 10</th>
<th>3E15 Division involving a given factor: 2; 10; 5</th>
<th>3E16 Division involving a given factor: 3; 4</th>
<th>3E17 Division involving a given factor: 9; 6; 8; 7</th>
<th>3E18 Division involving any two factors 0 to 10</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Array shown / flashed / screened Given number of rows and columns, determine total dots. Describe strategy.</td>
<td>Array flashed / screened Notate and refine strategies.</td>
<td>&quot;</td>
<td>&quot;</td>
<td>Array shown / flashed / screened Given number of rows and columns, determine total dots. Describe strategy.</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
</tr>
<tr>
<td></td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
</tr>
</tbody>
</table>

#### PHASE IV: Developing multiplicative strategies: a 1-digit and a 2-digit factor (Red Book chapter 7, instruction phase 6)

<table>
<thead>
<tr>
<th>Topic</th>
<th>3E19 Multiplication with a 1-digit and a 2-digit factor</th>
<th>3E20 Division with a 1-digit and a 2-digit factor</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Array shown Given number of rows and columns, determine total dots. Notate and refine strategies.</td>
<td>Array shown Given number of rows and columns, determine number of columns. Notate and refine strategies.</td>
</tr>
<tr>
<td></td>
<td>Array flashed / screened Notate and refine strategies.</td>
<td>Array flashed / screened Notate and refine strategies.</td>
</tr>
</tbody>
</table>
## Domain 3F: Multiplication Basic Facts

### MIDDLE NUMBER BAND

<table>
<thead>
<tr>
<th>Topic</th>
<th>Teaching Procedures</th>
<th>Green book</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>PHASE I: Habituating sequences of multiples</strong> <em>(Red Book chapter 7, ~instruction phase 2)</em></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
| 3F1 Sequences by 2s, up to 20 *(Forwards and backwards)* | N-tiles visible → face-down → screened  
Say cumulative number of dots.  
Verbal only  
Say NWS. | | |
| 3F2 Sequences by 5s, up to 50 | “ | “ | “ | “ | |
| 3F3 Sequences by 3s, up to 30 | “ | “ | “ | “ | |
| 3F4 Sequences by 4s, up to 40 | “ | “ | “ | “ | |
| **PHASE II: Habituating basic facts** *(Red Book chapter 7, instruction phase 5)* | | |
| 3F5 Level 1a: 2 x 2-5; 2-5 x 2 | Multiplication expression cards shown  
Say product, and strategy.  
Multiplication expression cards flashed  
Say product quickly. | | |
| 3F6 Level 1b: 2-5 x 2-5 | “ | (Progress to include all cards up to this level.) | “ | (Progress to include all cards up to this level.) | |
| 3F7 Level 2a: 1s, 10s | “ | “ | “ | “ | |
| 3F8 Level 2b: 2s, 5s | “ | “ | “ | “ | |
| 3F9 Level 3a: 0s | “ | “ | “ | “ | |
| 3F10 Level 3b: 3s, 4s | “ | “ | “ | “ | |
| 3F11 Level 4a: 9s, 6s | “ | “ | “ | “ | |
| 3F12 Level 4b: 8s, 7s | “ | “ | “ | “ | |
Appendix 5  -  Lesson 10 Analysis Chart (A3 size)

This reproduces Figure 5.1, in full A3 size.
### Lesson 10 Analysis Chart: Tasks, responses, and dimensions

#### Layer A Analysis: Observations

<table>
<thead>
<tr>
<th>Blair’s responses</th>
<th>Layer B Analysis: Local interpretations</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Tasks</strong></td>
<td><strong>RNG</strong></td>
</tr>
<tr>
<td><strong>10.5 Incrementing &amp; decrementing with 5-tiles</strong></td>
<td></td>
</tr>
<tr>
<td>5.1 Visible: One 5, two 5s...six 5s.</td>
<td>✓</td>
</tr>
<tr>
<td>How can you see 30 there?</td>
<td>✓</td>
</tr>
<tr>
<td>“10, 20, 30” coordinated with the three pairs of tiles.</td>
<td>Lo</td>
</tr>
<tr>
<td>5.2 Screened: another 5 (35)</td>
<td>✓</td>
</tr>
<tr>
<td>5.3 another 5 (40; 8 5s)</td>
<td>✓</td>
</tr>
<tr>
<td>5.4 another 5 (45; 9 5s)</td>
<td>× “It wouldn’t be even...” but answered 50.</td>
</tr>
<tr>
<td>5.5 another 5 (30; 5 5s)</td>
<td>✓</td>
</tr>
<tr>
<td>5.6 add two 5s (40: 8 cards). Let’s write this down, this is important.</td>
<td>Hi</td>
</tr>
<tr>
<td>Explained strategy that 30 would be six cards.</td>
<td>Hi</td>
</tr>
<tr>
<td>“Take the zero off the 30, then just double the 3.”</td>
<td>Hi</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>10.6 Division with 5-tiles: N÷F</strong></td>
<td></td>
</tr>
<tr>
<td>6.1 40 dots</td>
<td>✓ 8 cards</td>
</tr>
<tr>
<td>6.2 50 dots</td>
<td>✓ 10 cards. “See, it’s working!”</td>
</tr>
<tr>
<td>6.3 70 dots</td>
<td>✓ 140, no...14. Counted tiles in pairs “10, 15, 20...no 2, 4, 6...”</td>
</tr>
<tr>
<td>Check</td>
<td></td>
</tr>
<tr>
<td>6.4 60 dots</td>
<td>✓ 12 cards</td>
</tr>
<tr>
<td>What are you going to see under there?...</td>
<td></td>
</tr>
<tr>
<td>How many cards for 30?---for 60?</td>
<td></td>
</tr>
<tr>
<td>How could you get that 12 just from 60?</td>
<td></td>
</tr>
<tr>
<td>“Two cards here, 2 4 6 that’s 30, then double that.”</td>
<td>Hi</td>
</tr>
<tr>
<td>“Two, four, six...; “Two, four, six—twelve.”</td>
<td></td>
</tr>
<tr>
<td>“Take the zero off the end of the 60,...double the 6.”</td>
<td></td>
</tr>
<tr>
<td>6.5a 90 dots</td>
<td>✓ 18 cards</td>
</tr>
<tr>
<td>How many 10s? Don’t count.</td>
<td></td>
</tr>
<tr>
<td>How could you get that 12 just from 60?</td>
<td></td>
</tr>
<tr>
<td>Lifted screen: How many dots?</td>
<td></td>
</tr>
<tr>
<td>How many 5s?</td>
<td></td>
</tr>
<tr>
<td>6.5b How many 5s?</td>
<td>5, 10, 15...××</td>
</tr>
<tr>
<td>How many 10s?</td>
<td>Eight ××</td>
</tr>
<tr>
<td>Wrote: 90 → 18 × 5s → 9 × 10s</td>
<td></td>
</tr>
<tr>
<td>80 is nine 10s. Is that surprising?</td>
<td>⊘</td>
</tr>
<tr>
<td>“No...obviously it is 90.”</td>
<td></td>
</tr>
<tr>
<td>6.5c</td>
<td></td>
</tr>
<tr>
<td>7.1 80: find T, F. Lifted screen: Check T, F.</td>
<td>✓</td>
</tr>
<tr>
<td>7.2 60: find T, F.</td>
<td>✓</td>
</tr>
<tr>
<td>7.3 4 10s: find F, N. How did you just check that?</td>
<td></td>
</tr>
<tr>
<td>“I looked here [at arrow sentences]”</td>
<td></td>
</tr>
<tr>
<td>7.4 2 10s: find F, N.</td>
<td>✓</td>
</tr>
<tr>
<td>7.5 14 5s: find F, N. Lifted screen: Check N, F, T.</td>
<td>✓</td>
</tr>
<tr>
<td>7.6 Why is it?</td>
<td></td>
</tr>
<tr>
<td>“You can fit two 5s into one 10...So it’s basically doubling that number.”</td>
<td>Lo</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>10.7 Varied orientations with 5-tiles, recording in a table</strong></td>
<td></td>
</tr>
<tr>
<td>7.1 80: find T, F. Lifted screen: Check T, F.</td>
<td>✓</td>
</tr>
<tr>
<td>7.2 60: find T, F.</td>
<td>✓</td>
</tr>
<tr>
<td>7.3 4 10s: find F, N. How did you just check that?</td>
<td></td>
</tr>
<tr>
<td>“I looked here [at arrow sentences]”</td>
<td></td>
</tr>
<tr>
<td>7.4 2 10s: find F, N.</td>
<td>✓</td>
</tr>
<tr>
<td>7.5 14 5s: find F, N. Lifted screen: Check N, F, T.</td>
<td>✓</td>
</tr>
<tr>
<td>7.6 Why is it?</td>
<td></td>
</tr>
<tr>
<td>“You can fit two 5s into one 10...So it’s basically doubling that number.”</td>
<td>Hi</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**CODES**

<table>
<thead>
<tr>
<th>Codes as per Table 4.2.</th>
<th>NG: 0 = low 5s</th>
<th>HI: 50+ = beyond 50</th>
</tr>
</thead>
<tbody>
<tr>
<td>Terms in multiples of 5:</td>
<td>N: total</td>
<td>N = tens digit of numeral N</td>
</tr>
<tr>
<td>Proportional N-F-T strategies:</td>
<td>e.g. 2n→F = Doubling n gives F</td>
<td>1/2n→F = Halving n gives T</td>
</tr>
<tr>
<td>ORN Orientation:</td>
<td>e.g. N-F = given N, find F</td>
<td>inc-N&amp;F = given increment, find F and N</td>
</tr>
<tr>
<td>STR Structuring and strategies:</td>
<td>= no STR attention</td>
<td>* = some STR</td>
</tr>
</tbody>
</table>
Appendix 6  - High 5s Analysis Chart (A3 size)

This reproduces Figure 7.1, in full A3 size.
### High 5s Analysis Chart: Tasks, responses, and dimensions

<table>
<thead>
<tr>
<th>Layer A Analysis: Observations</th>
<th>Layer B Analysis: Local interpretations</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Task type and tasks</strong></td>
<td><strong>RNG</strong></td>
</tr>
<tr>
<td><strong>8.8 Multiplication with 5-tiles → bare numbers</strong></td>
<td></td>
</tr>
<tr>
<td>Bare, recording in a table as:</td>
<td>HI</td>
</tr>
<tr>
<td>2 × 5 = 1 × 10 = 10</td>
<td></td>
</tr>
<tr>
<td>2 × 5, 4 × 5, 6 × 5, 8 × 5, 10 × 5</td>
<td></td>
</tr>
<tr>
<td>16 × 5, 22 × 5, 26 × 5, 46 × 5</td>
<td></td>
</tr>
<tr>
<td><em>What’s the relationship between 12 × 6, 14 × 7, 16 × 8?</em></td>
<td>HI</td>
</tr>
<tr>
<td><strong>9.0 Incrementing &amp; decrementing with 5-tiles</strong></td>
<td></td>
</tr>
<tr>
<td>Visible: One 5, two 5s, ... six 5s.</td>
<td>HI</td>
</tr>
<tr>
<td>Screened: Seven 5s (55) +6(40) +5 (45)</td>
<td></td>
</tr>
<tr>
<td>take three 5s (30) + two 5s (40)</td>
<td></td>
</tr>
<tr>
<td>Of insight: Let’s write this down, this is important.</td>
<td>HI</td>
</tr>
<tr>
<td><strong>10.6 Division with screened 5-tiles (Given N find F)</strong></td>
<td></td>
</tr>
<tr>
<td>40, 50, 70</td>
<td>HI</td>
</tr>
<tr>
<td>60—What are you going to see under there?</td>
<td></td>
</tr>
<tr>
<td>90—How many 10s? Don’t count.</td>
<td></td>
</tr>
<tr>
<td>Wrote arrow sentence: 90 → 18 × 5 → 9 × 10s.</td>
<td></td>
</tr>
<tr>
<td><strong>10.7 Varied orientations with screened 5-tiles</strong></td>
<td></td>
</tr>
<tr>
<td>Recorded in a table as: 80 → 16 × 5s → 8 × 10s</td>
<td>HI</td>
</tr>
<tr>
<td>80; 60 (find T &amp; F); 410s; 210s (find F &amp; N); 145s (find T &amp; N). Why is it?</td>
<td></td>
</tr>
<tr>
<td><em>Inquiring about relations in the table.</em></td>
<td>HI</td>
</tr>
<tr>
<td><em>Checked against patterns in table.</em></td>
<td></td>
</tr>
<tr>
<td><em>Checked X, F, T in tiles. Everything is correct.</em></td>
<td></td>
</tr>
<tr>
<td><em>You can fit two 5s into one 10...it’s basically doubling.</em></td>
<td></td>
</tr>
<tr>
<td><strong>11.3 Multiplication with 5-tiles (Given F find N)</strong></td>
<td></td>
</tr>
<tr>
<td>Visible: 2s, 3s, 4s, 5s, 6s, 5s</td>
<td>HI</td>
</tr>
<tr>
<td>Screened: 8s 5s, 4s, 5s, 6s, 5s, 10s 9s, 5s, 8s, 10s, 5s, 6s, 5s, 9s</td>
<td></td>
</tr>
<tr>
<td><em>60.</em> Corrected after 6s 5s shown again in tiles.</td>
<td></td>
</tr>
<tr>
<td><em>25.</em> Corrected after shown pair of 5-tiles as 10.</td>
<td></td>
</tr>
<tr>
<td><em>9s confusion: ‘10 off 50.’</em></td>
<td></td>
</tr>
<tr>
<td><strong>11.4 Varied orientations with screened 5-tiles</strong></td>
<td></td>
</tr>
<tr>
<td>Recorded with parallel expressions notation.</td>
<td>HI</td>
</tr>
<tr>
<td>Find T &amp; N: 6s, 8s, 5s, 4s, 5s</td>
<td></td>
</tr>
<tr>
<td>Find N &amp; F: 510s, ...Here’s one of my 10s there.</td>
<td></td>
</tr>
<tr>
<td>Find T &amp; F: 80, 60, 145s</td>
<td></td>
</tr>
<tr>
<td>Find T &amp; N: 145s</td>
<td></td>
</tr>
<tr>
<td><strong>12.4 Division with bare numbers (Given N find F &amp; T)</strong></td>
<td></td>
</tr>
<tr>
<td>Parallel expressions. No 5-tiles: If I made 70...</td>
<td>HI</td>
</tr>
<tr>
<td>70, 40, 60</td>
<td></td>
</tr>
<tr>
<td>85, 95</td>
<td></td>
</tr>
<tr>
<td><strong>12.5 Multiplication with bare numbers (Given F - N)</strong></td>
<td></td>
</tr>
<tr>
<td>Posed and recorded in number sentences.</td>
<td>HI</td>
</tr>
<tr>
<td>6×5...Picked up the 5-tiles...</td>
<td></td>
</tr>
<tr>
<td>9×5...Challenge A: solve without counting from 1...</td>
<td></td>
</tr>
<tr>
<td>Challenge B: find a closer multiple</td>
<td></td>
</tr>
<tr>
<td>12×5, 15×5</td>
<td></td>
</tr>
<tr>
<td>18×5, 24×5 Can you get there without adding?</td>
<td></td>
</tr>
<tr>
<td>Find T...Last week you saw something...</td>
<td></td>
</tr>
<tr>
<td><strong>12.6 Multiplication with bare numbers (F - T &amp; N)</strong></td>
<td></td>
</tr>
<tr>
<td>16 × 5</td>
<td>HI</td>
</tr>
<tr>
<td>Assisted writing in parallel exp. notation.</td>
<td></td>
</tr>
<tr>
<td>We’ve got more...we’ve got 16 of these. [5-tile]</td>
<td></td>
</tr>
<tr>
<td>Parallel exp: 12×5, 14×5, ... Do you believe that?</td>
<td></td>
</tr>
<tr>
<td>Screened notation: 18×5</td>
<td></td>
</tr>
</tbody>
</table>

**Interpretations of Blair’s responses:**
- Counted pairs of 5-tiles as 10s.
- Recognised uneven, but relied on orien. and counting by 5s, 10 by 10s.
- Accepted concepts of 5-tiles and 10s.
- And more...
### Layer A Analysis: Observations

<table>
<thead>
<tr>
<th>Task type and tasks</th>
<th>Blair’s responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>14.3 Varied orientations, 5-tiles → bare numbers</td>
<td></td>
</tr>
<tr>
<td>Visible: 6 5s, find N&amp;F</td>
<td></td>
</tr>
<tr>
<td>Screened: 8 5s, find T&amp;N 60, find T&amp;F</td>
<td></td>
</tr>
<tr>
<td>Bare: 80, find F</td>
<td></td>
</tr>
<tr>
<td>✔ via five 5s + 5.</td>
<td></td>
</tr>
<tr>
<td>✔ Used parallel exp notation, ignored the tiles.</td>
<td></td>
</tr>
<tr>
<td>✔ Fluent.</td>
<td></td>
</tr>
<tr>
<td>15.3 Varied orientations with bare numbers</td>
<td></td>
</tr>
<tr>
<td>Screened: 6 5s, find N&amp;F</td>
<td></td>
</tr>
<tr>
<td>Bare: 8 5s, find N, 60, find F, 90, find F&amp;T.</td>
<td></td>
</tr>
<tr>
<td>✔ ‘Let me write this down.’ Wrote in parallel exp.</td>
<td></td>
</tr>
<tr>
<td>✔ Continued to use parallel exp notation.</td>
<td></td>
</tr>
<tr>
<td>✔ ‘For doing this type of thing, this strategy, you need an answer like 60, 40, or 30 or 4 10s.’</td>
<td></td>
</tr>
<tr>
<td>15.4 Varied orientations with bare numbers</td>
<td></td>
</tr>
<tr>
<td>Posed and recorded in number sentences.</td>
<td></td>
</tr>
<tr>
<td>10-5, 14-5; 60, find F: How would we write that normally?</td>
<td></td>
</tr>
<tr>
<td>_×5 = 60, _×5 = 80; Question: Could I also write it this way, writes 80 ÷ 5.</td>
<td></td>
</tr>
<tr>
<td>30-5, 50-5, 40-5.</td>
<td></td>
</tr>
<tr>
<td>14-5 via 10×5 + 4×5</td>
<td></td>
</tr>
<tr>
<td>Writes missing multiplier sentence: _×5 = 60.</td>
<td></td>
</tr>
<tr>
<td>‘Whaaaat? Hmm?’</td>
<td></td>
</tr>
<tr>
<td>40-5: pointed to some parallel exp notation, and said ‘I did that … I took 40, times it by 2–8’.</td>
<td></td>
</tr>
<tr>
<td>15.5 Discussion of strategy</td>
<td></td>
</tr>
<tr>
<td>Referring to solving 40÷5 by doubling 4:</td>
<td></td>
</tr>
<tr>
<td>Why does that trick work? Offered the tiles to help explain.</td>
<td></td>
</tr>
<tr>
<td>Explained, referring to the parallel exp notation.</td>
<td></td>
</tr>
<tr>
<td>15.6 Multiplication with bare numbers</td>
<td></td>
</tr>
<tr>
<td>5×9 posed as number sentence.</td>
<td></td>
</tr>
<tr>
<td>Try using a closer multiple.</td>
<td></td>
</tr>
<tr>
<td>Notated strategies with arrow sentences.</td>
<td></td>
</tr>
<tr>
<td>✔ via _×5 + 4×5 = 25 + 20 = 45</td>
<td></td>
</tr>
<tr>
<td>✔ via _×10 - 5 = 50 - 5 = 45</td>
<td></td>
</tr>
<tr>
<td>16.3 Multiplication with screened 5-tiles (F &amp; N)</td>
<td></td>
</tr>
<tr>
<td>Student to write in number sentences.</td>
<td></td>
</tr>
<tr>
<td>8 5s, find N.</td>
<td></td>
</tr>
<tr>
<td>Asked for another way. How many 10s?</td>
<td></td>
</tr>
<tr>
<td>‘There’s eight 5s’ [pointing to numeral 5].</td>
<td></td>
</tr>
<tr>
<td>‘I’ve got eight of these guys [holding up a 5-tile].</td>
<td></td>
</tr>
<tr>
<td>6 5s, find N. 7 5s, find N.</td>
<td></td>
</tr>
<tr>
<td>✔ via _×10 - 5 = 40</td>
<td></td>
</tr>
<tr>
<td>✔ ‘Eight 10s.’</td>
<td></td>
</tr>
<tr>
<td>✔ ‘Oh, There’s four 10s.’ → 40+</td>
<td></td>
</tr>
<tr>
<td>✔ 7.5s: ‘Three 10s plus one, plus 5.’</td>
<td></td>
</tr>
<tr>
<td>16.4 Varied orientations with a fixed table</td>
<td></td>
</tr>
<tr>
<td>including high 3s, 4s, and 5s</td>
<td></td>
</tr>
<tr>
<td>Recorded in number sentences.</td>
<td></td>
</tr>
<tr>
<td>Eight tasks, including three high 5s: 5-7, 30-5, 45-9.</td>
<td></td>
</tr>
<tr>
<td>✔ 45÷9: Checked table—‘Yeah, nine 5s equals 45.’</td>
<td></td>
</tr>
<tr>
<td>17.1 Varied orientations with a fixed table:</td>
<td></td>
</tr>
<tr>
<td>including high 3s, 4s, and 5s</td>
<td></td>
</tr>
<tr>
<td>Recorded in number sentences.</td>
<td></td>
</tr>
<tr>
<td>12 tasks, including: 6 _ _ &lt; 30</td>
<td></td>
</tr>
<tr>
<td>40 _ _ = 35 _ _ = … Here’s six 5s [3 pairs of 5-tiles]</td>
<td></td>
</tr>
<tr>
<td>27 _ _ =</td>
<td></td>
</tr>
<tr>
<td>✔ 6×5-30 as a known fact.</td>
<td></td>
</tr>
<tr>
<td>✔ Answered ‘5 times 5’, written as 40 ÷ 5 = 8.</td>
<td></td>
</tr>
<tr>
<td>35 ÷ _ _ = … [looking at six 5-tiles] → _ _ =</td>
<td></td>
</tr>
<tr>
<td>✔ ‘It can’t be 5 … by 3 equals 9.’</td>
<td></td>
</tr>
<tr>
<td>17.2 Varied orientations with varied n-tiles</td>
<td></td>
</tr>
<tr>
<td>Two tasks, including:</td>
<td></td>
</tr>
<tr>
<td>45 with 5-tiles, how many tiles?</td>
<td></td>
</tr>
<tr>
<td>✔ Wrote 45 ÷ 5 = 9. ‘I minused it off 10 times 5.’</td>
<td></td>
</tr>
</tbody>
</table>

### Layer B Analysis: Local interpretations

<table>
<thead>
<tr>
<th>Task type and tasks</th>
<th>Blair’s responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>14.3 Varied orientations, 5-tiles → bare numbers</td>
<td></td>
</tr>
<tr>
<td>Visible: 6 5s, find N&amp;F</td>
<td></td>
</tr>
<tr>
<td>Screened: 8 5s, find T&amp;N 60, find T&amp;F</td>
<td></td>
</tr>
<tr>
<td>Bare: 80, find F</td>
<td></td>
</tr>
<tr>
<td>✔ via five 5s + 5.</td>
<td></td>
</tr>
<tr>
<td>✔ Used parallel exp notation, ignored the tiles.</td>
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</tr>
<tr>
<td>✔ Fluent.</td>
<td></td>
</tr>
<tr>
<td>15.3 Varied orientations with bare numbers</td>
<td></td>
</tr>
<tr>
<td>Screened: 6 5s, find N&amp;F</td>
<td></td>
</tr>
<tr>
<td>Bare: 8 5s, find N, 60, find F, 90, find F&amp;T.</td>
<td></td>
</tr>
<tr>
<td>✔ ‘Let me write this down.’ Wrote in parallel exp.</td>
<td></td>
</tr>
<tr>
<td>✔ Continued to use parallel exp notation.</td>
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<tr>
<td>✔ ‘For doing this type of thing, this strategy, you need an answer like 60, 40, or 30 or 4 10s.’</td>
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</tr>
<tr>
<td>15.4 Varied orientations with bare numbers</td>
<td></td>
</tr>
<tr>
<td>Posed and recorded in number sentences.</td>
<td></td>
</tr>
<tr>
<td>10-5, 14-5; 60, find F: How would we write that normally?</td>
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<tr>
<td>_×5 = 60, _×5 = 80; Question: Could I also write it this way, writes 80 ÷ 5.</td>
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</tr>
<tr>
<td>30-5, 50-5, 40-5.</td>
<td></td>
</tr>
<tr>
<td>✔ via _×5 + 4×5 = 25 + 20 = 45</td>
<td></td>
</tr>
<tr>
<td>✔ via _×10 - 5 = 50 - 5 = 45</td>
<td></td>
</tr>
<tr>
<td>15.5 Discussion of strategy</td>
<td></td>
</tr>
<tr>
<td>Referring to solving 40÷5 by doubling 4:</td>
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</tr>
<tr>
<td>Why does that trick work? Offered the tiles to help explain.</td>
<td></td>
</tr>
<tr>
<td>Explained, referring to the parallel exp notation.</td>
<td></td>
</tr>
<tr>
<td>15.6 Multiplication with bare numbers</td>
<td></td>
</tr>
<tr>
<td>5×9 posed as number sentence.</td>
<td></td>
</tr>
<tr>
<td>Try using a closer multiple.</td>
<td></td>
</tr>
<tr>
<td>Notated strategies with arrow sentences.</td>
<td></td>
</tr>
<tr>
<td>✔ via _×5 + 4×5 = 25 + 20 = 45</td>
<td></td>
</tr>
<tr>
<td>✔ via _×10 - 5 = 50 - 5 = 45</td>
<td></td>
</tr>
<tr>
<td>16.3 Multiplication with screened 5-tiles (F &amp; N)</td>
<td></td>
</tr>
<tr>
<td>Student to write in number sentences.</td>
<td></td>
</tr>
<tr>
<td>8 5s, find N.</td>
<td></td>
</tr>
<tr>
<td>Asked for another way. How many 10s?</td>
<td></td>
</tr>
<tr>
<td>‘There’s eight 5s’ [pointing to numeral 5].</td>
<td></td>
</tr>
<tr>
<td>‘I’ve got eight of these guys [holding up a 5-tile].</td>
<td></td>
</tr>
<tr>
<td>6 5s, find N. 7 5s, find N.</td>
<td></td>
</tr>
<tr>
<td>✔ via _×10 - 5 = 40</td>
<td></td>
</tr>
<tr>
<td>✔ ‘Eight 10s.’</td>
<td></td>
</tr>
<tr>
<td>✔ ‘Oh, There’s four 10s.’ → 40+</td>
<td></td>
</tr>
<tr>
<td>✔ 7.5s: ‘Three 10s plus one, plus 5.’</td>
<td></td>
</tr>
<tr>
<td>16.4 Varied orientations with a fixed table</td>
<td></td>
</tr>
<tr>
<td>including high 3s, 4s, and 5s</td>
<td></td>
</tr>
<tr>
<td>Recorded in number sentences.</td>
<td></td>
</tr>
<tr>
<td>Eight tasks, including three high 5s: 5-7, 30-5, 45-9.</td>
<td></td>
</tr>
<tr>
<td>✔ 45÷9: Checked table—‘Yeah, nine 5s equals 45.’</td>
<td></td>
</tr>
<tr>
<td>17.1 Varied orientations with a fixed table:</td>
<td></td>
</tr>
<tr>
<td>including high 3s, 4s, and 5s</td>
<td></td>
</tr>
<tr>
<td>Recorded in number sentences.</td>
<td></td>
</tr>
<tr>
<td>12 tasks, including: 6 _ _ &lt; 30</td>
<td></td>
</tr>
<tr>
<td>40 _ _ = 35 _ _ = … Here’s six 5s [3 pairs of 5-tiles]</td>
<td></td>
</tr>
<tr>
<td>27 _ _ =</td>
<td></td>
</tr>
<tr>
<td>✔ 6×5-30 as a known fact.</td>
<td></td>
</tr>
<tr>
<td>✔ Answered ‘5 times 5’, written as 40 ÷ 5 = 8.</td>
<td></td>
</tr>
<tr>
<td>35 ÷ _ _ = … [looking at six 5-tiles] → _ _ =</td>
<td></td>
</tr>
<tr>
<td>✔ ‘It can’t be 5 … by 3 equals 9.’</td>
<td></td>
</tr>
<tr>
<td>17.2 Varied orientations with varied n-tiles</td>
<td></td>
</tr>
<tr>
<td>Two tasks, including:</td>
<td></td>
</tr>
<tr>
<td>45 with 5-tiles, how many tiles?</td>
<td></td>
</tr>
<tr>
<td>✔ Wrote 45 ÷ 5 = 9. ‘I minused it off 10 times 5.’</td>
<td></td>
</tr>
</tbody>
</table>

### CODES

<table>
<thead>
<tr>
<th>Codes as per</th>
<th>Responses</th>
<th>Terms in multiples of 5:</th>
<th>Proportional N-F-T strategies:</th>
<th>ORN Orientation:</th>
</tr>
</thead>
<tbody>
<tr>
<td>RNG Range:</td>
<td>00 - low 5s</td>
<td>N – total</td>
<td>e.g. 2n→F = Doubling n gives F</td>
<td>e.g. N-F = given N, find F</td>
</tr>
<tr>
<td>SET Setting:</td>
<td>[5] - 5-tiles visible</td>
<td>n – tens digit of numeral</td>
<td>IF→F = Halving F gives T</td>
<td>inc F&amp;N = given increment, find F and N</td>
</tr>
<tr>
<td>N &amp; T Notation:</td>
<td>30÷5,</td>
<td>– = no notation</td>
<td>STR Structuring and strategies:</td>
<td>– = no STR attention</td>
</tr>
</tbody>
</table>

| 4.2 |  

*F* = correct *v* = correct with confidence *x* = incorrect ... = thinking time