Use of grouping strategies to solve addition tasks in the range one to twenty by students in their first year of school: a teaching experiment

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August, 2014
I certify that the work presented in this thesis is, to the best of my knowledge and belief, original, except as acknowledged in the text, and that the material has not been submitted, either in whole or part, for a degree at this or any other university.

I acknowledge that I have read and understood the University’s rules, requirements, procedures and policy relating to my higher degree research award and to my thesis. I certify that I have complied with the rules, requirements, procedures and policy of the University (as they may be from time to time).

Andrea Dineen

27 August, 2014
Abstract

This study investigates a teaching approach based on grouping strategies to support students in habituating knowledge of the basic addition combinations. A literature review indicated that teaching students in the first year of school to solve one-digit addition tasks has focused mainly on the use of counting-based strategies. Only recently have studies explored the use of grouping strategies as either an alternative or a complementary approach to teaching students to solve one-digit addition tasks.

The participants in this study were a class of 20 first-year students, in a mid socioeconomic range school in a suburb of an Australian city. The study followed a design research methodology and comprised three phases. The first phase involved the development of a preliminary curriculum design. The second involved conducting a teaching experiment and the third involved a retrospective analysis.

Curriculum documents from 10 countries were reviewed regarding the timing of and emphasis on grouping strategies to solve addition tasks in the early years of school. The analysis of the documents found that reference to counting-based strategies is most prevalent in the Australian curriculum.

Pre- and post-assessments via clinical interviews were conducted with all 20 students. Each interview was videotaped and consisted of 12 task groups. The teaching sequence consisted of 24 one-hour lessons conducted across a seven-week period. The whole class components of each lesson were videotaped. The progression of three students was documented, and three case studies were constructed.

The pedagogical approach used during the teaching sequence focused on the development of students’ use of grouping strategies to solve addition tasks. Extensive use
was made of material settings designed to engender students’ additive reasoning. These settings included ten-frames, twenty-frames and the arithmetic rack.

The key outcomes of this study are: (a) students in the first year of school can show significant advancement in their number knowledge through a program focused on grouping strategies; (b) students’ part-part-whole knowledge of the structure of numbers can enhance their development of additive reasoning strategies; (c) the simultaneous focus on grouping strategies and formal arithmetic notation supports students’ conception of addition tasks as binary operations; and (d) the Phases of Early Grouping Strategies (PEGS) constitutes a viable learning progression from pre-numerical counting to efficient and flexible grouping strategies and thus PEGS provides a basis for mastery of addition tasks involving two one-digit addends.
Acknowledgements

I never could have predicted that a casual conversation on a balmy Thursday evening in Rundle Mall, Adelaide, could be the catalyst for this amazing journey. There are many people who need to be acknowledged and thanked for their support and encouragement.

Firstly, I must acknowledge the gratitude and deep respect I have for Dr Bob Wright. From beginning to end he has been generous with his time and sage advice, and a constant source of support and encouragement. It has been a privilege and a pleasure to work with him.

My thanks also to Dr Martin Hayden for his timely contributions and input; to Kev, for trusting in my ability to get it done, for keeping me sane, for picking up my slack in the busy life of a family, for the countless cups of coffee and words of encouragement, and for doing it all with a smile; to Abby, Georgia and Harry for tolerating me disappearing into the study, yet still being interested enough to ask “How many words left now, Mum?”; to my parents and all those friends who supported me in many ways and asked interested questions; and to fellow “Maths Edders” Dave, Pam, Vicki, Lucy, Noreen, Thi and Rumi for their encouragement.

Thanks also to Jill and the students of Prep F. They provided me with many opportunities to observe, reflect and learn how children think about addition strategies; and finally to my principals Paul and James who both supported me in every way they could.
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<td>ACARA</td>
<td>Australian Curriculum, Assessment and Reporting Authority</td>
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<td>FNWS</td>
<td>Forward Number Word Sequence</td>
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<td>PEGS</td>
<td>Phases of Early Grouping Strategies</td>
</tr>
<tr>
<td>RME</td>
<td>Realistic Mathematics Education</td>
</tr>
<tr>
<td>SEAL</td>
<td>Stages of Early Arithmetical Learning</td>
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<td>TIMSS</td>
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Chapter One: Introduction

The basic addition combinations, that is, addition tasks consisting of two addends in the range 1 to 10, have been considered important foundational knowledge since Babylonian times (Baroody, Bajwa, & Eiland, 2009). In this modern age, habituation of the basic addition combinations still features prevalently in the curriculum documents of many countries. Although in some contexts the basic number combinations are referred to as the “basic facts”, truly knowing an arithmetic combination means far more than knowing an isolated fact (Clements & Sarama, 2009). Therefore, the term addition combinations will be used in this study. This teaching experiment focused on students’ learning of the basic addition combinations, that is, additions of two numbers in the range 1 to 10.

During the 1980s, the advent of constructivism as an approach to thinking about how students learn mathematics changed the landscape of many classrooms across parts of the Western world. Constructivism as a pedagogical approach has, as its core, the teacher as an engineer, manipulating learning situations that encourage students to make sense of mathematical concepts and to be active in their construction of knowledge. However, even in classrooms based on constructivist pedagogy, fluency combined with knowledge of the basic addition combinations is considered to be an important building block of conceptual knowledge of arithmetic. Baroody et al. (2009) see this as involving adaptive expertise and describe it as: “well-understood knowledge that can be applied efficiently, appropriately, and flexibly to new, as well as familiar, tasks” (p. 70).

Prior to the advent of constructivist approaches to pedagogy, direct instruction was the prevalent method of instruction for students in most mathematics classrooms, and learning by rote was an expectation of virtually all students. Almost 100 years ago, Thorndike (1922) asserted that “it seems probable that little is gained by using any of the
child’s time for arithmetic before grade 2, though there are many arithmetic facts that they can [memorize by rote] in grade 1” (p. 198). This seems to be a reasonable characterisation of attitudes towards student learning in classrooms adopting direct instruction. However, in a constructivist classroom, the role of the teacher is to provide rich educational experiences, which afford students the opportunities to construct their own meaning from the activity and integrate this with their current mathematical knowledge. This study investigates an approach based on fostering grouping strategies to support students in habituating knowledge of the basic addition combinations. In this study, the term “grouping strategy” is used to describe a strategy to solve an addition task which does not involve counting-by-ones.

Counting is described as the “first and most basic and important algorithm” (Clements & Sarama, 2009, p. 22), and counting strategies are typically the first taught in the early years of school as a means of determining the sum of two numbers. The prevalence of counting-based addition strategies in curriculum documents is described in Chapter Two. However, Gray (2010) challenges the appropriateness of this focus on counting-based strategies in arguing that students in their third and fourth year of school may be reluctant to use alternative, more mathematically sophisticated approaches to solve simple addition tasks if the sole focus of their schooling prior to this has been on the use of counting strategies.

In determining the topic for this study, research conducted in the last 120 years regarding the teaching, learning and retention of the basic addition combinations was reviewed and is presented in Chapter Two. In reviewing the literature, it became evident that research into the teaching and learning of strategies to solve one-digit addition tasks has focused mainly on the use of counting-based strategies. It is only in recent years that
some studies have explored the use of grouping strategies as either an alternative or a complementary approach to count-by-ones strategies to solve one-digit addition tasks.

Researchers have written about the greater efficiency and higher level of mathematical sophistication used to solve addition tasks by way of grouping or non-counting strategies (Baroody et al., 2009; Carpenter & Moser, 1984; Clarke, 2005; Cobb, McClain, Whitenack, & Estes, 1995; Hatano, 1982; Putnam, deBettencourt, & Leinhardt, 1990; Sarama & Clements, 2009a; Steffe, 1979; Van den Heuvel-Panhuizen, 2008; Wright, Martland, & Stafford, 2006; Young-Loveridge, 2002). However, many of these studies focused on students with more than one year of formal schooling. Some researchers questioned the view that counting-based strategies should be the sole basis for advancing numeracy development (Bobis, 1996; Labinowicz, 1985; Van Luit & Schopman, 2000; Willis, 2002). According to Bobis (1996): “While counting is initially an important strategy used by children to deal with numbers, an emphasis only on counting does not allow children to develop a rich variety of number relationships” (p. 28).

In many early years mathematics classrooms, students are taught to solve a task such as 5 + 3 by following a process explained to them in terms such as “start at five, and count on three more”. This example illustrates the count-on-by-ones strategy which is the most sophisticated count-by-ones strategy. Prior to using this strategy, students typically use a count-by-ones-from-one strategy, in which they count each addend by ones first, and then count-by-ones from one to determine the total. Wright, Stanger, Stafford, and Martland (2006) describe this as “counting-forward-from-one-three-times” (p. 83). Baroody, Wilkins, and Tiilikainen (2003) describe both of these examples as using a unary approach to solving addition tasks. Similarly, Gray (2010) describes those students who perceive “5 + 3” as an instruction to start at five and count-on three as on the other side of the “proceptual divide”
from those students who can regard five and three as two parts of the whole, eight. The proceptual divide describes

those who cling to the comfort of counting procedures that, at best, enable them to solve simple problems by counting and those who develop a more flexible form of arithmetic in which the symbols can be used dually as processes or as concepts to manipulate mentally. (Gray & Tall, 2007, p. 26)

The second group of students Gray refers to regard $5 + 3$ as a binary operation. The focus of this study is to reduce the galvanisation of the proceptual divide for a group of students in their first year of school, and encourage them to perceive five and three, for example, as parts of the whole. This focus involves extensive use of visualisation strategies, formal arithmetic notation, and quinary- and ten-based material settings in a mathematics program with a focus on the use of grouping strategies to solve simple addition tasks.

This study was carried out in the tradition of design research (Gravemeijer, Cobb, Bowers, & Whitenack, 2000). It was implemented as a classroom-based teaching experiment (Steffe & Thompson, 2000) and conducted with a class of students in their first year of school. The experiment considered the effectiveness of a focus on grouping strategies in facilitating student knowledge of the basic addition combinations. A hypothetical learning trajectory (Gravemeijer, 2004a) was developed and enacted as a design experiment over the course of 24 lessons. During the teaching experiment, ongoing reflection informed future teaching in terms of the mathematical content knowledge to be addressed, and the selection of material settings and ways to record student thinking.

1.1 Research Questions

This study has aimed to answer five research questions:

1. What levels of student knowledge are prerequisite to the efficient use of grouping strategies to solve addition tasks involving two addends in the range 1 to 10?

2. How does a teaching focus on grouping strategies influence students’ methods of solving simple addition tasks?
3. To what extent is it appropriate and useful to introduce formal arithmetic notation to students in the first year of school who are reasoning additively in the material settings of quinary- and ten-based materials?

4. To what extent does knowledge of the part-part-whole structure of numbers support the use of grouping strategies to solve addition tasks involving two addends in the range 1 to 10?

5. What role does a teaching focus on visualisation have in supporting the use of grouping strategies to solve simple addition tasks?

1.2 Thesis Outline

This thesis documents the course of this study in the following way. Chapter Two reviews, from a historical perspective, the literature on the teaching and learning of the basic addition number combinations and the importance that has always been afforded counting-based strategies. Curriculum documents from Australia and other countries are then evaluated with regard to the emphasis placed on the use of counting strategies to solve addition tasks in the first few years of school. Literature regarding the role of concrete materials and formal arithmetic notation in the teaching and learning of mathematics is also examined. Finally, literature focusing on the use of non-counting-based strategies to solve addition tasks is reviewed.

Chapter Three summarises the research traditions that helped to shape the design research approach and teaching experiment methodology used in this study while Chapter Four outlines the methodology of the study.

Chapter Five presents the results from the pre- and post-assessments conducted with 20 students in their first year of school. Twelve task groups were presented in an interview situation to all students at the beginning and the end of the teaching sequence. Each of these interviews was videotaped to facilitate analysis of student responses.
Chapter Six discusses the progress in students’ results from the pre- to the post-assessments, and describes ways to account for the vicissitudes in the students’ ability to solve addition tasks. It outlines a model referred to as the Phases of Early Grouping Strategies (PEGS) which emerged from this study as a learning progression to describe students’ use of more mathematically sophisticated grouping strategies to solve simple addition tasks.

Chapter Seven describes three case studies that compare the results of three individual students from the pre- and post-assessments. Intensity sampling (Patton, 2002) was adopted: each of these students was representative of a level of growth in mathematical knowledge from the pre- to the post-assessments. Jack was selected as a representative of students from the low range, Bridget from the mid-range, and Tracey from the high range. Examples from the teaching sequences are described in an attempt to account for changes in student knowledge and in strategy use from the pre- to the post-assessments. The results of these three case studies are discussed in Chapter Eight. Chapter Nine summarises the findings of the study, draws conclusions and makes recommendations for future research.
Chapter Two: Literature Review

2.1 Overview

This chapter reviews the relevant research literature in order to inform the development of the sequence, structure, content and purpose of this study. The importance of arithmetical strategies in the early years of school will be established, particularly knowledge of the basic addition combinations as a foundation for future learning. Askew (2013) suggests that there are generally three approaches used to solve basic addition combinations: counting, decomposition and retrieval. Literature regarding each of these approaches is reviewed.

The literature will be reviewed regarding the primacy afforded counting strategies, both in pedagogical approaches and current curriculum documents. With a focus in this study on the first year of school, the literature regarding the importance of concrete materials in the foundational years is evaluated, alongside literature that examines the timing and desirability of the introduction of arithmetic notation. Finally, after the more conventional approaches to teaching and learning to solve one-digit addition tasks through the use of counting strategies have been assessed, literature regarding the use of grouping strategies to solve the same task types is reviewed. This sequence is explained in more detail below.

Section 2.2 describes the high volume of research conducted over the last century regarding strategies used to solve one-digit addition tasks, reflecting the importance that education, and society in general, attributes to the mastery of these basic addition combinations. A chronological summary of the theories about the way in which basic addition combinations are stored and accessed from memory, and a description of how these theories have informed pedagogical approaches to the teaching of this topic are
presented. The literature indicates that, historically, direct retrieval was the preferred way of “knowing” but, in the absence of this, the predominant strategies for solving one-digit addition tasks involved the use of counting-based strategies. “Strategies” refers to “cognitive operations that are effortful and subject to consciousness” (Bjorklund, Muir-Broaddus, & Schneider, 1990, p. 97).

Section 2.3 analyses curriculum documents produced by 10 countries for the inclusion of statements regarding the teaching and learning of one-digit addition tasks. These curriculum documents are reviewed with respect to the emphasis placed on counting-based and grouping strategies to solve addition combinations. Each of these countries is then considered in terms of their ranking in TIMSS (Trends in International Mathematics and Science Study) (Mullis, Martin, Foy, & Arora, 2012).

Section 2.4 reviews the research literature concerning the structure and challenges of using concrete materials to support the learning of early arithmetical concepts. The role of teachers and students, and the elements of the classroom culture that support effective use of concrete materials are considered.

Section 2.5 analyses the literature regarding the introduction and use of formal arithmetic notation and its role in supporting student learning in the early years of school. The link between formal mathematical recording and students’ interpretation of arithmetic notation for addition combinations as a process to be enacted or as a representation of the part-whole relationship is also reviewed.

Finally, Section 2.6 reviews the literature regarding the use of grouping strategies to solve addition tasks. This section explores the use of visualisation strategies to support part-part-whole knowledge of number, which in turn, can support grouping strategies, including quinary- and ten-based strategies.
2.2 A Historical Approach to Teaching and Learning of the Basic Addition Combinations

“Memorization of the basic number combinations, which include single-digit addition items (e.g., $9 + 3 = 12$) and related subtraction items (e.g., $12 - 9 = 3$), has been a central goal of elementary instruction since ancient Babylonian times” (Baroody et al., 2009, p. 69). Locuniak and Jordan (2008) argued that knowledge of number combinations is the strongest single predictor of calculation fluency in later grades. There is also little dispute that automatisation of correct solutions to the basic addition combinations is a goal of early arithmetical teaching and learning. However, debate concerning the most effective pedagogical approach to support the automatisation of these number combinations persists (Cowan et al., 2011), and is the focus of this section.

Whilst automatisation remains a goal of arithmetical learning, Campbell and Xue (2001) cautioned that even for adults “… exclusive reliance on retrieval for simple arithmetic probably is a rare achievement” (p. 314). Similarly, Imbo and Vandierendonck (2008) believe that “even skilled adults are not always able to retrieve simple-arithmetic facts from their memory” (p. 528). Therefore although a goal of teaching, automatisation of addition combinations is not achieved by all students.

Across many decades, debate has persisted regarding how the basic addition combinations are most efficiently memorised, how they are recalled, and what is the most effective way to teach students in order to expedite the memorisation and efficient reproduction of these combinations.

Terms such as recall, reproduce, remember, habituate, memorise, automatise and retrieve are commonly used in the research literature to define the way the basic addition combinations are stored and accessed from memory. “We define automaticity as correct and accurate knowledge and fluency as automaticity plus complex understandings that
provide the foundations for that automaticity and for adaptive expertise [emphasis in original]” (Sarama & Clements, 2009a, p. 139). Henry and Brown (2008) defined fluency as “a combination of derived-fact strategies and retrieval from long-term memory” (p. 180).

The terms “automatised” and “direct retrieval” are often used synonymously in the literature. Direct and immediate retrieval of the basic addition combinations is the least cognitively taxing and most time-saving strategy for solving one-digit addition tasks, placing less demands on working memory, and allowing attention to be focused on more complex tasks (Baroody et al., 2009; Bjorklund et al., 1990). Working memory has been described as “the ability to maintain explicitly a mental representation of some amount of information, while being engaged simultaneously in other mental processes” (Geary, Hoard, Bryd-Craven, & DeSoto, 2004, p. 124).

In the context of this thesis, the term automatised will be used to refer to direct, expedient retrieval of addition combinations with minimal cognitive effort, whilst retrieval will indicate that the addition task is solved via the use of either grouping or counting strategies. It is interesting to note that, Verschaffel, Greer, and De Corte (2007) cautioned that it can be extremely difficult to know whether addition combinations are automatised or simply computed using a strategy that is executed with great speed: “Neither self-report nor speed will distinguish automatic retrieval from rapid non-conscious computation or application of a rule” (p. 564).

In some contexts, the basic addition combinations are referred to as the “basic facts”. Clements and Sarama (2009) observed that truly knowing an arithmetic combination means far more than knowing an isolated fact. Therefore, as stated earlier, for the purposes of this study, the term addition combinations will be used in preference to addition facts. The addition combinations are defined as one-digit addition tasks with both addends in the range 1 to 10.
Most research and writing about the teaching and learning of the basic addition combinations from the early part of the twentieth century focus on the use of memorisation strategies, primarily facilitated by drill and rote learning. Young children can learn to recite the oral counting sequence by memorising the number word sequence, independently of being able to coordinate this oral count with perceptual items (Steffe, 1992). Similarly, students can memorise correct responses to the basic addition combinations without the accompanying conceptual understanding. This process is often described as rote learning. Gray and Tall (2007) cautioned that rote learning can be used to respond successfully to standard problems presented in a familiar context but, in order to be able to solve novel problems, connections must be made and a level of flexibility in thinking attained. Proponents of rote learning assert that, in order to develop strong associations between two numbers and their sum, the key to successful memorisation is repetitious practice. However, Baroody et al. (2009) refuted this assertion on the basis that it is not clearly supported by empirical evidence. If solutions to addition combinations are stored only as isolated pieces of data, success in the use of a memorisation strategy can be limited. If drill and rote learning form the primary method of instruction, there is no allowance for students to use a strategy other than automatic recall to determine the solution, and when this direct retrieval fails, the answer is incorrect. Correspondingly, if some practice sessions result in the production of correct and incorrect solutions, then weak associations develop, which can result in students having low confidence in their knowledge (Henry & Brown, 2008). However, students who learn derived strategies alongside memorisation are more likely to master fluency (Sarama & Clements, 2009a).

Siegler (1988) compared the advantages of solving basic addition tasks via direct retrieval with the use of strategies:
Both retrieval and use of backup strategies have clear, though different, advantages. Retrieval can be executed much faster; backup strategies often yield high accuracy rates on problems where retrieval cannot. Ideally, children would use retrieval where it could be executed accurately and would use backup strategies where they were necessary for accurate performance. (p. 834)

2.2.1 Drill Versus Generalisation in the First Part of the Twentieth Century

In Western societies during the early twentieth century, the “drill method” based on repetitive drill and rote learning was the predominant instructional method used to teach the basic addition combinations. Also in existence, although far less popular and often ill-considered, was a “new conception of the pedagogy of arithmetic” (Thiele, 1938, p. 16) known as the “generalisation method”. This method was advocated by meaning theorists and encouraged the use of relationships and connections between basic addition combinations as a scaffold to learning (Brownell & Chazal, 1935; Swenson, 1949; Thiele, 1938). However, at the time, many members of the mathematics education community discredited the generalisation method as an “objectionable dependence upon reasoning or a chain of thinking” (Studebaker, Knight, & Findley, 1929, p. 20).

Drill theorists, who proclaimed the superiority of the rote learning method, believed that errors were often due to a “lack of sufficient drill to fix the correct responses” (Smith, 1921, p. 767). If students of this era used counting or derived strategies to determine the solution to a task, they were described as evading the important tasks of memorising addition combinations (Smith, 1921; Wheeler, 1939).

Smith (1921) gave context to the classroom environment of the time when he criticises the use of strategies other than memorisation: “We should be careful about letting pupils acquire forms or roundabout schemes for securing a result in the lower grades which will prove a handicap to them in the upper grades” (p. 765). Describing a similar approach, Wheeler (1939) referred to classroom practices in which “speed is regulated in an effort to control undesirable habits of computation” (p. 296).
Opposing pedagogical approaches regarding the teaching and learning of the basic addition combinations were put forward by drill theorists and meaning theorists. However, although preference for drill and rote learning was apparent in most classrooms at the time, some researchers advocated the use of strategies to solve addition combinations. Brownell and Chazal (1935) lamented that, “in many a classroom, instruction in arithmetic has become virtually synonymous with the administration of drill” (p. 18). As one of the first teams of researchers to challenge the primacy of drill and rote learning, they documented students’ solution strategies. Prior to Brownell and Chazal’s seminal study, research into the basic addition combinations focused predominantly on the frequency of correct responses and/or the time taken to respond, irrespective of the strategy used by the student to determine the solution. At this time, and for many years, research studies assumed that a slow response time indicated the use of a slow, non-direct retrieval strategy. Brownell and Chazal questioned the validity of using response time to attribute the same level of cognitive functioning to a student who solved an addition task using a count-by-ones strategy, and a student who used a grouping or derived strategy, or direct retrieval, just because their response time was the same.

As described, Brownell and Chazal (1935) were considered controversial due to the fact that they did not support drill and rote learning as the principal method for the teaching of basic addition combinations. In an attempt to further understand students’ use of cognitive strategies, Brownell and Chazal (1935) were among the first researchers to interview students about their solution strategies. This interview approach was unusual at the time. They concluded that “to be more effective drill must be preceded by sound instruction … Learning, not drill, is the important question in arithmetic [emphasis in original]” (Brownell & Chazal, 1935, p. 26).
As described above, early in the twentieth century, advocates of the drill theory were far more prevalent than advocates of meaning theory. However, in spite of the popular theory of the time, Thiele (1938) and Swenson (1949) supported Brownell and Chazal (1935) in promoting meaning theory. In his thesis which analysed data from more than 500 students, Thiele (1938) found that “the objective evidence seems to indicate that the achievements of the pupils taught by the generalization method, as determined by the testing program, are greatly superior to those attained by the pupils taught by the drill method” (p. 77).

Swenson (1949) supported Thiele’s (1938) finding when she studied the acquisition and retention of the 100 basic addition combinations by 332 students, taught via three different instructional methods she named the generalisation method, the drill method and the drill-plus method. Based on her results, she concluded emphatically that students taught by the generalisation method, which valued the relationships and connections between sets of addition combinations, far outperformed students taught by the drill or drill-plus method. The drill-plus method of instruction was similar to the drill method with two concessions: 1) students were encouraged to verify new addition combinations by manipulating concrete objects; and 2) addition combinations were presented in groups according to the size of the sum, rather than in miscellaneous order, as was the case when instruction followed the drill method.

Despite the convincing data from these and similar studies, drill and rote learning persisted for many years as the prevalent pedagogical approach to the teaching and learning of basic addition combinations. Described as the traditional view (Cowan et al., 2011) which “equates proficiency with having the solutions to basic calculations stored in long term memory so that they can be readily retrieved” (p. 786), this type of instruction still exists in many classrooms today. With this traditional view as a focus, memorisation and
rote learning take priority to the detriment of providing students with opportunities to find patterns and relationships within and between addition combinations (Baroody et al., 2009).

### 2.2.2 The Problem-size Effect in Solving Basic Addition Tasks

By the second half of the twentieth century, debate had shifted away from drill versus meaning to focus on the way in which addition combinations are stored and retrieved from memory. Numerous studies explored solution approaches used to solve one-digit arithmetic tasks and many of these focused on the ways in which solutions to basic addition combinations are accessed from memory (Ashcraft, 1982, 1983; Ashcraft & Battaglia, 1978; Baroody, 1985; Campbell, 1995; Campbell & Xue, 2001; Geary, 1996; Gelman & Gallistel, 1978; Ginsburg, Klein, & Starkey, 1998; Groen & Parkman, 1972; Imbo & Vandierendonck, 2008; LeFevre, DeStefano, Penner-Wilger, & Daley 2006; Locuniak & Jordan, 2008; Siegler, 1987; Siegler & Shrager, 1984; Smith, 1921; Steffe, 1979; Svenson & Broquist, 1975; Swenson, 1949; Wheeler, 1939; Wright, 1991).

A range of models were developed to account for students’ knowledge of basic addition combinations and most of these include reference to the problem-size effect. The problem-size effect is “the virtually ubiquitous phenomenon that the difficulty of simple-arithmetic problems increases as problem size increases” (Campbell & Xue, 2001, p. 299) and have been described by many mathematics education researchers and cognitive psychologists (Ashcraft, 1982, 1983; Ashcraft & Battaglia, 1978; Baroody, 1985; Dehaene, 1992; Gallistel & Gelman, 1991; Goldman, Pellegrino, & Mertz; 1988; Goldman, Mertz, & Pellegrino, 1989; Gould, 2000; Groen & Parkman, 1972; Imbo & Vandierendonck, 2008; Siegler & Shrager, 1984; Svenson & Broquist, 1975). Essentially, the problem-size effect is that basic addition tasks involving larger numbers (e.g., 8 + 7) take longer to solve than tasks involving smaller numbers (e.g., 3 + 2).
Although an advocate of drill and rote learning, Wheeler (1939) may have been referring to students using a strategy other than direct retrieval when he stated that “psychologically the child should be able to learn \(5 + 4 = 9\), as easily as \(2 + 3 = 5\), but this is not the case according to the investigations of combinations difficulties” (p. 311). Although it had not yet been named as such, this observation could be accounted for by the problem-size effect. Groen and Parkman (1972) were the first researchers to formally identify the existence of this effect and in the literature there are generally two explanations of how addition combinations are retrieved from memory, both of which account for the problem-size effect. Firstly, the solution is retrieved directly from memory or, secondly, a counting-based strategy is used to solve the task (Geary, 1996). If direct retrieval is used to find the sum of two addends, the problem-size effect accounts for slower response time to addition combinations with larger addends because the association of the addends with the sum is weaker due to less time allocated to the practice of these combinations, when compared to combinations involving two smaller numbers (Geary, 1996). Groen and Parkman (1972) presented counting-based strategies as an alternative to direct retrieval or automatisation to solve addition combinations with both addends in the range 1 to 5. Their “min model” describes students as counting-on-by-ones from the larger addend to find the total of two addends, hence “min”-imising the number of counts to be made. In their study, the problem-size effect was explained by the magnitude of the smaller addend, as this determines the number of counts to be made to calculate the solution. Groen and Parkman (1972) postulated that identifying the larger addend required a constant amount of time, and therefore the time to solve the task was dependent upon the size of the smaller addend and the minimum number of counts to be made, hence the title “min model”. Groen and Parkman (1972) found that doubles (e.g., \(7 + 7\)) did not fit this min model. It is claimed that these doubles or ties are easier to retrieve from memory than the other addition
combinations and all doubles are accessed at the same rate, independent of the magnitude of the addends (Ashcraft, 1997; Groen & Parkman, 1972).

Unlike earlier studies, LeFevre, Sadesky, and Bisanz (1996) proposed that, alongside direct retrieval and counting-based strategies, the use of transformation procedures (grouping and partitioning numbers to simplify computation e.g., building through ten) should also be considered as an explanation for the problem-size effect. In their study, most tasks were solved using retrieval strategies (71.2%), and only a small number of tasks were solved using counting strategies (8.6%). However, 13 of the 16 participants used a transformation strategy on 16.5% of the tasks, and 65% of these transformation strategies were described as using facts that summed to 10. Other transformation strategies used included decomposition by using a known fact, use of doubles and conversion of addition to multiplication (e.g., \(5 + 6 = 2 \times 5 + 1\)) (LeFevre et al., 1996).

2.2.3 Explanatory Models for Retrieving Addition Combinations from Memory

A review of the literature reveals two predominant theories to explain how addition combinations are accessed from memory. One theory posits a reproductive approach and the other a reconstructive approach. Groen and Parkman (1972) defined these two approaches as “reproductive processes, which are concerned with the retrieval of stored facts, and reconstructive processes, which are concerned with the generation of facts on the basis of stored rules” (p. 329).

When an addition combination is automatised, retrieval of this combination from memory is considered to be via a reproductive process – the answer is reproduced from memory with speed and ease. However, if a combination is yet to be committed to memory, access to it may still be available via a reconstructive process. As previously described, the
reproductive process requires less cognitive effort and is less taxing on working memory, as the focus is solely on recall, whether the addition combination is known or not.

When reconstructive strategies are used to determine the sum of a basic addition combination they typically fall into two main categories: counting-based strategies and grouping strategies. Counting-based strategies include those strategies that involve counting-by-ones. By contrast, grouping strategies include those strategies that do not involve counting-by-ones. Although the term grouping strategies is used in this thesis, it is important to acknowledge that there are many other terms used by researchers to describe non-counting strategies. These include calculating by structuring (Van den Heuvel-Panhuizen, 2008), part-whole relationships (Young-Loveridge, 2002), derived strategies (Clarke, 2005; Putnam et al., 1990), thinking strategies (Steffe, 1979), grouping solutions (Cobb et al., 1995), derived facts (Carpenter & Moser, 1984), regrouping strategies (Hatano, 1982), reasoning strategies (Baroody et al., 2009) and derived-combination strategies (Sarama & Clements, 2009a).

Baroody et al. (2009) proposed that students progress through three phases in learning the basic addition combinations. Phase 1 involves the use of counting-based strategies to solve tasks; Phase 2 involves the use of strategies to reason about known combinations to find the sum of unknown combinations; and Phase 3 is considered to be mastery, whereby addition combinations are efficiently produced from memory.

Similarly to Groen and Parkman’s (1972) reconstructive and reproductive processes described above, Baroody et al. (2009) proposed two perspectives on the relative importance of each of these three phases. The first perspective is the Passive Storage View, which suggests that Phase 1 and Phase 2 can facilitate the memorising of addition combinations, but are not essential in order for memorisation to occur. Proponents of the
Passive Storage View believe it is quite possible for students to bypass Phase 1 and Phase 2 and progress straight to Phase 3.

The second perspective is the Active Construction View, which suggests that Phase 1 and Phase 2 support meaningful memorisation, the development of numerical relationships, and number sense, all of which are an important part of mastery in Phase 3. Proponents of the Active Construction View believe that Phase 1 and Phase 2 are “critical to creating the rich network of factual, relational, and strategic knowledge that is the basis for mastery with fluency” (Baroody et al., 2009, p. 70).

Prior to Baroody et al.’s (2009) description of the Passive Storage View and the Active Construction View, other theories were developed. The remainder of Section 2.2.3 includes a summary of the three other main theories developed to explain the way basic addition combinations are stored and accessed from memory. These theories are:

- Network Retrieval Model (Ashcraft, 1982)
- Distributions of Associations Model (Siegler & Shrager, 1984)
- Network Interference Model (Campbell, 1995).

Whilst Siegler and Shrager (1984), Geary (2006) and Groen and Parkman (1972) described solutions to one-digit addition tasks as being accessed by both reconstructive and reproductive processes (depending upon the individual task), others described access occurring as a predominantly reproductive process. The most prominent of these is Ashcraft’s (1982) Network Retrieval Model. Ashcraft argued that third grade is the approximate level at which transition from counting to direct retrieval and automatisation occurs. Whilst acknowledging that Groen and Parkman’s (1972) Min Model accurately described the performance of first graders, Ashcraft’s (1982) Network Retrieval Model suggested that, for students beyond the fourth grade, addition combinations are stored in a
declarative framework. Ashcraft and Battaglia (1978) described this declarative network as an addition table:

The network representation for addition resembles a printed addition table, that is a square, with entry nodes for the digits 0–9 on two adjacent sides. The correct sum of any two numbers is located in the square matrix at the point of intersection of the two entry node vectors. Thus, retrieval time here would simply be the time required for the search down the column to intersect with the search across the row. (p. 536)

As Ashcraft (1985) described, the solutions to addition tasks are represented in cells (or nodes) where the row and columns intersect. “Nodes vary with respect to accessibility, so that the more difficult problems are less accessible during memory search” (p. 100). Ashcraft did clarify that, in conjunction with the declarative network described as an addition table above, there exists a parallel procedural component, which allows for reconstructive processes to provide access to solutions if reproductive declarative knowledge fails. In Ashcraft’s (1982) model, reduced accessibility to the more difficult problems, presumably those which consist of larger addends, accounted for the problem-size effect.

Whilst Ashcraft (1985) recommended the reproductive process of direct retrieval as the most desirable strategy for adults to use, he conceded that “… an arithmetic curriculum that stresses only memorization or drill is not truly teaching arithmetic” (p. 102). In summary, Ashcraft is likely to endorse the use of counting strategies to support the mastery of basic addition combinations in preference to rote learning for younger students. Nevertheless, he asserted that, by about year four of school, these addition combinations should be automatised.

In 1997, Ashcraft developed a computer simulation of the Network Retrieval Model, a modified version of his 1982 model. This version described two processes that occur in parallel to solve basic addition tasks. One process involved the direct retrieval of
the addition combination, and the other required procedure or strategy use to calculate the solution. The two processes “compete” until access to the solution is determined by one or the other. Ashcraft hypothesised that procedure as a strategy is engaged only if direct retrieval or automatisation does not provide the answer. The only strategy allowed for in this computer simulation, however, was a counting-on strategy. Ashcraft (1997) recommended: “More complex forms of procedural knowledge, both informal and formal (e.g., estimation strategies, rules for carrying in complex addition or multiplication), represent an important avenue for further development of the simulation model” (p. 323).

As previously stated, however, Ashcraft did not place much credence on procedure or strategy use for students beyond year four, which is a stance that was challenged by the more recent research referred to in Section 2.2 from Campbell and Xue (2001), and Imbo and Vandierendonck (2008). As described above, according to these two studies, many adults do not have automatised retrieval of all one-digit addition combinations, and therefore rely on reconstructive strategies.

The second model to be considered is the Distributions of Associations (DOA) Model (Siegler & Shrager, 1984). This model acknowledged the use of reproductive and reconstructive strategies to solve basic arithmetic tasks. This model focused only on addition tasks with addends in the range 1 to 5 and described reproductive strategies as superior to reconstructive strategies. It is only if students are unable to automatically retrieve the answer to a task, that they fall back “on successively more time-consuming overt approaches” (Siegler & Shrager, 1984, p. 249). Similarly to Groen and Parkman’s (1972) model, the only reconstructive strategies considered in the DOA Model are count-by-ones with or without students representing one or both addends on their fingers.

The DOA Model (Siegler & Shrager, 1984) described an increasing association between the addition task and the solution as students have more experience with particular
addition tasks. Students become more likely to retrieve the answer rather than need to use a procedure to solve it. Siegler and Shrager (1984) stated that “our basic assumption about how children acquire these distributions is that each time they answer a problem, the associative strength linking that answer to the problem increases” (p. 264).

Geary (2006) supported Siegler and Shrager’s (1984) model and used the metaphor of overlapping waves to describe the sporadic use of strategies at a lower level of sophistication, occurring concurrently with an overarching development of increasingly sophisticated strategy use. This “three steps forward, one step back” approach was contrasted with a lock-step movement through a hierarchical list of increasingly sophisticated strategies. Geary’s (2006) overlapping waves metaphor described the interplay between the use of reproductive and reconstructive strategies by students:

The waves represent the strategy mix, with the crest representing the most commonly used problem-solving approach. Change occurs as once dominant strategies, such as finger counting, decrease in frequency, and more efficient strategies, such as memory retrieval, increase in frequency. (p 779)

This overlapping waves characterisation was further supported by Cowan et al.’s (2011) study.

A third model, which focused on the use of reproductive processes, is Campbell’s (1995) Network Interference Model. This model considered number-fact retrieval to be influenced by two aspects of memory – a magnitude code and a physical code:

The magnitude code represents the approximate numerical size of the answer to a problem and primes the associated physical codes that represent exact answers. Physical codes for problems are assumed to be visual or verbal associative units consisting of the operand pair, operation sign, and answer. (Campbell, 1995, p. 122)
The problem-size effect was considered in that, as the problem-size increases, there is a larger number of possible incorrect solutions that “interfere” (hence the name) with the identification of the correct solution.

As described above, researchers who have investigated strategy use to solve addition combinations aligned themselves with one of two primary approaches. The first approach focused on strategies that are described as reproductive processes (Groen & Parkman, 1972), or the Passive Storage View (Baroody et al., 2009), and has direct retrieval as its main goal. The second approach focused on strategies that are described as reconstructive processes (Groen & Parkman, 1972), or the Active Construction View (Baroody et al., 2009), and had the use of counting or grouping-based strategies and the construction of a cohesive network of interrelated facts as its goal.

In contrast to Ashcraft (1982), who viewed procedure or strategy use as at least a distraction if not an impediment to the ultimate goal of direct retrieval of the basic addition combinations, other researchers perceived strategies as a way of scaffolding students’ understanding as they work towards direct retrieval. Siegler (1988) believed that “teaching children to execute backup strategies more accurately affords them more opportunities to learn the correct answer (i.e., to build distributions with strong peaks at the correct answer)” (p. 850). However, the backup strategy Siegler recommended is counting based, with the support of fingers to keep track of the counts as necessary.

Baroody (1985) was one of the first to challenge the aspiration of automatisation as the pinnacle for accessing addition combinations. He wrote that “[a]ccording to current theories, efficient production of number combinations is exclusively a reproductive process” (p. 86), but he advocated the preferential use of reconstructive strategies. Baroody’s view was that students’ learning of basic addition combinations involves “discovering, labeling, and internalizing relationships – processes encouraged by teaching
thinking strategies” (p. 83). He argued that “using stored procedures, rules, or principles to quickly construct a range of combinations is cognitively more economical than relying exclusively on a network of individually stored facts” (Baroody, 1985, p. 89).

Writing at the same time as Baroody (1985), Steinberg (1985) also endorsed the benefits of students using grouping strategies to solve addition tasks:

Using many DFSs (derived fact strategies) might help children structure the facts, make connections among different facts, and thus generate associations with them … The use of DFSs might also influence how the facts are represented in, and retrieved from, long-term memory. (p. 350)

The approach described above by Baroody (1985) and Steinberg (1985) is reflective of current thinking in terms of mathematics pedagogy. In Western societies since the late 1970s, pedagogical approaches have placed a greater emphasis on the use of thinking strategies (Steffe, 1979) to support reconstructive processes that serve as scaffolds until facts can be accessed via direct retrieval. However, this use of reconstructive strategies in preference to reproductive strategies was already a feature of pedagogical approaches in Japanese classrooms (Hatano, 1982). Hatano advocated that “we should not teach addition-subtraction number facts as if children were learning paired associates. Nor should we force children to learn rote-counting solution procedures and to practice them mechanically” (p. 213).

Groen and Parkman (1972) had counting-based strategies as the alternative reconstructive process to the reproductive process of direct retrieval, and similarly Steffe’s (1979) work focused primarily on the use of counting-based strategies to solve addition tasks. He attested that “in any study dealing with thinking strategies, the counting behaviour of children must be considered as a foundation for the instruction” (Steffe, 1979, p. 373). However, in contrast to Groen and Parkman, for whom counting-based strategies are the only reconstructive process considered, others believed that these counting strategies should
be regarded as a means to developing more mature thinking strategies. Steffe (1979) suggested that “the evidence is strong that children’s failure to develop thinking strategies is because they have not yet developed critical counting strategies” (p. 372).

Research in the last thirty years has progressed from analysis of the ways in which addition combinations are stored and retrieved to determining the best pedagogical approach to support students in automatising these combinations. Whilst students’ mastery of the basic addition combinations is a goal in itself, it must be viewed as one element of a pedagogical approach which seeks to develop students as numerate members of society who are able to use arithmetical knowledge to assist them to solve tasks as they encounter them, both within and beyond the classroom setting. This approach to teaching and learning was commented upon by Clements and Sarama (2009): “Practice should not be ‘meaningless drill’ but should occur in a context of making sense of the situation and the number relationships” (p. 83). Studies have shown that students are more likely to be able to solve addition tasks presented in a variety of settings and in a variety of ways if they have been taught reconstructive strategies rather than simply drill and practice or rote learning (Baroody, 1985; Brownell & Chazal, 1935; Carpenter & Moser, 1984; Garnett, 1992; Steinberg, 1985; Tournaki, 2003). Articulating correct responses is an important goal of instruction, but flexible and facile use of reconstructive strategies should also be a goal (Steffe, 1979).

The current focus on reconstructive strategies as the preferred pedagogy was summarised by Jordan and Kaplan (2009): “It is difficult to memorize arithmetic facts by rote without understanding how combinations relate to one another on a number line (e.g., 3 + 2, 2 + 3, 5 – 2, and 5 – 3)” (p. 851).
2.2.4 Addition Combinations Presented as Word Problems

Most of the research into students’ strategies to solve simple addition tasks before the 1980s focused on tasks presented in bare number format. However, in the late 1980s, research was conducted into students’ strategies to solve simple addition tasks when they were presented as word problems. This focus on research into students’ strategies for solving simple word problems led to an early number professional development program known as Cognitively Guided Instruction (CGI), which was designed to help teachers to use students’ thinking to inform future teaching (Carpenter & Fennema, 1992).

CGI identifies children’s solution strategies based on a detailed analysis of the problem space, which separates addition and subtraction word problems into basic types according to whether they describe joining or separating actions, comparing situations, or part-whole relations. This categorisation is then augmented by combining it with what is unknown. (Fennema, Franke, Carpenter, & Carey, 1993, p. 557)

Based on their research into children’s solutions of word problems, Carpenter and Moser (1984) described the following progression of reconstructive strategies: counting-all, counting-on from first, counting-on from larger, recall, and derived facts.

In CGI, word problems involving addition and subtraction are categorised as either join, separate, part-part-whole or compare. Each of these problem types can be further categorised into three subtypes: result unknown, change unknown or start unknown (Fennema et al., 1993). Fennema et al. described the underlying principle of CGI:

> When young children initially solve such problems, they directly model the action or relationships in the problems; those problems that cannot be modeled cannot be solved. Children advance in their problem-solving skills from use of direct modeling to various counting strategies, which involve some modeling, to use of relationships between number facts, and finally to use of memorized facts. (p. 557)

The following are examples of problem types considered in CGI. An example of a join result unknown problem is: “Connie had 5 marbles. Jim gave her 8 more marbles.
How many does Connie have altogether? An example of a separate start unknown problem is: “Connie had some marbles. She gave 5 to Jim. Now she has 8 marbles left. How many marbles did Connie have to start with?” An example of a part-part-whole problem is: “Connie has 13 marbles. Five are red and the rest are blue. How many blue marbles does Connie have?” (Fennema et al., 1993, p. 558).

On face value, CGI (Carpenter & Fennema, 1992) and the explanatory models described above seem to have little in common. CGI describes worded addition combinations presented as a worded task, whilst the explanatory models refer to tasks presented using formal arithmetic notation. However, similarly to Baroody et al.’s (2009) description of the three phases towards mastery of the basic addition combinations, the underlying principle of CGI describes the use of counting strategies, followed by reasoning strategies, and finally direct retrieval. It appears that the underlying principles of reconstructive processes to describe the acquisition of addition combinations are similar for both CGI and the explanatory models.

2.3 The Emphasis on Count-by-ones Strategies in a Range of Curriculum Documents

2.3.1 “Counting”: A Polysemantical Term

Section 2.3 highlights the importance placed on basic addition combinations in the curriculum documents of 10 countries. For example, the Australian curriculum states that seven-year-old students should “solve simple addition and subtraction problems using a range of efficient mental and written strategies” (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2011). Whilst the literature review will reveal that mastery of the basic addition combinations is evident in the curriculum documents of all 10 countries, this section will explore the inclusion of counting as a key strategy in achieving this mastery.
Counting was described as the first and most important algorithm (Clements & Sarama, 2009) and has traditionally formed an important component of the mathematics curriculum in the first years of school, internationally and in Australia. However, what constitutes the act of counting is not always clearly defined.

When “counting” is listed as an outcome or a skill in curriculum documents, it is often used to describe one or both of two distinct behaviours. These behaviours were described by Wright, Martland, Stafford, and Stanger (2006) as: (1) the recitation of the forward or backward number word sequence, with no link to the numerosity of a collection; and (2) as a strategy to solve an addition task.

Recitation of the number word sequence is an important sub-skill of the facile use of counting to solve addition tasks. However, it is a skill which (a) can be learnt independently of counting (i.e., counting with the purpose of determining the numerosity of a collection); and (b) is initially learnt through imitation (Clements & Sarama, 2009; Munn, 2010; Threlfall, 2010; Van den Heuvel-Panhuizen, 2008; Wright, 2008). Students mimic others in reciting numbers in counting order in much the same way they learn to sing nursery rhymes. It cannot be assumed that students make links between number names and quantity. Clements and Sarama (2009) suggested that, when students first learn to recite the number word sequence, they do not even regard the number names as independent labels:

They [students] learn to count verbally by starting at the beginning and saying a string of words, but they do not even “hear” counting words as separate words. Then, they do separate each counting word and they learn to count up to 10, then 20, then higher. (p. 21)

As explained above, the term counting is used to describe two quite distinct behaviours. The recitation of a number word sequence without an associated link to quantity is sometimes referred to as “rote counting”, although some researchers recommended against the use of this term as it implies a lack of understanding by the students (Clements & Sarama, 2009; Wright, 2008). Alternatives to the term rote counting
have been listed as “acoustic” counting (Van den Heuvel-Panhuizen, 2008), recitation of the number word sequence (Wright, Martland, Stafford, & Stanger, 2006) or “verbal” counting (Clements & Sarama, 2009).

Wright, Martland, Stafford, and Stanger (2006) described the distinction between counting and the recitation of the number word sequence:

“Counting” is used in situations where we assume the child has a cognitive goal, for example to determine the numerosity of a collection. Thus the child is attempting to solve a problem and in doing so is conceptualizing items of some kind, rather than merely reciting the number word sequence. (p. 20)

Learning to use counting to determine the numerosity of a collection is a key goal in early arithmetical knowledge and its importance has been acknowledged by many researchers (Gelman & Gallistel, 1978; Gelman & Meck, 1986; Steffe, Thompson, & Richards, 1982; Threlfall, 2010; Wright, 2008). Clements and Sarama (2009) stated that “the capstone of early numerical knowledge is connecting the counting of objects in a collection to the number of objects in that collection” (p. 21). This conceptual understanding is labelled “cardinality” and is described in more detail in Section 2.3.2.

Most of the curriculum documents reviewed prescribe counting in the sense of reciting the number word sequence as an integral component of the early arithmetical learning of students. However, these documents typically do not distinguish between counting in the sense of reciting the number word sequence and counting in the sense of using one of an increasingly sophisticated progression of counting strategies to solve problem-based addition tasks. The term counting is used interchangeably to describe both actions.

In Sections 2.3.3 and 2.3.4 Australian curriculum documents are compared and contrasted with nine international documents, in terms of the importance afforded the use of counting as a strategy to solve addition tasks.
2.3.2 Counting as a Strategy to Solve Addition Tasks

Many researchers have investigated students’ use of counting strategies to solve addition tasks (Ashcraft, 1982; Baroody, Brach, & Tai, 2006; Dehaene, 1992; Flexer, 1986; Geary, 2006; Gelman & Gallistel, 1978; Gray, 2010; Groen & Parkman, 1972; LeFevre et al., 2006; Maclellan, 2010; Siegler, 1987; Siegler & Shrager, 1984; Svenson & Broquist, 1975; Thompson, 2010; Threlfall, 2010; Van Luit & Schopman, 2000; Wright, 1994). Counting to determine the number of items in a collection requires students to know that, in conjunction with being a label for the last item counted in a collection, the last number word uttered is also representative of the total number of items in the collection. Gelman and Gallistel (1978) labelled this the cardinal principle and define it as “the tag applied to the final item in the set represents the number of items in the set” (p. 80). To achieve mastery of counting in this sense, five principles need to be satisfied. Gelman and Meck (1986) summarised these principles:

(a) one-one correspondence – every item in a set must be assigned a unique tag; (b) the stable order principle – the tags used must be drawn from a stably ordered list; (c) the cardinal principle – the last tag used in a count has a special status; it represents the cardinal value of the set; (d) the item-difference (or irrelevance) principle – there are no restrictions on the collection of items that can be counted; and, (e) the order-indifference principle – the order in which items are tagged is irrelevant. (p. 30)

These five counting principles are informed by Piagetian theory which highlights the central role of the concepts of cardinality and ordinality in the student’s construction of number (Piaget, 1941). Many researchers have written about the notion of cardinality (Dehaene, 1992; Geary, 2006; Hunting, 2003; Maclellan, 2001, 2010; Munn, 2010; Piaget, 1941; Threlfall, 2010; Tolchinsky, 2003; Wright, 2008; Young-Loveridge, 2002). However, Piaget’s theory of cardinality has been interpreted as implying that, from an instructional perspective, cardinality needs to be established independently and separately for each number (1 to 20) before a student can be said to have learnt cardinality.
Whilst Gelman and Meck (1986) used the term cardinal principle in a sense that differs from Piaget’s notion of cardinality, their approach is similar to what Steffe (1992) referred to as perceptual counting. While Gelman and Meck described five principles, Steffe described successful perceptual counting as consisting of three components: (1) “the ability to produce a perceptual collection of units that can be counted” (p. 86); (2) knowledge of the forward number word sequence in the necessary range; and (3) the ability to coordinate the first two components so that each oral (or sub-vocal) count corresponds to one unit.

Once the initial stage of counting to establish cardinality is mastered, research shows that students advance in the complexity of their use of counting strategies. They learn to combine collections and determine the total of the two collections in progressively more sophisticated ways. Five theoretical frameworks which inform the advancement in complexity of students’ use of counting strategies to solve addition tasks are described below:

1. “Count by Min” Model (Groen & Parkman, 1972)
2. Addition Strategies (Carpenter & Moser, 1984)
3. Stages in the Construction of the Number Sequence (Steffe & Cobb, 1988)
4. Sum and Min Models (Goldman, Mertz, & Pellegrino, 1989)

Each of these five theoretical frameworks will be elaborated upon in the following paragraphs.

Groen and Parkman’s (1972) seminal study identified five counting models. Each of these models described strategies to find the total of two collections. Although not the labels used by Groen and Parkman (1972), these models can be described as: (1) count...
all (counting the two addends at least once each); (2) count on from first addend; (3) count on from second addend; (4) count on from smallest addend; and (5) count on from largest addend (p. 331). As described in Section 2.2.2, this study also identified the problem-size effect associated with solving basic addition combinations. Apart from the ties or doubles combinations (e.g., \(4 + 4\)), use of Groen and Parkman’s (1972) Model 5 was found to result in the least time spent on solving addition combinations and therefore was considered to be the most efficient.

The second theoretical framework is described by Carpenter and Moser (1984) and had three basic levels of addition strategies, two of which include counting strategies. The first level is described as a direct modelling strategy and includes the use of objects to support a count-all strategy. The second level is described as counting strategies and includes counting-on from first, and as conceptual understanding develops is replaced by counting-on from larger. The third level of Carpenter and Moser’s addition strategies describes the use of more sophisticated non-counting strategies and refers to recall and derived facts to solve addition combinations.

Steffe and Cobb’s (1988) theoretical framework included five stages. The first four relate to counting-based strategies, and the fifth assumes students are beginning to use non-counting strategies such as grouping to solve addition tasks. The five stages are labelled: (1) perceptual counting; (2) figurative counting; (3) initial number sequence; (4) tacitly nested number sequence; and (5) explicitly nested number sequence. As with Carpenter and Moser’s (1984) model, the first stages presumed that items are available for students to perceptually touch (Stage 1) or to visualise (Stage 2) as they re-produce the count. Stage 3 is distinguished by the use of more sophisticated counting strategies as in the Model 5 count-on from largest addend of Groen and Parkman (1972) or the Level 2 counting-on from larger of Carpenter and Moser (1984). Steffe (1992) attributed Piaget’s
notion of cardinality to the student who has attained Stage 3. Steffe explained that a student has reached Stage 3 (at which they are able to use a count-on strategy) when, in the mind of the student, the single number word comes to symbolise a count.

In contrast to the other counting models, which only focus on addition tasks, Stage 4 of Steffe and Cobb’s (1988) model described strategies which can facilitate solutions to subtraction tasks as well as addition. For instance, a student at Stage 4 may use a count-up to strategy to solve the subtraction task 15 – 12. As with Carpenter and Moser’s (1984) third level, Steffe and Cobb’s Stage 5 describes the use of non-counting-based strategies to solve addition and subtraction tasks. The student at Stage 5 has developed an understanding of part-whole operations and comprehends subtraction as the inverse of addition (Wright, 1991).

The fourth theoretical framework described by Goldman, Mertz, and Pellegrino (1989) is similar in many aspects to the other frameworks. It describes two strategies: (1) the count-all strategy which is labelled the Sum Model; and (2) the count-on model (count on from largest addend, as this minimises the number of counts to be made) which they labelled the Min Model (p. 491).

The final theoretical framework described is the Stages of Early Arithmetical Learning (SEAL) and is informed by the work of Steffe and Cobb (1988). Counting strategies feature heavily in the first four stages of SEAL but, when a student attains Stage 5, grouping strategies are used to solve addition and subtraction tasks. The SEAL theory has its empirical basis in students’ use of progressively more sophisticated counting strategies and ultimately grouping strategies to solve simple addition and subtraction tasks (Steffe & Cobb, 1988). Thus SEAL can account for students’ progression from perceptual counting (Stage 1), that is, counting objects that are seen (or heard or touched), to using grouping strategies to solve simple addition and subtraction tasks (Stage 5). Table 2.1
presents the SEAL model which describes the progression of counting used in increasingly more sophisticated ways to solve addition tasks.

Table 2.1
Model for Stages of Early Arithmetical Learning (SEAL) (Wright, Martland, & Stafford, 2006, p. 22)

<table>
<thead>
<tr>
<th>Stage</th>
<th>Early Arithmetical Strategies</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stage 0:</td>
<td>Cannot count visible items. The child either does not know the number words or cannot coordinate the number words with items.</td>
</tr>
<tr>
<td>Emergent Counting</td>
<td></td>
</tr>
<tr>
<td>Stage 1:</td>
<td>Can count perceived items but not those in screened (that is, concealed) collections. This may involve seeing, hearing or feeling items.</td>
</tr>
<tr>
<td>Perceptual Counting</td>
<td></td>
</tr>
<tr>
<td>Stage 2:</td>
<td>Can count the items in a screened collection but counting typically includes what adults might regard as redundant activity. For example, when presented with two screened collections, told how many in each collection and asked how many counters in all, the child will count from ‘one’ instead of counting-on.</td>
</tr>
<tr>
<td>Figurative Counting</td>
<td></td>
</tr>
<tr>
<td>Stage 3:</td>
<td>Child uses counting-on rather than counting from ‘one’, to solve addition and missing addend tasks (e.g. 6 + [] = 9). The child may use a count-down-from strategy to solve removed items tasks (e.g. 17 – 3 as 16, 15, 14 – answer 14) but not count-down-to strategies to solve missing subtrahend tasks (e.g. 17 – 14 as 16, 15, 14 – answer 3).</td>
</tr>
<tr>
<td>Initial Number Sequence</td>
<td></td>
</tr>
<tr>
<td>Stage 4:</td>
<td>The child counts-down-to to solve missing subtrahend tasks (e.g. 17 – 14 as 16, 15, 14 - answer 3). The child can choose the more efficient of count-down-from and count-down-to strategies.</td>
</tr>
<tr>
<td>Intermediate Number Sequence</td>
<td></td>
</tr>
<tr>
<td>Stage 5:</td>
<td>The child uses a range of what are referred to as non-count-by-ones strategies. These strategies involve procedures other than counting-by-ones but may also involve some counting-by-ones. Thus in additive and subtractive situations, the child uses strategies such as compensation, using a known result, adding to ten, commutativity, subtraction as the inverse of addition, awareness of the ‘ten’ in a teen number.</td>
</tr>
<tr>
<td>Facile Number Sequence</td>
<td></td>
</tr>
</tbody>
</table>

The significance of counting in early arithmetic is evident in the quantity of research and the various theoretical frameworks developed to account for students’ advancing knowledge in the use of counting strategies to solve addition tasks. The
remainder of Section 2.3 will examine the role counting plays in the curriculum documents of 10 countries.

2.3.3 Counting as an Additive Strategy in Australian Curriculum Documents

In this section, the learning statements in Australian curriculum documents will be reviewed with regard to the inclusion of counting-based strategies to solve addition tasks.

The five theoretical frameworks previously discussed highlight the significance of counting-based strategies in the early years of arithmetical development. Analogously, counting-based strategies have always been given primacy in early years mathematics curriculum documents. Wright (2008) described “… [t]he progression to counting-on is almost universally regarded as the major advancement in early number learning, in the first 2 years of school” (p. 211).

However, many researchers have suggested that early years mathematics programs typically underestimate the abilities of children (Hunting, Mousley, & Perry, 2012; Mulligan, English, Mitchelmore, Welsby, & Crevensten, 2011; Papic, Mulligan, & Mitchelmore, 2011; Van Luit & Schopman, 2000; Wright, 1994; Young-Loveridge, Peters, & Carr, 1997). Gould (2012) examined the number knowledge of more than 65,000 students entering their first year of school in New South Wales. He found that over 50% of students were able to match the number word sequence with a collection of objects to state the total of the collection prior to any instruction. Based on these results, he challenged the low benchmarks set for Australian students in mathematics:

The expectation that, by the end of the Foundation year, students can make connections between the number names, numerals and quantities up to ten would be a low expectation for at least half of the students in NSW public schools. (Gould, 2012, p. 109)

The Australian curriculum (ACARA, 2011) included an achievement standard to be reached by the end of each school year, and content descriptors that outline the outcomes
that will support the students in achieving these standards. Table 2.2 lists the statements from the achievement standards for the first three years of school which refer to counting, either in terms of oral counting or using counting strategies to solve addition and subtraction tasks.

Table 2.2
Achievement Standards for the First Three Years of School in the Australian Curriculum (ACARA, 2011)

<table>
<thead>
<tr>
<th>School Level</th>
<th>Achievement Standard</th>
</tr>
</thead>
<tbody>
<tr>
<td>Foundation Level</td>
<td>Students make connections between number names, numerals and quantities up to 10. Students count to and from 20 and order small collections.</td>
</tr>
<tr>
<td>(5 year olds)</td>
<td></td>
</tr>
<tr>
<td>Level One</td>
<td>Students describe number sequences resulting from skip counting by 2s, 5s, and 10s. Students count to and from 100 and locate numbers on a number line. They carry out simple additions and subtractions using counting strategies.</td>
</tr>
<tr>
<td>(6 year olds)</td>
<td></td>
</tr>
<tr>
<td>Level Two</td>
<td>Students recognize increasing and decreasing number sequences involving 2s, 3s, and 5s. Students count to and from 1000. They perform simple addition and subtraction calculations using a range of strategies.</td>
</tr>
<tr>
<td>(7 year olds)</td>
<td></td>
</tr>
</tbody>
</table>

As indicated by Gould (2012), the Australian curriculum achievement standards expect students to make connections between names and quantity in the range 1 to 10 only by the end of their first year of school. Table 2.2 also indicates that the curriculum expects students to use grouping or counting strategies to solve addition and subtraction tasks at the end of their third year of schooling. With most students starting school at five years of age, this means that seven-year-olds are still being encouraged to use counting-based strategies to solve addition and subtraction tasks. As part of the curriculum document, content descriptors are used to support teachers in interpreting the achievement standards and translating these into learning outcomes. In Level One, the achievement standard explicitly describes only the use of counting strategies, but the content descriptors suggest that
students can use “a range of strategies including counting on, partitioning and rearranging parts” (ACARA, 2011). Section 2.3.4 investigates references to counting in other international curriculum documents.

2.3.4 Counting as an Additive Strategy in Curriculum Documents from Other Countries

The primacy afforded the teaching of counting strategies to solve addition tasks is not unique to the Australian curriculum. Cotter (2000) contrasted the importance placed on counting in the curriculum in the United States compared with Japan:

In the United States, counting is considered the cornerstone of arithmetic: children engage in various counting strategies: counting all, counting-on and counting back. Japanese teachers have a different view of counting. Starting in first grade, students in Japan are discouraged from using one-by-one counting procedures. (p. 111)

This section will review the inclusion of counting strategies in the curriculum documents from 10 countries; however, firstly, it is important to take into consideration the age at which students commence school. Table 2.3 lists the various countries whose documents are being examined and the ages at which their students commence their formal schooling.

Table 2.3
Chronological Age at Which Students Commence School in Ten Countries (National Foundation for Educational Research, 2013)

<table>
<thead>
<tr>
<th>5 years</th>
<th>6 years</th>
<th>7 years</th>
</tr>
</thead>
<tbody>
<tr>
<td>Australia</td>
<td>Canada</td>
<td>Sweden</td>
</tr>
<tr>
<td>England</td>
<td>Hong Kong</td>
<td>Ireland</td>
</tr>
<tr>
<td>The Netherlands</td>
<td>Singapore</td>
<td>United States</td>
</tr>
<tr>
<td>New Zealand</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Tables C1–C4 in Appendix C list the curriculum statements for curriculum documents from these 10 countries that were accessible via the internet. Only statements from curriculum documents which indicate the use of counting or grouping strategies to
solve addition tasks in the range 1 to 20 were included. Statements which used the term counting to describe recitation of a number word sequence were omitted.

Each country has its own system for naming the school year. For example, across various countries the first year of school is referred to as Kindergarten, First Grade, Primary 1, Prep or Infants Class. However, regardless of the school year, the curriculum statements in Tables C1–C4 in Appendix C are comparable in terms of the ages of the students to which they refer. For this reason, not all countries are referred to in each of the tables. Where possible, the content descriptors have been referenced in preference to standards statements, as the content descriptions are generally more specific in their inclusion of detail. The curriculum statements for England, Hong Kong, the Netherlands and Sweden are written to account for stages of learning across more than one academic year of school, and therefore across two or more age groups. The statements for these four countries are shown separately in Table C4. The year in which the curriculum document was published is referenced under the name of each country. (Any direct use of the term “counting” in Tables C1–C4 is highlighted in bold to expedite ease of reference.)

In the following section, four key points regarding the reference to counting strategies in the curriculum documents are highlighted.

(i) Table C2 indicates that in the second year of school (six-year-old students) the Australian curriculum recommended that students “solve simple addition problems using a range of strategies including counting on, partitioning and rearranging parts” (ACARA, 2011). This statement could be interpreted as teachers should place equal emphasis on the use of counting-on, partitioning and rearranging parts as strategies to solve addition tasks. This appears to be in contrast to three of the four theoretical frameworks which inform how students advance in the complexity of their use of counting strategies to solve addition tasks outlined in Section 2.3.2 (Groen &
Parkman, 1972; Carpenter & Moser, 1984; Steffe & Cobb, 1988). In each of these frameworks, the use of grouping strategies such as partitioning and rearranging parts is considered to be indicative of strategies at a higher level of sophistication than counting-based strategies.

(ii) Table C2 indicates that, of the six countries with statements specifically regarding six-year-old students, Australia, Canada and Ireland are the only three that advocate the use of counting strategies to solve addition tasks. However, for the Canadian and Irish six-year-olds, this is their first year of formal schooling, whereas it is the second year at school for Australian students. The New Zealand, Singapore and United States curriculum documents do not use the term counting to describe strategies to solve addition tasks for six-year-old students.

(iii) The United States curriculum uses the term counting in its recommendations for seven-year-old students when it suggests counting-on together with decomposing and using known relationships to solve tasks. The Australian curriculum recommends using counting-on to find the missing element in an additive problem, however, seven-year-olds in Australia are in their third year of formal schooling, in the United States they are only in their second year. The decomposing and using known relationships in the Unites States curriculum is similar to the recommendations for seven-year-old students in the Australian curriculum which describe using grouping strategies such as doubles and building to ten.

(iv) Recall that the English, Hong Kong, Dutch and Swedish curriculum statements are written to cover stages of learning, not individual year levels. Table C4 in Appendix C indicates that the use of counting strategies is not referred to in the curriculum statements of any of these four countries. This could be interpreted that, as with the theoretical frameworks for advancing addition and subtraction strategies described
earlier, mastery of counting-based strategies is considered to be an important aspect of the learning progression, but not the ultimate goal. The use of counting strategies is therefore assumed but not specified as a directive in the curriculum statements. Wright, Ellemor-Collins, and Lewis (2007) argued that when students are ready, the instructional approach should transition from a focus on count-by-ones to more efficient grouping strategies.

From the analysis of the curriculum documents from 10 countries, it appears that reference to counting-based strategies is most prevalent in the Australian curriculum documents.

### 2.3.5 Comparison of Curriculum Documents and Trends in International Mathematics and Science Study (TIMSS) for 10 Countries

Every four years, many countries across the world participate in the Trends in International Mathematics and Science Study (TIMSS) (Mullis et al., 2012). TIMSS is a measure of mathematical content knowledge and is administered to grade four students. Table 2.4 lists the 2011 TIMSS ranking for nine of the countries whose curriculum documents were discussed previously in Section 2.3.4, as well as the age of the students for whom counting and grouping strategies are described in the curriculum documents. (Canada’s results are not included as they were not part of TIMSS 2011.)

<table>
<thead>
<tr>
<th>TIMSS Ranking</th>
<th>Country</th>
<th>Reference to counting strategies</th>
<th>Reference to grouping strategies</th>
</tr>
</thead>
<tbody>
<tr>
<td>1&lt;sup&gt;st&lt;/sup&gt;</td>
<td>Singapore</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3&lt;sup&gt;rd&lt;/sup&gt;</td>
<td>Hong Kong</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9&lt;sup&gt;th&lt;/sup&gt;</td>
<td>England</td>
<td>6 &amp; 7 years</td>
<td>5, 6 &amp; 7 years</td>
</tr>
<tr>
<td>11&lt;sup&gt;th&lt;/sup&gt;</td>
<td>United States</td>
<td>6 &amp; 7 years</td>
<td>6 &amp; 7 years</td>
</tr>
<tr>
<td>12&lt;sup&gt;th&lt;/sup&gt;</td>
<td>The Netherlands</td>
<td></td>
<td></td>
</tr>
<tr>
<td>17&lt;sup&gt;th&lt;/sup&gt;</td>
<td>Ireland</td>
<td>6, 7 &amp; 8 years</td>
<td>6, 7 &amp; 8 years</td>
</tr>
<tr>
<td>19&lt;sup&gt;th&lt;/sup&gt;</td>
<td>Australia</td>
<td>5, 6 &amp; 7 years</td>
<td>6 &amp; 7 years</td>
</tr>
<tr>
<td>26&lt;sup&gt;th&lt;/sup&gt;</td>
<td>Sweden</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Hong Kong and Singapore have both ranked either first or second in the last three TIMSS (Mullis et al., 2012) assessments for fourth grade students (2003, 2007 and 2011). Lam (2002) suggested the high results in TIMSS for Hong Kong are reflective of rote learning and that a negative attitude towards learning mathematics is typical in Hong Kong schools:

Under this atmosphere of examinations and strong competition, schools tended to consider the attainment of good examination results as their top priority. In order to achieve this objective, students have been exposed to constant drills on skills and content in order to hone their abilities in writing examinations. (p. 204)

This suggests that, although in Hong Kong results are high on assessments such as TIMSS, societal attitudes towards the learning of mathematics are rather negative.

In Singapore, whilst it is not explicitly stated in the curriculum documents, teachers are given directives regarding the time to be spent on topics, the pedagogical approaches to be used and assessment strategies. Curriculum materials are approved by the Ministry of Education (Dindyal, 2005). By the end of the first year of school, six-year-old students are expected to have direct retrieval of all one-digit addition combinations.

The Singapore Ministry of Education’s website (2012) featured the motto “Teach Less, Learn More”, and stated that “we should encourage more active and engaged learning in our students, and depend less on drill and practice and rote learning”. The phrase “building up the addition bonds up to 9 + 9” in the Singapore curriculum documents for six-year-olds indicates that strategies may be taught in a progression, building upon prior knowledge. This seems to suggest an approach focusing on the use of grouping strategies rather than counting-based strategies.
Table 2.4 indicates that two of the four lowest ranked countries considered in this analysis place the highest emphasis on the use of counting-based strategies to solve addition tasks in the range 1 to 20. Ireland (ranked 17th) and Australia (ranked 19th) both explicitly describe the use of counting strategies in each of the first three years of school more frequently than any of the other countries. In Australia, counting-based strategies are still being recommended for students up to seven years of age, and in Ireland they are recommended for students up to eight years of age. Whilst the intention is not to suggest that knowledge of the basic addition combinations is the sole indicator of performance on mathematical assessments, it is interesting that, with the exception of Hong Kong (which pays the price of poor student attitudes towards mathematics), many of the higher performing nations considered do emphasise the use of grouping strategies from the first year of primary education.

In light of the statements from the curriculum documents presented in Table 2.2, Table C1 and Table C2, it can be assumed that, in their first year of school, most students in Australia, New Zealand, Canada, Ireland and the United States are taught to solve addition tasks via the use of counting strategies. Gray (2010) cautioned against the implicit message we pass on to our students with such a strong emphasis on counting in the mathematics curriculum: “Perhaps it seems natural that if so much energy and time is expended upon the development of sound counting skills within Key Stage 1, some children within Key Stage 2 appear to be reluctant to use alternative approaches” (p. 83).

As described in the five theoretical frameworks in Section 2.3.2, children typically use count-by-ones as their initial strategy to solve addition tasks. However, some researchers cautioned against the overuse of this as the sole strategy (Askew, 2013; Bobis, 1996; Van Luit & Schopman, 2000). Conversely, Fuson and Fuson (1992) advocated counting as a viable strategy well beyond the early years of schooling:
Counting methods that use fingers are not necessarily crutches that later interfere with more complex tasks … counting on and counting up with one-handed finger patterns [are] accurate and fast enough for use in four-digit addition and subtraction with regrouping. (p. 77)

Diametrically opposed to the argument made by Fuson and Fuson (1992) is the approach to solving addition tasks adopted in Japanese classrooms that encourages the use of grouping strategies in preference to counting strategies. The grouping strategies allow the student to consider quantities as groups or collections and encourage part-whole thinking, rather than only reproducing the procedure of one count for one item. Hatano (1982) stated:

Japanese early mathematics education is dependent on a non-counting-based model. Counting-up and counting-down strategies are not recommended in contemporary Japanese mathematics education … because these strategies tend to be slower, to be prone to errors, and according to Sugaku-kyoiku-kyogikai (Council of Mathematics Education), to lead students to operate on “abstract numbers that are separated from actual quantity”. (p. 221)

In summary, there appear to be significant references to counting-based strategies in some of the international curriculum documents well beyond the first year of school. Whilst, Steffe (1979) stated that “one goal of thinking strategies is to take addition and subtraction beyond counting” (p. 372), many curriculum documents advocate this transition beyond counting occurs across a number of years. Van den Heuvel-Panhuizen (2008) acknowledged the need to move students beyond counting strategies when she recommended, that “the structure of numbers should also be given a lot of attention in mathematics teaching, because it is structuring that makes it possible to go beyond the level of calculating by counting” (p. 44).

2.4 The Use of Concrete Materials to Support Instruction

This section of the literature review discusses concrete materials, and examines the research regarding the benefits and challenges of their use as a pedagogical tool.
2.4.1 Why Use Concrete Materials?

“Concrete materials” is the phrase commonly used in Australia to describe manipulatives that are used by teachers and/or students in their mathematics lessons. Other terms also include manipulatives, concrete objects or “hands-on” materials. While the definition of a manipulative is still open to interpretation (Moyer, 2001), the term “concrete materials” has come to describe a piece or pieces of equipment that are typically selected by teachers to “introduce new concepts or to extend previously learned concepts” (McNeil & Jarvin, 2007, p. 310). In recent years this has also come to include a growing range of virtual manipulatives (see Section 2.4.7).

Whilst “the use of manipulatives to teach mathematics is often prescribed as an efficacious teaching strategy” (Carbonneau, Marley, & Selig, 2013), Moyer’s (2001) study found that some teachers “viewed their use of manipulatives for mathematics instruction as playing, exploring, or a change of pace” (p. 188).

Sowell (1989) defined manipulatives as being either concrete or pictorial representations. She defines concrete manipulatives as materials that children can work with directly, and pictorial manipulatives as demonstrations provided by the teacher, animations or pictures in printed materials.

In Australian mathematics classrooms, concrete materials are typically used more regularly in junior primary lessons than upper primary lessons. A study of 820 West Australian teachers found that almost 83% of teachers of five-year-old students used manipulatives daily, compared to only 4.5% of teachers of 12-year-old students (Swan & Marshall, 2010).

Moch (2001) suggested that the use of concrete materials is now internationally accepted as a best practice pedagogical technique. This was not always the case, however, with Friedman (1978) asserting that, “on the basis of recent research, it appears that after
the first grade, where the manipulative strategy has been effective in several situations, an instructional strategy that gives preeminence to the use of manipulative materials is unwarranted” (p. 79).

Most recently, the use of concrete materials has been found to benefit, hinder and neutrally affect student performance (Carbonneau et al., 2013). The impact has been described as dependent upon “the level of instructional guidance, type of manipulative, age of learners, and other characteristics of a learning environment” (Carbonneau et al., 2013, p. 381). These factors in the use of concrete materials will be considered in the following sections.

2.4.2 Limitations of the Use of Concrete Materials

Counters are one example of concrete materials that are regularly used in Australian classrooms. However, students’ use of counters should not be limited to following rote procedures. Kamii, Lewis, and Kirkland (2001) believe that, “while counters can be very helpful, they can also be used in overly prescriptive ways that interfere with children’s thinking” (p. 23).

Carbonneau et al. (2013) suggested that the level of instructional guidance offered by teachers to their students falls somewhere along a continuum ranging from student-controlled learning (i.e., a discovery learning environment) to teacher-controlled learning (i.e., direct instruction). Martin (2009) cautioned that too much instructional guidance can impede learning in that students are unable to transfer their knowledge and understanding to assist them in solving novel tasks. Similarly, insufficient instructional guidance in the form of discovery learning does not lead to deep learning and results in lower levels of student performance (Mayer, 2004).

Researchers believe that concrete materials do not embody mathematical meaning and they must be used judiciously and carefully for good results (Baroody, 1989;
Hiebert & Carpenter, 1992; Moyer, 2001; Uttal, Scudder, & DeLoache, 1997). Ball (1992) maintained that concrete materials are not magic, nor an insurance policy for authentic student learning: “Although kinaesthetic experience can enhance perception and thinking, understanding does not travel through the fingertips and up the arm” (p. 47). Put more succinctly, “manipulatives are not a panacea” ( Uttal et al., 1997, p. 50).

Research results indicate that the effectiveness of concrete materials is strongly influenced by the way teachers use them ( Bjorklund, 2013; Carbonneau et al., 2013). Stein and Bovalino (2001) cautioned that “simply using manipulatives does not guarantee a good mathematics lesson” (p. 356). On the other hand, Green, Flowers, and Piel (2008) noted that some teachers consider manipulatives to be “enjoyable diversions but do not believe they are essential to teaching and understanding” (p. 235). The crucial role of the teacher is advocated by researchers who believe that establishing a strong connection between concrete materials and concepts of formal mathematics is the key to a successful lesson ( Ball, 1992; Gravemeijer, 2004b; Maclellan, 2001; McNeil & Jarvin, 2007; Uttal et al., 1997). Also crucial is the teacher’s ability to make strategic decisions about the need for some sort of intervention at any stage during the lesson (Brown, 1992). The teacher has an obligation to ensure that students can recognise the mathematics inherent in the task (Ball, 1992; Gravemeijer, 2004b; Uttal et al., 1997). Students do not have the perspective of the teacher, and it is the teacher’s responsibility to support them in making the connections ( Zhou & Peverly, 2005).

In his study of students’ use of manipulatives, Holt (1982) found that students who were unable to perform tasks without the support of concrete materials were similarly unable to complete them with the materials: “They found the blocks … as abstract, as disconnected from reality, mysterious, arbitrary, and capricious as the numbers that these blocks were supposed to bring to life” (p. 219).
The choice of appropriate materials can greatly influence students’ ability to “see” the inherent mathematics and make the desired connections. Hiebert and Wearne (1992) offered the example of unifix cubes assembled into quasi-permanent bars of 10 compared with the “longs” in base-10 blocks which cannot be disassembled into 10 ones. They suggested that, depending upon the student’s ability to see 10 as both a composite of 10 ones and a unit of one 10, this simple difference in the choice of concrete materials could affect the connections that the student is able to make in their mathematical understanding.

As well as being unaware of the subtle differences in the way students think about concrete materials, teachers may simply choose to replicate the procedures for teaching these concepts that they have always followed. Many teachers teach in the same way they were taught (Green et al., 2008) which may result in teachers always using the same concrete material without questioning their suitability, and the way in which their inclusion supports the purpose of the lesson. Moyer (2001) cautioned that, if incorrect choices are made, this can sometimes make the teacher’s task of introducing and connecting mathematical concepts more difficult.

Teachers who favour direct instruction are often criticised because teaching in this way can result in knowledge of the procedural steps of an algorithm without a conceptual understanding of the underlying mathematics (Baroody, 1989; Clarke, 2005; Kamii et al., 2001). Similarly, Stein and Bovalino (2001) expressed concern that direct instruction in the use of concrete materials may result in students using the materials in a procedural or rote way. Without the opportunity for reflection and discussion about the underlying mathematical concepts, deep knowledge and understanding is unlikely to occur. To develop deep understanding, teachers need to create a classroom culture that values student thinking and promotes reflection on strategy use. These expectations or norms of
student behaviour should be an assumed element of the classroom culture and come to be “taken-as-shared” (Cobb & Whitenack, 1996; McClain, 2002).

Some teachers will incorrectly assume that a student’s ability to complete tasks, with or without the scaffolding of concrete materials, is indicative of their mastery of a concept because they are able to replicate the algorithm independently. Unfortunately, however, if the complexity of the task is increased or if the context of the task is changed, these students are often unable to solve the new task. For these students, their knowledge of mathematical concepts remains at a superficial level because they have only learnt to reproduce an algorithm that directs their manipulation of concrete materials to solve the task. Martin (2009) described three elements of physically distributed learning which result in changes in cognitive systems and support understanding in a novel context. These three critical elements were observed as students solved number problems with manipulatives. “First, action is beneficial but insufficient without interpretations. Second, actions and interpretations coevolve. Third, children can transition to solving problems mentally” (Martin, 2009, p. 141). These three elements highlight the importance of the level of instructional guidance needed for reflection, as described earlier by Carbonneau et al. (2013).

Typically, once students demonstrate their mastery of a task type using concrete materials, they are then expected to complete an algorithm at the level of formal arithmetic notation without materials. Often, when students are asked to use mental strategies and mathematics at a more formal level to solve similar tasks, they are unsuccessful because they do not have the deep knowledge and understanding necessary to solve the task (Dowker, 2005). The solution has no context for the student beyond concrete materials and they have little sense of the feasibility of their solutions. Visualisation as a strategy can be used to remedy this inability to solve novel tasks (see Section 2.4.6).
2.4.3 Students’ Interactions with Concrete Materials

The research above clearly indicates that teacher behaviour is critical in the effective use of concrete materials to support student learning. This section examines the importance of students’ interactions with concrete materials. A student involved in the physical manipulation of concrete materials might appear to be actively engaged in their learning and to have successfully mastered the mathematical concept being taught; however, the student may simply be mimicking a procedure performed by the teacher, with little conceptual understanding of the mathematics underlying the task (McLellan & Dewey, 1895). Baroody (1989) cautioned that “just as with symbols, pupils can learn to use manipulatives mechanically to obtain answers” (p. 4). Expecting students to think about, and reflect upon, their actions when using concrete materials is imperative to conceptual development (Higgins, 2005).

Resnick (1992) recommended that talking about mathematical relationships and ideas, and reflecting on thinking should be an expectation in every mathematics classroom. This is supported by other researchers (Ambrose, 2002; Ball, 1992; Clements & McMillen, 1996; Clements & Sarama, 2009; Gravemeijer, 2004b; Higgins, 2005; Moyer, 2001; Uttal et al., 1997) who recommended that, when using concrete materials, children need to talk about and reflect on their actions, as this allows them to clarify their own ideas, and to re-establish their understandings in light of what they have learnt.

Clements and McMillen (1996) identified two types of concrete knowledge: sensory-concrete knowledge and integrated-concrete knowledge. Sensory-concrete knowledge is defined as students using concrete materials to assist them in making sense of an idea; however, “Integrated-concrete knowledge is built through learning. It is knowledge that is connected in special ways … Integrated-concrete thinking derives its strength from the combination of many separate ideas in an interconnected structure of
knowledge [emphasis in original]” (Clements & McMillen, 1996, p. 271). This suggests that concrete materials can facilitate the growing together of a student’s understanding of mathematical concepts – the integrating of previously learnt knowledge with new ideas and the application of these ideas in new settings.

2.4.4 The Dual Roles of Concrete Materials

Many researchers (DeLoache, Kolstad, & Anderson, 1991; McNeil & Alibali, 2005; Meira, 1995; Uttal et al., 1997) referred to the challenge of dual representation when students are working with concrete materials. Dual representation can be described as the challenge of students having to perceive familiar concrete materials (e.g., blocks) as simultaneously representing or symbolising a mathematical concept. McNeil and Jarvin (2007) described three obstacles to dual representation: “(a) the non-transparent mappings between manipulatives and the concepts or procedures they symbolize; (b) children’s limited cognitive resources; and (c) children’s tendency to resist change” (p. 313).

Dual representation explains the difficulties of making links between the concrete object, the hypothetical situation and the symbolic notation that mathematicians assign to embody the situation (Dowker, 2005). “Whether manipulatives are effective teaching tools depends upon whether children interpret them as representations of something else and understand the nature of the representational relation” (Uttal et al., 1997, p. 45).

Many students face challenges in translating between arithmetic problems posed in concrete, verbal and numerical formats.

Some children do not understand that the same operation is being used in adding two counters to three counters; in solving a problem such as “John had three sweets and his mother gave him two more sweets, so now he has five sweets”; and in doing the sum “$2 + 3 = 5$”. (Dowker, 2005, p. 98)
Crucial to the student’s knowledge of the representational relationship is the teacher’s ability to make those links explicit and meaningful for the student. This further supports the belief expressed earlier (Section 2.4.3) that reflection on the use of concrete materials and discussion between students and teacher are imperative to ensure understanding.

The following section examines the literature on the characteristics of effective concrete materials.

2.4.5 Characteristics of Effective Concrete Materials

Several researchers have written about their perception of the most important characteristics of effective concrete materials (Clements & Sarama, 2009; Hatano, 1982; Hiebert & Wearne, 1992; McNeil & Jarvin, 2007; Sowell, 1989; Uttal et al., 1997; Wright, Martland, Stafford, & Stanger, 2006). In Japanese classrooms, the same manipulatives are used throughout the first years of primary school: “The manipulatives become highly familiar and hence less interesting as objects in their own right. Becoming accustomed to the same manipulative set might free the children to focus instead on what the manipulatives represent” (Uttal et al., 1997, p. 50).

Sowell’s (1989) study found that consistent use of manipulatives over long periods of time had the greatest positive effect on student learning, and similarly Hiebert and Wearne (1992) found prolonged and meaningful experience with one manipulative to be more beneficial than similar experiences with a variety of manipulatives. Clements and Sarama (2009) concurred: “A synthesis seems to indicate that multiple representations are useful (e.g., a manipulative, drawings, verbalizations, symbols), but many different manipulatives may be less useful” (p. 279). These findings challenge Dienes’ principle of multiple embodiment (as cited in Moyer, 2001) which, in the 1970s, led to the use of
several different manipulatives as representations of a concept in order to maximise children’s chances of understanding.

Concrete manipulatives can be described as perceptually rich or bland (Carbonneau et al., 2013). Perceptually rich materials are manipulatives which are designed to mimic realistic items (e.g., play money, pizza) whereas bland materials are described as nondescript (e.g., counters, blocks). In reference to bland materials, McNeil and Jarvin (2007) proposed that, “when these simpler manipulatives are used, it may be easier for students to view the manipulatives as mathematical tools and to focus on the underlying concepts” (p. 314). However, a recent meta-analysis reported that there are studies which suggest that perceptually rich manipulatives may both enhance and inhibit student learning (depending upon the focus of the learning). The authors suggested that further research is needed in order to be conclusive (Carbonneau et al., 2013).

2.4.6 Using Concrete Materials to Support Visualisation

Concrete materials are traditionally used as items to be manipulated by students and teachers “with the aim of helping students to understand abstract concepts through concrete, kinesthetic, and visual experiences” (Özgün-Koca & Edwards, 2011, p. 389). However, as described earlier, the transition from concrete materials to verbal and numerical formats is not always easy (Dowker, 2005). An intermediary step between concrete materials and more abstract formal mathematical notation is the use of visualisation.

The use of concrete materials is important, but rather than moving directly from physical representations to the manipulation of abstract symbols to explain the traditional abstract procedures of algorithms, it is suggested that the emphasis be shifted to using visual imagery prior to the introduction of more formal procedures. (Bobis, 1996, p. 21)

One approach to combining the use of concrete materials and visualisation to support the transition to formal arithmetic notation is described below. Wright, Martland,
Stafford, and Stanger (2006) used the term “settings” to describe the situation in which an arithmetical task is posed to students.

Settings can be (a) material (i.e., a physical situation); (b) informal written; (c) formal written; or (d) verbal. The term setting refers not only to the material, writing or verbal statements but also to the ways in which these (material, writing or verbal statements) are used in instruction and feature in students’ reasoning. Thus the term setting encompasses the often implicit features of instruction that arise during the pedagogical use of the setting. (R. J. Wright, personal communication, August 30, 2013)

Wright, Martland, Stafford, and Stanger (2006) recommended that, when a setting is first introduced, preliminary discussions should aim to familiarise students with the elements of the setting, and feature explicit instruction in how these elements support strategy use to solve mathematical tasks. They believe repeated use of concrete materials in the same setting reduces the need for students to become accustomed to using a range of concrete materials, and increases familiarity with the structure of the manipulative and therefore advances the focus on the mathematical concepts.

In order to enhance students’ ability to visualise the use of concrete materials as a strategy to strengthen the connection between informal and formal settings, Wright, Ellemor-Collins, and Tabor (2012) described the practice of “distancing the setting”. The purpose of this approach is to support students to “increasingly develop their visualization and mental organization, until they can solve tasks independently of any setting” (p. 17).


The pedagogical approach of distancing the material setting is most effective in teacher-controlled situations (Carbonneau et al., 2013) and involves four stages. In the first stage, the materials are visible or unscreened. The teacher and students freely manipulate
the materials. In the second stage, the materials are briefly shown to the students, and then screened. This process is known as flashing. This can result in students creating a mental image of the materials and then being encouraged to visualise the strategy. Once the task is solved, the solution is checked by unscreening the materials, and re-enacting the strategy. In the third stage, the materials are screened from vision. The scaffold of the materials remains and can be accessed if an impasse occurs, but the students are expected to use mental strategies to solve the task. At any of these stages, arithmetic notation and symbols may be used strategically to record students’ strategies. The final stage of distancing the material setting is when tasks are presented using symbolic notation as bare number tasks (i.e., formal arithmetic notation) without the scaffold of concrete materials.

2.4.7 “Virtual” Concrete Materials

In recent years, discussion regarding the use of concrete materials has come to include a focus on virtual manipulatives. Virtual manipulatives can be considered to be representations of concrete materials on a device such as a computer, an interactive whiteboard or a tablet. One of the most commonly referred to definitions of a virtual manipulative is as an “interactive, Web-based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge” (Moyer, Bolyard, & Spikell, 2002, p. 373). Similar to Sowell’s (1989) description of manipulatives as either pictorial or concrete, virtual manipulatives have been described as either static or dynamic. Static manipulatives are considered to be virtual images or pictures of concrete materials. Conversely, dynamic virtual manipulatives are objects that can be manipulated, and therefore are of greater pedagogical value (Moyer et al., 2002).

At this stage, research indicates that the success of virtual manipulatives in supporting student learning is very closely tied to the pedagogy which frames their use (Zevenbergen & Lerman, 2008). Burns and Hamm’s (2011) study found that student
learning was unchanged whether concrete or virtual manipulatives were used. Suggested advantages for virtual manipulatives include that they can be closely linked to symbolic representations (Sarama & Clements, 2009b), offer immediate feedback (Burns & Hamm, 2011; Sarama & Clements, 2009b) and provide high levels of engagement for students if their use is supported by strong pedagogy (Zevenbergen & Lerman, 2008).

In the following section, the literature is reviewed with regard to the role of formal arithmetic notation in supporting student learning.

2.5 Formal Arithmetic Notation

If teachers are asked to justify the use of manipulative materials in their mathematics classroom, a typical explanation is that they facilitate a connection between concrete and abstract mathematics. “There is a common belief in mathematics education that children progress from the ‘concrete’ to the ‘semi-concrete’ level of pictures and then to the ‘abstract’ level of numerals and mathematical symbols” (Kamii, 2001, p. 207). In describing mathematics as abstract, Kamii believes teachers are typically referring to the use of formal symbolic arithmetic notation to solve arithmetic tasks. These labels of concrete, semi-concrete and abstract are reminiscent of Bruner’s (1964) modes of representation: enactive, iconic and symbolic.

Bruner (1964) described these modes of representation as appearing in the life of a child in the order described, with each mode depending on the previous one for its development. He described the enactive mode with representation being associated with a motor response, and the iconic mode with representation associated with images. Finally, the symbolic mode of representation includes “remoteness and arbitrariness” (Bruner, 1964, p. 2) in the use of symbols and/or language. The interplay among these three modes of representation and the importance of oral language (Haylock & Cockburn, 2003) is visually represented in Figure 2.1.
Askew (2013) challenged this Piagetian notion of developmental stages and moving from concrete to abstract representations by suggesting that “the structures for thinking abstractly are in place from an early age. What distinguishes young children’s thinking is not the quality of their ability to reason per se, but the limits of their experiences” (p. 2).

2.5.1 Inducting Students Into the Use of Formal Arithmetic Notation

The introduction of formal arithmetic notation can be a contentious issue in the early years of mathematics education. According to Ginsburg et al. (1998), “codified mathematics is enormously powerful, but few children easily understand it or find it useful” (p. 418). The challenge of successful symbol use in mathematics is stated more strongly by Gray and Tall (1994): “Mathematical symbolism is a major source of both success and distress in mathematics learning” (p. 4). They believe that the source of this success and distress is because

those who are successful in mathematics … employ the simple device of using the same notation to represent both a process and the product of that process … The symbol $5 + 4$ represents both the process of adding through counting all or counting on and the concept of sum ($5 + 4$ is 9) [emphasis in original]. (p. 4)
The notion of symbolism as representing a process or a product will be further explored in Section 2.5.3.

Societal expectations play a part in the introduction of formal mathematical symbols in early years classrooms. In traditional mathematics classrooms, a key indicator of success is the completion of voluminous collections of written “sums”, all marked as correct. “Parents are encouraged when they see worksheets … they see such papers as indicators that their children are learning” (Ransom & Manning, 2013, p. 188).

There is a spectrum of opinion regarding the timing and pedagogical approaches to the introduction of formal mathematical symbols. At one end of the spectrum, some educators recommend allowing students extended periods of time to experiment with their own symbolic recording of mathematical concepts and knowledge, whilst at the other end, educators recommend introducing formal mathematical symbols as soon as formal schooling commences. Research opinion is divided and often lies somewhere between these two extremes. The following section will review the relevant literature.

Thompson and Thompson (1990) contend that “premature introduction of symbols actually interferes with the children’s development of the concept involved” (p. 10). Similarly, Carruthers and Worthington (2010) advocated extended time for student exploration of informal recording: “Children need extended periods of time in which to explore symbols in their own ways before they are ready to use standard symbolic operations with small numbers, with understanding [emphasis in original]” (p. 130).

In contrast, Ginsburg et al. (1998) advocated instruction in the use of formal symbolism for students from the early years of schooling. They based their argument on the premise that children cannot reinvent the conventional arithmetic symbols and methods of recording.
Codified arithmetic needs to be taught through a process of formal instruction – that is, through organized teaching in the classroom or in tutorial sessions. Written arithmetic is a cultural legacy; it represents the accumulated wisdom of the race, put in written form so as to be available to all, and it is obviously far superior to children’s informal arithmetic [emphasis in original]. (Ginsburg et al., 1998, p. 418)

In the same way that students must learn the number word sequence, there are some conventions of arithmetic notation that must, at some stage, be taught to students via instruction, as activities alone will not provide them with the opportunity to discover this knowledge for themselves. Ginsburg et al.’s (1998) approach must be considered with some caution, however, because whilst there are some symbolic conventions which must be universally adopted, such as the digits (0 – 9) and operation signs (+, −, ×, ÷), recent research suggests that there are multifarious informal, yet valid, ways of recording the process of using these operations to solve arithmetic tasks, that do not necessarily follow the traditional formal methods (Clarke, 2005; Gravemeijer, 2004b; Treffers, 1991; Van den Heuvel-Panhuizen, 2008; Wright, Martland, Stafford, & Stanger, 2006).

Hughes (1986) recommended a greater initial emphasis on students’ own recording methods when he suggested that “work could be done with children’s own representations of addition and subtraction before introducing them to the conventional plus and minus signs” (p. 177). However, this approach is criticised when it is likened to withholding standard letters from children who are learning to write. “What is important is that we provide children with the whole picture – and for addition and subtraction this will include the standard symbols [emphasis in original]” (Carruthers & Worthington, 2010, p. 125).

Gravemeijer et al. (2000) described the attempt made in the Realistic Mathematics Education (RME) approach to balance the use of informal and formal symbols:
On the one hand, the dialectical relation between symbolizing and mathematical sense-making indicates the value of student initiative. On the other hand, the eventual goal of enabling students to reason powerfully with conventional symbolizations points to the need to introduce symbolizations developed in advance by the designer. (p. 236)

This accords with the claim of Hiebert and Carpenter (1992) that “helping students connect their intuitive knowledge of mathematics with written symbols is one way of helping them connect new knowledge with prior knowledge” (p. 83). The role of the teacher in making connections between students’ informal knowledge and formal mathematical symbols is critical. Teachers can use formal mathematical notation to describe children’s mathematical intuitions (Resnick, 1992), to model their own thinking processes (Carruthers & Worthington, 2010) and to relate them to student’s mathematical activity (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997).

This approach of integrating thinking, conceptual knowledge and formal notation is described by Wright et al. (2012) as follows: “Ideally, when notation is introduced, we want students to experience this as an invitation to partake in shared tools, which can illuminate their own thinking, rather than as an imposition of someone else’s way, which obscures their own thinking” (p. 18).

However, teachers need to be mindful of when to introduce formal notation to students. Although the teacher delivers a verbal message to encourage students to record their thinking in a manner that makes sense to them, once the formal recording is shown to the students, the implicit message received by the students may be that this is the best way to record, regardless of whether or not it is a true reflection of their thinking. Carruthers and Worthington (2010) contend that “they want to comply and interpret the request to ‘put something down to show’ as meaning ‘do what I have just shown you’. This leads to compliance and conformity without understanding” (p. 208). Similarly, this type of occurrence was described by Gifford (1997): “In group situations, once one child had
thought to use symbols, the others usually followed with alacrity rather than ‘universal reluctance’” (p. 79).

2.5.2 Connecting Formal Arithmetic Notation, Concrete Materials and Real World Applications

From their earliest experiences in school mathematics, students need to understand that formal arithmetical symbols do have real world meaning (Maclellan, 2001). However, rather than linking real world meaning to symbolic representations, teachers can construct pedagogical experiences that allow students to build symbolic representations out of their experiences. This process is known as mathematisation (Gravemeijer, 2004b; Wright et al., 2012).

Both horizontal and vertical mathematisation are described in the tradition of RME: “Horizontal mathematization involves going from the world of life into the world of symbols, while vertical mathematization means moving within the world of symbols” (Van den Heuvel-Panhuizen & Wijers, 2005, p. 288).

In connecting real world applications with the use of concrete materials and formal mathematical notation, the process of mathematisation can see informal mathematical activity begin as a “model of” a situation and become a “model for” other, equivalent situations (Gravemeijer, 1999; Van den Heuvel-Panhuizen & Wijers, 2005).

There is no consensus about the most effective use of concrete materials and symbolic notation to support student learning. Gravemeijer et al. (2000) believe that “the challenge for the designer (and the teacher) is to anticipate a developmental route for the classroom community that culminates with the powerful use of conventional symbolizations” (p. 238).
2.5.3 Formal Arithmetic Notation as Both a Process, and a Product of the Process

As discussed in Section 2.5.1 above, Gray and Tall (1994) attributed successful mathematical thinkers with the ability to interpret formal arithmetic notation as representative of two concepts: the process and the product of that process. Research suggests that, rather than regarding the equal symbol as an indicator of equivalence, many students interpret it as either a syntactic indicator which shows where the answer should be written, or as an operator sign or stimulus to do something (Saenz-Ludlow & Wammable, 1998; Warren & Cooper, 2009).

Gray (2010) described eight-year-old Joseph’s attempts to add 4 + 3. As he struggles to solve this task using a counting strategy, his teacher recommends he use counters or his fingers to assist. Joseph and children like him are counting, because they are unable to do anything else: “Faced with a problem such as 4 + 3, they translate it into a counting action” (p. 84). Repeated exposure to tasks such as this, with well-intentioned advice to use manipulatives to support the counting process, leads to addition tasks being solely viewed as operations or procedures to be performed. Without guided reflection and an explicit focus on making connections between known addition combinations, Gray and Tall (1994) believe that “[teachers’] persistence in emphasising procedures leads many children inexorably into a cul-de-sac from which there is little hope of future development” (p. 18).

When a student interprets the “+ 3” in 4 + 3 as an operation to be performed, they are perceiving addition as a unary operation. When perceived as a unary operation, 4 + 3 and 3 + 4 are different expressions. One begins with a collection of four items, to which three more items are added. The other begins with a collection of three items, to which another four items are added. Whilst the result is the same, the operations are quite different (Weaver, 1982). However, perceiving 4 + 3 as a binary operation means that four and three
are combined to make a total of seven, or conversely that four can be removed from seven and leave a result of three. It is only when an expression is conceived in this way, that it can be said that commutativity is understood.

Baroody et al.’s (2003) interpretation of these operations was illustrative: “A unary operation is an operation on one number that results in another … A binary operation is an operation on two amounts that results in a third. Part-part whole problems … embody this meaning” (p. 151). The widely accepted view is that students firstly view addition as a unary operation, and later are able to perceive it as a binary operation (Baroody et al.).

As stated in Section 2.4, it is the responsibility of teachers to design activities to support students in making connections between tasks and building upon prior knowledge. Many students seem to be aware of the implicit messages teachers send through their choice of tasks, strategies and settings. Gray (1991) noted that an emphasis on procedure can result in students valuing memory as more important than conceptual knowledge. Students may feel that remembering the procedure to be followed provides them with security in their ability to solve each task. However, there is also the possibility that, when using counting strategies, students may need to invest so much cognitive effort into determining the sum whilst also keeping track of the two addends that, in the mind of the student, the sum and the expression may not be connected (Baroody et al., 2009). Whilst Hughes (1986) stated that “fingers can play a crucial role in linking the abstract and the concrete” (p. 51) Gifford (1997) expressed a similar caution to Baroody regarding the way students perceive arithmetical tasks: “One concern is that children see plus and minus signs merely as prompts to count bricks or fingers, without any understanding of them as symbols representing the operations of addition and subtraction” (p. 79).

Gray and Tall (1994) labelled a student’s ability to flexibly make strategy choices to solve tasks presented in bare number format notation as proceptual thinking. The
term “proceptual” is a combination of the terms process and concept. “We characterize proceptual thinking as the ability to manipulate the symbolism flexibly as process or concept, freely interchanging different symbolisms for the same object [emphasis in original]” (p. 7). In the same way as Weaver (1982) described unary and binary operations, Gray and Tall describe the ability to think proceptually as the distinction between students who can perceive tasks such as $4 + 3$ as (1) the result of three added to four; (2) the combination of four and three; and (3) a total of seven, compared to students who can perceive it only as a unary operation. The difference between students who can and cannot think in this way is described as the “proceptual divide” (Gray, 2010).

Gray and Tall (1994) did not disregard entirely the use of procedures to solve tasks. Rather, they argued that students need to be able to make informed choices about which procedures are appropriate, and when they are appropriate. It is this ability to make an informed choice that empowers students to be proceptual thinkers:

Proceptual thinking includes the use of procedures. However, it also includes the flexible facility to view symbolism either as a trigger for carrying out a procedure or as a mental object to be decomposed, recomposed and manipulated at a higher level [emphasis in original]. (Gray & Tall, 1994, p. 11)

Once students are able to regard addition tasks at a proceptual level, the concept of commutativity becomes accessible to them. However, this is not to say that students who are able to recognise commutativity in tasks are thinking proceptually. Students working at the level of proceptual thinking are described by Resnick (1992) as using the “mathematics of operators”. Resnick (1992) described the two highest levels of mathematical thinking as “mathematics of numbers” and “mathematics of operators”. “In the mathematics of numbers, operations are like transitive verbs. They describe actions that can be performed on numbers … In the mathematics of operators, operations do behave like nouns” (Resnick, 1992, p. 405).
It appears that what Resnick (1992) described as the mathematics of numbers, Gray and Tall (1994) described as procedural thinking, and what Resnick described as the mathematics of operators, Gray and Tall described as proceptual thinking.

Baroody et al. (2003) proposed that using a subitising or mental model may lead students to separate and combine the same two small collections in different orders (e.g., OO + O and then O + OO) and notice that the result is the same. Opportunities for students to notice and reflect on such occurrences can serve to support them in becoming proceptual thinkers (Gray & Tall, 1994) and perceiving addition tasks as binary operations (Weaver, 1982).

More recently, Askew (2013) has advocated treating basic addition combinations, in Vygotskian terms, as scientific concepts rather than spontaneous concepts. By scientific concepts he means that knowledge of one addition combination can (and should) be connected to other addition combinations; a systematic organisational structure can be used to support learning. However, Askew argued that basic addition combinations can also be learnt as spontaneous concepts in the way that is often associated with rote learning. For example, 6 + 4 can be learnt as a separate combination, independently of other combinations such as 6 + 5, and the relationship between the two combinations ignored. Askew argued that it is the role of the teacher to mediate learning and advance students’ perception of the addition combinations as scientific concepts.

In Sections 2.2 and 2.3 the literature was reviewed with a focus on the use of counting strategies to solve addition tasks. In Section 2.6, the literature regarding instruction in the use of grouping strategies to solve addition tasks will be reviewed.

**2.6 Using Grouping Strategies to Solve Addition Tasks**

As indicated by the five theoretical frameworks described in Section 2.3.2 and the voluminous research available, counting is an integral part of the development of early
arithmetical strategies. However, counting in isolation is not considered adequate for learning the basic addition combinations (Van Luit & Schopman, 2000).

In the last thirty years, research has explored the possibility that the arithmetical knowledge of students may also be advanced through the use of non-counting or grouping strategies. Grouping strategies promote the use of facile and robust mental calculation strategies (Bobis, 1996, 2008; Cotter, 2000; Fischer, 1990; Flexer, 1986; Gravemeijer et al., 2000; Hatano, 1982; Labinowicz, 1985; Murata, 2004; Putnam et al., 1990; Fosnot & Dolk, 2001; Van den Heuvel-Panhuizen, 2008; Willis, 2002; Wright, Martland, Stafford, & Stanger, 2006). The terms “structuring numbers” and “structure of numbers” are also used to describe the use of grouping strategies. Van den Heuvel-Panhuizen (2008) argued that “the structure of numbers should … be given a lot of attention in mathematics teaching, because it is structuring that makes it possible to go beyond the level of calculating by counting” (p. 44).

A focus in instruction on the use of grouping strategies to solve addition tasks eliminates the need to learn addition combinations in isolation and allows the focus to be on making connections, integrating new knowledge with current knowledge, and constructing a mental system of interrelated addition combinations which makes these ideas easier to apply and remember (Baroody, 2006; Putnam et al., 1990; Tait-McCutcheon, Drake, & Sherley, 2011).

It is conceivable that students might progress to solving basic addition combinations without the use of counting strategies, if instruction includes situations that are designed to encourage numerical reasoning. Cotter (2000) stated that “to help a young child associate the correct number with a small collection, we should refer to the whole collection by its number and avoid having children focus on individual objects through the counting ritual” (p. 109). Researchers have speculated that this may be the case for some
Aboriginal children. “While children frequently use counting-all and counting-on strategies, and also counting-back, to solve addition and subtraction tasks, Aboriginal children might be less likely to employ strategies which rely on verbalization and sequential enumeration” (Wright & Steffe, 1986, p. 128). This different learning path could be an alternative to explain how some students “who could not ‘count to six’ nevertheless could say that there were seven sweets or eight pebbles” (Willis, 2002, p. 119). Thus, consideration also needs to be given to the possibility that a student may develop the notion of cardinality without counting (Threlfall, 2010). This possibility will be further explored in Section 2.6.1.

2.6.1 Visualisation and Subitising

The ability to visualise is an important strategy which can support the development of students’ conceptual knowledge of number (Bishop, 1989; Bobis, Mulligan, & Lowrie, 2009; Hughes, 1986; Mulligan, Mitchelmore, Outhred, & Bobis, 1996; Labinowicz, 1985; Papic et al., 2011; Wheatley, 1991). Wheatley (1998) reported that “evidence is provided that thinking strategies used by young children in adding and subtracting is often based on images” (p. 75). When used regularly and accompanied by reflection on the process (Yackel & Wheatley, 1990), instruction in visualisation can be a powerful tool (Bobis, 1996) to build upon students’ natural strategies and provide a strong foundation for students’ conception of number (Von Glasersfeld, 1982) as well as support the use of grouping approaches to solve addition tasks.

In Section 2.4.3, the importance of students talking about and reflecting on their thinking when working with concrete materials was discussed. When explicitly teaching visualisation strategies to students, similar focus on student reflection is highly beneficial. Yackel and Wheatley (1990) found that “pupils apparently improved in their ability to elaborate or alter their visual image as they reflected on their classmates’ descriptions” (p. 57).
As described earlier, counting-by-ones is one strategy to determine the numerosity of a collection; this section will look at subitising as an alternative strategy. Subitising is a grouping approach which can be used to support student development of the concept of cardinality, as “things are quantified by looking” (Warren, DeVries, & Cole, 2009, p. 47). The literature regarding subitising is reviewed in more detail below.

Kaufman, Lord, Reese, & Volkmann (1949) were the first to use the term “subitising”. It was offered as an alternative to the term “estimating”, when referring to collections smaller than six:

The term estimating is commonly applied to a report of numerosness … We propose that it be reserved for the discrimination of stimulus-numbers greater than 6 … a new term is needed for the discrimination of stimulus-numbers of 6 and below … The term proposed is subitize. (p. 520)

The concept of subitising has been a source of debate since the term was first suggested by Dr Cornelia Coulter (Kaufman et al., 1949), and has been written about by many researchers (Clements, 1999; Dehaene, 1992; Gelman & Gallistel, 1978; Gould, 2012; Hughes, 1986; Kauman et al., 1949; Klein & Starkey, 1988; Labinowicz, 1985; Mandler & Shebo, 1982; Penner-Wilger, 2004; Piazza, Mechelli, Butterworth, & Price, 2002; Tolchinsky, 2003; Trick & Pylyshyn, 1988; Von Glasersfeld, 1982; Warren et al., 2009; Wright, Martland, Stafford, & Stanger, 2006).

The usefulness of subitising and also estimation as an instructional topic has been contentious. Beckwith and Restle (1966) defined subitising as “a somewhat mysterious but very rapid and accurate ‘perceptual’ method” (p. 443), which has been “criticized for lacking a precise theoretical characterization” (Dehaene, 1992, p. 14).

Based on testing of children between the ages of three and five, Gelman and Gallistel (1978) hypothesised that the act of subitising must be preceded by one-to-one counting. They believed that over time children increase the speed with which they are able
to count the items and reduce their utterances of the numbers to sub-vocalisation, until the process is so rapid that it is interpreted as subitising. In contrast to Gelman and Gallistel’s hypothesis, researchers (Ginsburg et al., 1998; Young-Loveridge, 2002) believe that, as a strategy to determine quantity, subitising is distinct from, but no less important than, counting-based strategies.

Subitising has been defined in many ways, but most definitions include the following characteristics. “Subitizing is thought to be a perceptual process that operates accurately and very quickly … in determining exactly how many objects are contained in a small set of objects (less than or equal to four or five)” (Klein & Starkey, 1988, p. 9).

Mandler and Shebo (1982) hypothesised that canonical patterns, that is, arrangements of items in a standard or regularly arranged pattern, are subitised faster than non-canonical arrangements. Examples of canonical patterns include regular dice or domino patterns (see Figure B1).

Expanding on the findings of Mandler and Shebo (1982), Cotter (2000) made four recommendations to guide the presentation of visual items to support subitising strategies: (1) groups should not be embedded in pictures; (2) items should not be pictorial; (3) regular arrangements of items are preferable; and (4) there should be a strong contrast in colour between the background and the items.

However, Von Glasersfeld (1982) believes there is a distinction between the process students use to ascribe number to canonical patterns and subitising. He cautioned against attributing conceptual knowledge of numerosity to students who are able to label dot patterns with what adults perceive to be terms describing quantity, such as “five”. To young students, the number word “five” might not refer to quantity. It may simply be a label for a familiar visual image of dots arranged as in Figure B1 in Appendix B. Von Glasersfeld (1982) drew a parallel with the example that young children may generalise a
label such as “dog” to describe any four-legged furry animal such as a possum, goat or cat, until conceptual knowledge of the specific attributes of a dog are developed. He suggested that “in subitizing, the child associates figural patterns with number words by a *semantic* connection and not because of the number of perceptual units of which they are composed [emphasis in original]” (Von Glasersfeld, 1982, p. 8).

In order to use visual spatial patterns to support additive reasoning, students must move beyond naming the patterns and connect the semantic name with a conception of number or numerosity. This process was described as reflective abstraction (Von Glasersfeld, 1982). It is only through this process of reflective abstraction that students can come to associate the knowledge that the counting sequence for an arrangement of items ends with the number word that also describes the visual pattern, occasioning “the result of the child’s counting scheme … the same as the result of the naming scheme” (Von Glasersfeld, 1982, p. 18).

Clements and Sarama (2009) made a distinction between two different types of subitising: perceptual subitising and conceptual subitising. They noted that one uses perceptual subitising “when you ‘just see’ how many objects in a very small collection” (p. 9). This description accords with Kaufman et al.’s (1949) original notion of subitising. In contrast, conceptual subitising involves “seeing the parts and putting together the whole” (Clements & Sarama, 2009, p. 9). For example, recognising eight dots is beyond the limits of perceptual subitising (Beckwith & Restle, 1966); however, being able to put four dots and four dots together to make eight can happen very quickly, and can be considered conceptual subitising.

The implication of a distinction between perceptual and conceptual subitising for pedagogy is that “for most children, perceptual subitizing occurs quite naturally without
specific instruction, but conceptual subitizing (usually) must be learned” (Bobis, 2008, p. 6).

On initial reading, the definition of conceptual subitising (Clements & Sarama, 2009) appears to describe an addition strategy. However, when a student states they can see two “fours” in an “eight”, it cannot be assumed with any certainty that this knowledge corresponds to a formal additive concept of $4 + 4 = 8$. One aspect of additive reasoning involves the student perceiving two addends as combining to make the sum, in the sense that four and four make eight. In the example of conceptual subitising described above, the student may not be reasoning additively, and may simply be recognising one familiar visual pattern (four dots) repeated twice within another familiar visual pattern (eight dots) (Labinowicz, 1985). Engaging students in conceptual subitising as described by Clements and Sarama (2009) is a worthwhile activity; however, the principle of parsimony demands caution because this behaviour is not necessarily indicative of more sophisticated conceptual knowledge, that is, formal additive reasoning.

2.6.2 Part-part-whole Knowledge of Number

Fischer (1990) argued that “a major portion of the primary mathematics curriculum includes skills that are directly or indirectly dependent on the child’s understanding of part-part-whole relations” (p. 214). The importance of being able to consider a number in terms of the whole and its parts is an early arithmetical skill, which has received greater focus in recent years (Bobis, 2008; Ellemor-Collins & Wright, 2009; Fosnot & Dolk, 2001; Papic et al., 2011; Resnick, 1983; Steffe, 1992; Treacy & Willis, 2003; Willis, 2002; Young-Loveridge, 2002; Zhou & Peverly, 2005). Sarama and Clements (2009a) believe that “composing and decomposing are combining and separating operations that help children develop generalized part-whole relations, one of the most important accomplishments in arithmetic” (p. 129).
As described in Section 2.6.1, visualisation is a very powerful strategy and, when supported by judicious questioning, it can be used to encourage students to use additive reasoning as a precursor to formal addition strategies (Von Glasersfeld, 1995). Expecting students to articulate their thinking and justify their reasoning in the context of visualised images can serve to advance the knowledge of the class as a community of learners (Yackel & Wheatley, 1990). The role of the teacher in facilitating reflection, discussion and interaction with others as a classroom community of learners is critical (Steffe, 1990). The use of visualisation strategies in settings which are carefully structured to engender additive reasoning strategies can promote students’ thinking, talking and reconceptualising about numbers in terms of their part-part-whole structure (Bobis, 1996).

Baroody et al. (2009) described a lack of emphasis on composing and decomposing numbers in early childhood mathematics instruction in the United States. Typical addition tasks in early years’ lessons are of the “join” or “combine” type described by Carpenter and Moser (1984), and focus on addition as a unary operation (Weaver, 1982). However, Thompson (2010) suggested that exploring part-part-whole combinations through activities such as “finding all the partitions of a number is considered at least as important as finding sums” (p. 102). Part-part-whole number knowledge is a fundamental numerical concept as it encourages students to regard all numbers as composites of other numbers and encourages students to “see” relationships between numbers, leading to greater flexibility in mental strategies (Bobis, 2008).

According to Putnam et al. (1990) “the part-whole schema specifies the additive relationships among three numbers, a whole, and two parts … Such complementary transformations applied to part-whole triples form the basis for the derived-fact strategies” (p. 249). Therefore, a strong focus on part-part-whole knowledge in early arithmetical learning provides a strong foundation for using grouping strategies to solve addition tasks.
Knowledge of the part-part-whole structure of number can also be referred to as unitising (Olive, 2001; Steffe, 1992; Wright et al., 2012). Often referred to in the context of place value knowledge with 10 ones being considered as one 10, unitising or part-part-whole knowledge is critical to a student’s ability to move beyond the use of count-by-ones strategies to solve tasks. Fosnot and Dolk (2001) described unitising as follows:

The whole is thus seen as a group of a number of objects. The parts together become the new whole, and the parts … and the whole … can be considered simultaneously … For learners, unitizing is a shift in their perspective … Unitizing these ten things as one thing … requires almost negating their original idea of number [emphasis in original]. (p. 11)

Similarly, Steffe (1992) labelled the fourth learning stage as “tacitly nested number sequence” and used this to describe the student who has reached the stage when “number words symbolize … the operations involved in making a unit of units” (p. 84).

Many researchers highlight the link between part-part-whole knowledge and the strategies used to solve addition and subtraction tasks (Bobis, 1996, 2008; Fischer, 1990; Hunting, 2003; Labinowicz, 1985; Verschaffel et al., 2007; Young-Loveridge, 2002; Zhou & Peverly, 2005) Part-part-whole number knowledge in the context of learning to add and subtract is important. As Hunting (2003) pointed out:

Just as part-whole reasoning and integration operations serve as foundations for solving informal addition and subtraction problems … facility with part-whole reasoning enables children to conceptualize, and be successful solvers of addition and subtraction problems. (p. 232)

Emphasis on the use of part-part-whole strategies in early arithmetical instruction is an attempt to avoid students’ dependence on lengthy, error-prone count-by-ones strategies (Wright et al., 2007). As shown in Section 2.3, early arithmetical teaching often focuses primarily on counting strategies and “this emphasis on counting means that many students remain dependent on immature and often inefficient counting strategies to perform even the most basic computations” (Bobis, 1996, p. 19).
The following section is a review of the literature regarding teaching strategies and material settings designed to facilitate the development of strong part-part-whole knowledge as a precursor to robust mental strategies for solving addition tasks.

2.6.3 Using Quinary- and Ten-based Strategies in Various Settings to Solve Addition Tasks

The term quinary-based refers to arithmetical strategies and concrete materials which use the number five as a base, and ten-based refers to arithmetical strategies and concrete materials which use 10 as a base. Instruction in early arithmetic has a long history of building on base-ten strategies, primarily for the development of students’ knowledge of place value. Place value knowledge has traditionally been regarded as an important base for computational strategies in the four operations of addition, subtraction, multiplication and division.

Van den Heuvel-Panhuizen (2008) advocated the teaching of additive strategies through a grouping approach and particularly through using quinary- and ten-based strategies: “Structuring numbers by doubles (or almost doubles) and by groups of five and ten does not prove difficult, and certainly not after a period of sustained practice” (p. 46).

The use of quinary-based strategies in conjunction with ten-based strategies in early number instruction has the potential to reduce the need to use counting strategies. Five is included as an additional base to 10, and is given particular emphasis in instructional settings (Wright, Martland, Stafford, & Stanger, 2006).

Asian countries have a long history of using quinary- and ten-based strategies to support arithmetical operations (Murata, 2004; Verschaffel et al., 2007). However, the use of five as a base for calculations is not new. Cotter (2000) noted that “the Romans constructed their numerals in groups of fives 2500 years ago, originally they wrote IIII to represent 4, V for 5, VI for 6 and VIII for 9” (p. 109). An emphasis on quinary-based
strategies to support additive reasoning has only recently become more prevalent in Western countries.

In this section, six sets of concrete materials which feature a quinary- and ten-based structure are described. Their use in carefully designed pedagogical settings is aimed to facilitate part-part-whole knowledge and the use of grouping and additive reasoning strategies to support the solution of basic addition tasks. The six material settings are: (1) finger patterns (including bunny ears); (2) the arithmetic rack; (3) Japanese TILEs; (4) five-frame; (5) ten-frame; and (6) twenty-frame (see Appendix B).

Finger patterns constitute a material setting in which students’ part-part-whole knowledge of numbers in the range 1 to 10 can be developed (Sugarman, 1997). Encouraging students to represent a whole number by making a part of that number on each hand facilitates the development of additive reasoning. Activities such as “bunny ears” (Clements & Sarama, 2009) involve students raising fingers simultaneously behind their head to represent a total. For example, if the teacher asks the students to make bunny ears for 5, a student could partition five by raising three fingers on one hand and two fingers on the other hand. Judicious use of questioning can elicit from the students all partitions of five, and facilitate rich discussion about combining and partitioning numbers in the range 1 to 5. Sequential recording of partitions can also serve to scaffold student learning about arithmetical patterns and function as a support for the construction of relational knowledge.

The arithmetic rack is a material setting that is used extensively in RME and facilitates additive reasoning in terms of the quinary- and ten-based structure of numbers in the range 1 to 20. It is described as a combination model (Van den Heuvel-Panhuizen, 2008) as numbers can be represented in both a line model (i.e., 8 is represented as 5 beads and 3 beads in a linear arrangement) and a group model (i.e., 8 is represented as 4 beads on the upper row and 4 beads on the lower row).
In Japan, TILEs (see Figure 2.2) are used as a quinary-based material setting to teach ascribing number, and partitioning and combining strategies in the range 1 to 10 (Hatano, 1982, p. 213).

The semidecimal-binary system of number is used instead of the decimal system at the initial stage of learning … The use of five as an intermediate unit, adopting unsegmented TILEs of five instead of connected or isolated single TILEs, makes it possible to represent the numbers 6, 7, 8, 9 as 5 + 1, 5 + 2, 5 + 3 and 5 + 4. (Hatano, 1982, p. 215)

A five-frame consists of five empty boxes in a horizontal line, that is, a rectangular grid with one row and five columns. A ten-frame is essentially two five-frames, organised as an upper row of five boxes and a lower row of five boxes. A twenty-frame is essentially two ten-frames, organised as an upper row of 10 boxes and a lower row of 10 boxes, with a line separating the first five boxes in each row from the second five boxes. Labinowicz (1985) believes that “organizing the objects into patterns facilitates our ability to recognize and give number names to groupings larger than five” (p. 107).

The material settings of finger patterns, arithmetic rack, and five-, ten- and twenty-frames, support the use of quinary- and ten-based strategies due to the emphasis placed on the structure of all of these settings into groups of 5 and 10 (Van den Heuvel-Panhuizen, 2008; Verschaffel et al., 2007; Wright et al., 2012).
2.7 Summary

In summary, this chapter reviewed the literature relevant to the teaching and learning of strategies to solve one-digit addition tasks. The existence of a variety of theoretical frameworks and the prolific amount of literature regarding the use of counting-based strategies to solve the basic addition combinations highlight the focus on this particular area over the years. Although prevalent in some Asian countries for many years, it has only been in the last 30 years in the Western world that there has been a move towards research into the use of grouping strategies to solve addition combinations. Most of this research has considered the use of grouping strategies as a more sophisticated strategy than counting to solve addition tasks; and therefore to be considered after a student has mastered the skilful use of counting-based strategies. This approach is supported by the curriculum documents of 10 countries which were reviewed for this study and which referred to the use of grouping strategies, but this approach typically only occurs after the first few years of school and after the mastery of counting-based strategies. This study seeks to trial the use of grouping strategies with students in their first year of school, prior to the formal teaching of counting-based strategies to solve simple addition tasks.

With a focus in this study on students in their first year of school, the literature review also analysed previous research with regard to two other important aspects of the teaching and learning of early arithmetic, that is, the use of concrete materials to support learning, and the timing and approach to the introduction of formal arithmetic notation. Conducted as a teaching experiment, this study will use quinary- and ten-based concrete materials in an attempt to support students in reasoning additively in the context of the materials and then in using grouping strategies to solve addition tasks. In the effective use of concrete materials, the importance of teacher behaviour and reflection on the part of the
students was described as critical in providing a pedagogically sound approach. Both teacher behaviour and student reflection constitute key aspects of a teaching experiment.

Research indicated that students often regard the equal sign as a syntactic indicator to “do something” (Gray, 2010; Saenz-Ludlow & Walgamuth, 1998; Warren & Cooper, 2009) and expressions as unary operations which denote which operation to perform. This study has aimed to encourage students to perceive formal written arithmetic notation as indicative of a binary operation, describing a part-part-whole relationship between the three elements (Baroody et al., 2003).

Chapter Three describes the theoretical framework under which this study was conducted.
Chapter Three: Theoretical Framework and Research

Plan

The purpose of this chapter is threefold. Firstly, a chronology of approaches to mathematics teaching is summarised, secondly a theoretical framework is described that draws on design research in the teaching and learning of mathematics and, thirdly, a teaching experiment approach to research is outlined. The methodology of design research informed the planning of this study, which was enacted as a teaching experiment. An important component of this experiment was the opportunity to observe students’ behaviour in their regular learning environment. This allowed for micro-adjustment of the teaching sequence, material settings and tasks to provide a dynamic response to knowledge and skills demonstrated by the students. Undertaking this teaching experiment through the methodology of design research provided an opportunity for the researcher to consider and reflect upon the reflexive connection between research and the reality of the classroom. The primary aim was to observe students’ interactions with tasks and learn from these observations in preference to applying a set of teaching procedures and instructing students in a particular way of teaching. Implementing the teaching experiment in an authentic classroom community required ongoing consideration of the challenge of enacting a hypothetical learning trajectory whilst meeting the learning needs of 20 students operating at a variety of cognitive ability levels.

3.1 The Advent of Constructivism in Mathematics Education

Constructivism is one approach to thinking about the learning of mathematics that came to the fore during the 1980s. As its essence, it has the teacher as an engineer, manipulating learning situations that encourage students to make sense of mathematical
concepts and to be active in their construction of knowledge. The teacher provides rich educational experiences which are often open-ended, and students investigate these with the expectation that they will construct their own meaning from the activity and integrate this with their current mathematical knowledge and understanding. Freudenthal (1991) modified this idea of students reconstructing their own knowledge and described the approach as “guided reinvention”. The reinvention principle is described by Gravemeijer (1999):

The idea is to allow learners to come to regard the knowledge they acquire as their own private knowledge, knowledge for which they themselves are responsible. On the teaching side, students should be given the opportunity to build their own mathematical knowledge store on the basis of such a learning process. (p. 158)

Cobb (2000b) describes constructivism as based on the “assumption that people actively build or construct their knowledge of the world and of each other” (p. 277). Referring to discovery learning as being based on the tenets of constructivism, Brown (1992) cautioned: “Discovery learning is often contrasted with didactic instruction … There is considerable evidence that didactic teaching leads to passive learning, by the same token unguided discovery can be dangerous too” (p. 169). Constructivism was perceived by many in the field of mathematics education as being too unstructured compared with the rigidity of didactic instruction. This shift from didactic instruction to constructivism is elucidated by Cobb (2000b), when he describes the shift from students acting as “processors of information” to students intentionally making choices in the cognitive reorganisation of their own mathematical understanding.

Constructivism is central to the approach to teaching and learning mathematics referred to as Realistic Mathematics Education (RME) which Van den Heuvel-Panhuizen and Wijers (2005) describe as a theory “… that is based on a view of mathematics as a subject, a view on how children learn mathematics and a view on how mathematics should
be taught” (p. 289). The RME approach has been implemented widely in the Netherlands.

Van den Heuvel-Panhuizen and Wijers (2005) describe its central tenets:

The present form of RME is strongly influenced by Freudenthal’s (1977) view on mathematics. He felt mathematics must be connected to reality, stay close to children’s experience and be relevant to society, in order to be of human value. Instead of seeing mathematics as ready-made knowledge to be transmitted, Freudenthal stressed the idea of mathematics as a human activity. Mathematics lessons should give students the “guided” opportunity to “re-invent” mathematics by doing it. (p. 288)

Horizontal and vertical mathematization are key aspects of RME that influence the way formal arithmetic notation is introduced and were briefly referred to in Section 2.5.2. Horizontal and vertical mathematization are described by Van den Heuvel-Panhuizen and Wijers (2005) as follows:

In horizontal mathematization, the students come up with mathematical tools that can help to organize and solve a problem set in a real-life situation. Vertical mathematization is the process of reorganization within the mathematical system itself, for instance, finding shortcuts and discovering connections between concepts and strategies and then applying these discoveries. (p. 288)

In RME, the term “realistic” is used in the sense that the context is imaginable for students.

3.1.1 The Emergent Perspective

On the continuum of educational theories, constructivism is quite remote from the theories of instructional design that were in favour in the 1960s and 1970s. The model of teacher as instructor views the teacher as the fount of all knowledge who imparts this wisdom to the students, who are then required to demonstrate their knowledge through the completion of a multitude of tasks. Using this approach to instructional design, experts began with the end point in mind and determined the important mathematical concepts for students to master. Gravemeijer (2004a) described this task analysis approach:

The performance of the expert is taken apart and laid out in small steps, and a learning hierarchy is constituted that describes what steps are prerequisite and in what order these steps should be acquired. The result is
a series of learning objectives that can make sense from the perspective of the expert, but not necessarily from the perspective of the learner. (p. 106)

Educators who acknowledge that students have a right to be engaged in their own learning embrace the need for reform in mathematics teaching. However, Gravemeijer (2004a) acknowledged that “the central problem in reform in mathematics teaching is the well-known tension between the openness towards the students’ own constructions and the obligation to work towards certain endpoints” (p. 106).

The following summarises, in rudimentary terms, a chronology of approaches to mathematics teaching over the last 100 years or so. As one of the earliest influential researchers, John Dewey is credited with the initiatory stance of regarding human inquiry as an active, rather than a passive, practice. This era of research in foundational arithmetic knowledge included an emphasis on subitising and counting as important early number concepts (Sarama & Clements, 2009a). Significant in the next phase were Piaget’s theory of number learning and Vygotsky’s theory of instruction.

According to Vygotskyian theory, development follows learning, that is, the experiences that children participate in, enable them to develop. This contrasts with the view that the learner has to reach a developmental level before learning can occur. Vygotsky also stressed the contribution of social factors to intellectual development (Ginsberg et al., 1998, p. 409). Askew (2013) outlines Vygotsky’s claim that learning precedes development in terms of the concept of addition:

… children do not develop into a “stage” whereby understanding the nature of addition becomes possible, it is working “as though” they understand addition that allows the development to occur. This is the essence of working in the Zone of Proximal Development – the adult and child working together to create a (metaphorical) space where more can be achieved jointly than by the child alone. (p. 7)

The central tenets of Piaget’s theory of learning are that knowledge is actively constructed by the child, and children progress through a series of developmental stages as
they mature. This led to the widely accepted belief that “children could not reason logically about quantities until the early elementary school years” (Sarama & Clements, 2009a, p. 31). However, the notion that children reason according to their developmental stage has since been undermined (Goswami & Bryant, 2010). It is now believed that children are able to reason; but the range of things they are able to reason about is affected by their lack of experience, rather than by their developmental stage (Askew, 2013).

In the “modern period” of research on mathematical thinking dating from the 1960s (Ginsberg et al., 1998), Gelman and Gallistel’s (1978) theories challenged those of Piaget’s, and research investigated the cognitive capabilities of students in their early years. History shows that “psychologists’ way of looking at the world influences how educators conceptualize the teaching and learning of mathematics” (Ginsberg et al., 1998, p. 410), and teacher-directed instructional design theories were prevalent throughout most of the twentieth century.

In the 1980s, the dominant views were very different and were based on constructivism with students being supported by teachers in their discovery of mathematical concepts and understandings. Currently, in reform mathematics education, the emergent perspective (Cobb, 1995) is distinguished from both strong psychological and strong social versions of constructivism. From the emergent perspective, neither individuals nor the classroom setting are of primary importance. Rather, they must be considered to be in a relationship and each should be viewed from the perspective of the other. The emergent perspective has as a basic assumption “that neither individual students’ activities nor classroom mathematical practices can be accounted for adequately except in relation to the other” (Cobb, 2000b, p. 310).
3.2 The Teaching Experiment Methodology

The reform mathematics education movement began in the late 1970s. Around the same time, the teaching experiment as a research methodology was first used in the United States (Steffe & Thompson, 2000). One of the first experiments was a yearlong sequence of teaching episodes (Steffe, 1983) conducted with first-grade students. The knowledge gained from these experiments underlies the basic tenets of the current teaching experiment methodology. Prior to this, educational experiments focused on data gathered at the beginning and conclusion of the experiment, with no real concern for the teaching and learning that occurred in between. Students were not active participants in the experiment. Rather, the experiment was performed on them. There was no conceptual analysis (Von Glasersfeld, 1995) of the cognitive reorganisations made by the students during the experimental process, and no attempt was made to scrutinise students’ behaviours, actions, inscriptions or interactions with other members of the classroom community.

As researchers continue to use the methodology of the teaching experiment as a means of studying the mathematical development of students in the context of a classroom, the methodology will continue to evolve. In line with its exploratory nature, a teaching experiment has as its basic goal the construction of models of students’ mathematics and an understanding of the way students assimilate new learning into their current cognitive schemes. Steffe and Thompson (2000) distinguish between students’ mathematics and the mathematics of students: “We use the phrase ‘students’ mathematics’ to refer to whatever might constitute students’ mathematical realities; we use the phrase ‘mathematics of students’ to refer to our interpretations of students’ mathematics” (p. 268). It is important to acknowledge that, as researchers, we can only observe changes in students’ interactions, behaviours, dialogues and any recordings they may make. Cognitive reorganisations cannot be seen, but observing behaviours allows inferences to be made. Steffe and Thompson
(2000) refer to this as being similar to the inner workings of a watch. We can see the hands move and the time change, but we can only make presumptions about what is happening inside the watch. It is neither possible nor appropriate to say what a student “knows”. Rather, we are only able to develop a model that makes sense from the teacher/researcher perspective at the time of the observation (Steffe & Thompson, 2000).

3.2.1 Assessment Interviews

Researchers use interviews to document the mathematical knowledge of students prior to a teaching experiment. These are referred to as clinical interviews (Brown, 1992; Steffe & Thompson, 2000), one-to-one, task-based assessment interviews (Clarke, 2001), one-to-one, task-based, video recorded assessment interviews (Wright, Martland, & Stafford, 2006), or individual interviews (Lamberg & Middleton, 2009). This initial collection of data on individual students via interview provides researchers with a snapshot of the current knowledge of the students. The teaching experiment allows the researcher to understand what progress students have made over an extended period of time. The use of clinical interviews in this way is not new in mathematics education research or practice. Individual assessment interviews form an integral part of the Mathematics Recovery Program (Wright, Martland, & Stafford, 2006) where they are used to develop an initial teaching sequence. Wright, Martland, Stafford, and Stanger (2006) described the purpose of the assessment interview as being to “provide rich and precise detail about the particular strategies currently used by the child and their current knowledge” (p. 28). This rich and precise detail is used to inform the initial teaching sequence, similar to the way in which an anticipatory thought experiment (Cobb, Stephan, McClain, & Gravemeijer, 2001; Gravemeijer, 2004a) is used during a teaching experiment to develop conjectures about an appropriate instructional path.
When beginning with an anticipatory thought experiment of the learning sequence for a teaching experiment, the teacher/researcher must be careful not to insist that students learn what the teacher/researcher knows; nor should the teacher/researcher resort to planning from an instructional design approach. The researcher’s own mathematical concepts can be orienting (Steffe & Thompson, 2000) in terms of developing conjectures, but should not form a checklist for what students need to know. The use of initial interviews to determine students’ prior knowledge can prove insightful, as the more comprehensively researchers and teachers can understand the ways and means by which students express and develop their mathematical understandings, the better they can support them in this development.

In order to create a model of the mathematics of students (Steffe & Thompson, 2000), it is useful to follow the development of other students in the classroom community. If a student is demonstrating incorrect understandings, rather than discounting their ideas as wrong, the researcher should place the knowledge the student does have in a frame of reference so as to be able to develop the student’s understandings further. Steffe and Thompson (2000) explained that:

> We understand better what students can do if we understand what they cannot do. We understand what students can understand better if we can understand what they cannot understand. It also helps to understand what a child can do if we understand what other students, whose knowledge is judged to be at a higher or lower level, can do. (p. 278)

### 3.3 Design Research

Conducting teaching experiments (Steffe & Thompson, 2000) falls under the umbrella of what is known as design research (Bannan-Ritland, 2003; Collins, Joseph, & Bielczyze, 2004; Gravemeijer et al., 2000), design-based research, developmental research (Cobb et al., 1997; Gravemeijer, 1994; Cobb, Wood, & Yackel, 1991), design experiment (Collins et al., 2004), design experimentation (Brown, 1992) or design study (Shavelson,
Phillips, Towne, & Feuer, 2003). Design research is essentially the integration of curriculum design and research. It focuses on the relationship between theory and practice where neither is primary in the relationship. There is a reflexivity between the two (Cobb et al., 2001).

Design research is an evolving practice, initially developed by Ann Brown in 1992. Kelly and Lesh (2002) described her formulation of design experimentation as “a hybrid cycle of prototyping, classroom field-testing, and laboratory study” (p. 1). Shavelson et al. (2003) acknowledged that, despite the fact design researchers are part of a small community (and they are attempting to develop excellent, defensible practices in studying students’ development through education research), they have “by no means reached consensus on terminology or the warrants for their work” (p. 25). This is evidenced in the many terms used to describe the closely interrelated approaches listed above. The model of design research acknowledges the importance of the “messiness” and the complexity of the classroom environment (Cobb, 2000a; Collins et al., 2004) in designing instructional approaches to the teaching of mathematics. This model of design research attempts to acknowledge real teachers teaching in real classrooms with real students and real constraints, and to work within these limitations rather than against them in order to test conjectures and, in the final phase, reconstruct an optimal instructional sequence (Gravemeijer, 1999).

Cobb, Confrey, diSessa, Lehrer, and Schauble (2003) used the term “ecology” to describe the multifaceted nature of the interactions between teachers and students in the classroom setting that is being studied: “Design experiments ideally result in greater understanding of a learning ecology – a complex, interacting system involving multiple elements of different types and levels – by designing its elements and by anticipating how these elements function together to support learning [emphasis in original]” (p. 9).
The instructional goal of design research is not to teach a set of strategies “but for students to develop a framework of number relations that offers building blocks for flexible mental computation” (Gravemeijer, 2004a, p. 114). While students are developing this framework of number relations, researchers are developing a model of the students’ learning. The whole process is a constant cycle of testing and revising conjectures, and can be contrasted with a traditional, linear, laboratory-type experiment. Design research keeps teachers’ contexts and meanings at the forefront at all times. Shavelson et al. (2003) suggested the strengths of design studies

- lie in testing theories in the crucible of practice; in working collegially with practitioners, co-constructing knowledge; in confronting everyday classroom, school and community problems that influence teaching and learning and adapting instruction to these conditions; in recognizing the limits of theory; and in capturing the specifics of practice and the potential advantages from iteratively adapting and sharpening theory in its context. (p. 25)

Barab and Squire (2004) referred to design-based research not as an approach but a series of approaches “with the intent of producing new theories, artifacts, and practices that account for and potentially impact learning and teaching in naturalistic settings” (p. 2). As design research is conducted in a naturalistic setting as opposed to a laboratory setting, its focus is cognition in context and it has the greatest potential for generating experience-based claims which endeavour to further knowledge in the field.

Middleton, Gorard, Taylor, and Bannan-Ritland (2008) developed The ‘Compleat’ Cycle of Design Research which is a cycle of inquiry that moves through seven phases from conception of the problem to dissemination to the wider research community. Middleton et al. propose the need to follow an explicit chain of reasoning from beginning to end, from developing conjectures, operationalising them and then further developing these into curricular offerings. The seven phases of The ‘Compleat’ Cycle of Design Research are: 1) Grounded Models; 2) Development of Artifact; 3) Feasibility Study; 4) Prototyping
These seven phases are described as a “cycle of inquiry practices involved in conceptualizing and conducting design research from inception of an idea, to creation of products and artifacts, to testing and upscaling innovations for a broader audience” (Lamberg & Middleton, 2009, p. 233).

Each of the three phases used in this study, that is, Development of a Preliminary Design, Implementation of the Teaching Experiment and Retrospective Analysis, corresponds to one or more of the phases in The ‘Compleat’ Cycle of Design Research. Phase 1 in this study corresponds to Phases 1–3 of The ‘Compleat’ Cycle of Design Research; Phase 2 corresponds to Phase 5 of The ‘Compleat’ Cycle of Design Research; and Phase 3 corresponds to the Impact component of Phase 7 of The ‘Compleat’ Cycle of Design Research. Phases 4 and 6 of The ‘Compleat’ Cycle of Design Research are not relevant to this study. Each of the three phases used in this study is described below.

### 3.3.1 First Phase – Development of a Preliminary Design

The design research approach begins with what is referred to as an anticipatory thought experiment (Gravemeijer, 2004a). This anticipatory thought experiment has similarities to the Teaching and Learning Cycle described by Wright, Martland, Stafford, and Stanger (2006). This cycle has four elements:

1. Where are the students now?
2. Where do I want them to be?
3. How will they get there?
4. How will I know when they get there? (p. 52)

Both an anticipatory thought experiment and this Teaching and Learning Cycle require the designer to begin by considering the current state of the students’ knowledge, to have in mind a goal (albeit a very fluid goal) for the conclusion of the teaching sequence,
and to have a proposed bank of educational experiences that the teacher will use to support the students’ learning along the way.

Simon (1995) used the term learning trajectories to describe an envisioned learning route as part of a “mathematics teaching cycle”. Simon’s definition of a hypothetical learning trajectory described a proposed learning process, and a prediction of how students’ thinking and understanding will evolve. Sarama and Clements (2009a) described learning trajectories as comprising three parts: “a goal (that is, an aspect of a mathematical domain children should learn), a developmental progression, or learning path through which children move through levels of thinking, and instruction that helps them move along that path” (p. 17).

Gravemeijer (2004a) has refined Simon’s definition of a learning trajectory to describe an overarching local instruction theory, which encompasses a hypothetical learning trajectory.

I reserve the term hypothetical learning trajectories for the planning of instructional activities in a given classroom on a day-to-day basis, and I use the term local instruction theories to refer to the description of, and rationale for, the envisioned learning route as it relates to a set of instructional activities for a specific topic (e.g., addition and subtraction up to 20) [emphasis in original]. (p. 107)

Gravemeijer (2004a) likened the local instruction theory to a travel journey from one place to another. Although the departure point and the destination are known, the exact route is not finalised. Rather, it is regularly revised dependent upon how long will be spent in each place along the way and exactly which roads will be travelled. The hypothetical learning trajectory (Cobb et al., 2001; Gravemeijer, 1999, 2004a; Lamberg & Middleton, 2009; Simon, 1995; Fosnot & Dolk, 2001) is the hypothesised detail of the route, the roads to be travelled and the towns to be stayed in. In design research, the researcher is firm on the direction in which they want to take the students, but is not locked into any particular
route. Gravemeijer (2004a) described the envisioned learning route of the local instruction theory as having three components:

1. Learning goals for students
2. Planned instructional activities and tools to be used
3. A conjectured learning process which anticipates students’ thinking and understanding (Gravemeijer, 2004a).

Underpinning all of these is a rationale for why the research is being conducted. This development of a preliminary design or local instruction theory is the first phase of design research.

When considering the relationship between a local instruction theory and the hypothetical learning trajectory, “the connection between the two is that the local instruction theory entails a framework against which teachers can construe hypothetical learning trajectories that fit the actual situation in their classrooms” (Gravemeijer, 1999, p. 157).

Earlier, the emergent perspective was described as a modified constructivist view of mathematics education: “This framework coordinates a social perspective on communal activities with a psychological perspective on the reasoning of the participating students” (Cobb et al., 2001, p. 114). An important element of the emergent perspective not yet discussed is the consideration of any design research or teaching experiment through a sociocultural lens. A learner can only be considered as one member of a learning community comprising all stakeholders – adults and students alike. These community members and their interactions all serve to shape the cognitive reorganisations of the learner. This sociocultural lens is of critical importance in design research. Cobb and Whitenack (1996) in discussing their research, stated: “The viewpoint taken when conducting the case studies was that students’ mathematical constructions have an
intrinsically social aspect in that they are both constrained by the group’s taken-as-shared basis for communication and contribute to its further development” (p. 215). The terms, “taken-as-shared” (Cobb & Whitenack, 1996; McClain, 2002) and “agreement to agree” (Confrey, 1994) are used to encapsulate sociomathematical norms (Gravemeijer, 2004a; McClain & Cobb, 2001) that are established as part of the classroom community. “Examples of such sociomathematical norms include what counts as a different mathematical solution, a sophisticated mathematical solution, an efficient mathematical solution, and an acceptable, mathematical explanation and justification” (Cobb, 2000a, p. 323). Thus, proponents of the emergent perspective on the teaching and learning of mathematics advocate that, rather than simply inducting a community into an already established way of reasoning and communicating, it is imperative to allocate time for the establishment of these norms at the beginning of any teaching experiment, and to also ensure that they are revisited throughout the duration of the teaching experiment.

From a sociocultural perspective, as members of a community of mathematicians, all participants, including students and teachers, have rights and responsibilities. Fosnot and Dolk (2001) referred to this community as a “community of discourse” or a “math congress”. All members determine what is the essence of a well-structured argument that will be willingly accepted by the other members of this learning community. However, the teacher must be careful to “walk the edge” (Fosnot, 1989) of the discussion and be mindful of not falling into the trap of steering discussion too much in one direction or another.

The use of individual interviews as an instrument to gather data about the current knowledge and strategies used by students was referred to earlier. Whilst important in the context of design research, this information gathered about individuals needs to be synthesised in order to develop an overarching picture of the needs of the classroom
community as a whole. This overarching picture, combined with the consideration of the sociocultural nature of the classroom community, informs the development of the local instruction theory and the hypothetical learning trajectory which form the first phase of conducting a design research teaching experiment. Cobb et al. (2001) proposed the learning trajectory should be about the “collective mathematical development of the classroom community” (p. 117).

3.3.2 Second Phase – Implementation of the Teaching Experiment

The second phase of design research is the implementation of the teaching experiment and the progression from a hypothetical learning trajectory to an enacted teaching sequence. Crucial to the success of the design research approach to a teaching experiment is the “cyclic process of thought experiments and instruction” (Gravemeijer, 2004a, p. 112) This cycle is described as the backbone of the design research method and is summarised in Figure 3.1.

![Figure 3.1 Reflexive Relationship Between Theory and Experiments (Gravemeijer, 2004a, p. 112)](image)

At the conclusion of each teaching session, the researcher/teacher evaluates the effectiveness of the instructional activities. Did the students make the anticipated connections? What evidence was there of this cognitive reorganisation? What implications does this have for tomorrow’s session? The researcher/teacher must then analyse this
information in light of student–teacher and student–student interactions and use this information in planning for the next day’s session. As Cobb (2000a) observed: “What is required is a systematic analytical approach in which provisional claims and conjectures are open to continual refutation” (p. 328). The Design-Based Research Collective (2003) described the dichotomy of this approach as follows: “By trying to promote objectivity while attempting to facilitate the intervention, design-based researchers regularly find themselves in the dual intellectual roles of advocate and critic” (p. 7).

In order to maintain the integrity of the design research process, the researcher must be prepared to accept that their hypothetical learning trajectory is not being realised, put this aside and conduct an objective analysis of the day’s proceedings. Acknowledgement of the limitations of one’s own conjecture can be challenging, but Cobb et al. (2003) put this in perspective:

If this conjecture is refuted, alternative conjectures can be generated and tested … As conjectures are generated and perhaps refuted, new conjectures are developed and subjected to test. The result is an iterative design process featuring cycles of invention and revision. Of course, to design iteratively demands systematic attention to evidence about learning … The intended outcome is an explanatory framework that specifies expectations that become the focus of investigation during the next cycle of inquiry. (p. 10)

The iterative nature of this type of research is seen as one of its key strengths (Barab & Squire, 2004; Cobb, 2000a; The Design-Based Research Collective, 2003; Gravemeijer, 2004a; Steffe & Thompson, 2000). Shavelson et al. (2003) referred to this as a “design-analysis-redesign cycle”. This redesign may include adjustments to the language modelled for the class, the inscriptions modelled by the teacher/researcher, the settings used and sociocultural factors such as the way students are grouped to work on a task. This constant revision must always have “the goal of promoting the greatest progress possible in all participating students” (Steffe & Thompson, 2000, p. 277). This ongoing tinkering with the hypothetical learning trajectory was likened to bricolage by Gravemeijer (2004a): “The
manner in which a conjectured local instruction theory is construed can be described as ‘theory guided bricolage’ (Gravemeijer, 1994) because it resembles the manner of working of an experienced ‘tinker’ or ‘bricoleur’” (p. 110).

Design research is developed to test a hypothetical model in a real situation. The reality of a teaching experiment in a classroom is that not all students progress at the same rate. As the hypothetical learning trajectory is developed with the classroom community in mind, part of this cyclic, ongoing evaluation in the second phase is the need to consider the learning needs of individual members of the classroom as well as supporting the learning of the class group as a whole. Lamberg and Middleton (2009) conceded the need for ongoing review during the implementation of the teaching sequence:

Thus if students need help, or if the model does not predict particular behaviours, in Phase 5 of the Compleat model of design research it is imperative to make the learning happen and use the record of that process as the basis for modifying the original theory. (p. 241)

Cobb et al. (2001) also acknowledged the complexities of the classroom when they argue that the teaching experiment cannot possibly be about the learning trajectory of each and every student as there “are significant qualitative differences in their mathematical thinking at any point in time” (p. 117). Their proposed resolution is to develop and revise conjectures about the collective mathematical development of the classroom community as a whole and use these to inform the teaching experiment.

Lamberg and Middleton (2009) concurred with this approach of attempting to meet the needs of the individual by catering to the needs of the group:

The bottom line is that we can create effective instructional tools and theories simultaneously, and although we may not be able to predict precisely what an individual will do in the course of working with others in a classroom context, we can use knowledge of individual development at least to hedge our bet and make it more probable that any given individual in the class will have the opportunity to learn and grow in a productive manner [emphasis in original]. (p. 243)
The role of the researcher/teacher in the implementation of the teaching experiment is a complex one. Barab and Squire (2004) stated that “researchers are not simply observing interactions but are actually ‘causing’ the very same interactions they are making claims about” (p. 9). According to Steffe and Thompson (2000):

The teacher-researcher formulates an image of the student’s mental operations and an itinerary of what they might learn and how they might learn it. Learning may be provoked by instruction, providing an impetus for students’ attention and reflection, but it is not caused by instruction [emphasis in original]. (p. 283)

These two perspectives serve to highlight the crucial roles of researchers and teachers, and the need for clarity in these roles. Both need to be cognisant of their ability to manipulate situations to stimulate interactions and, when objectively reflecting on these, remain mindful of the influence of the parameters they have established. Similarly, from a constructivist point of view, any cognitive reorganisation that has taken place, has not been caused by the teacher. Rather, the student has increased the depth of their knowledge in direct response to the learning experience provided by the teacher.

Prior to the commencement of the teaching cycle, the researcher has to decide whether to act as a researcher only or as a researcher who also acts as a teacher. Cobb (2000a) suggested:

The decision of whether or not to act as the teacher involves a trade off … researchers who collaborate with teachers probably have less flexibility in pursuing particular visions of reform than do researchers who act as teachers. In the latter case, the researcher can follow up immediately on his or her own conjectures and intuitions in the classroom. (p. 330)

In the literature review, the importance of guided reinvention (Freudenthal, 1991) was described. Guided reinvention involves supporting students to construct concepts via participation in carefully constructed learning experiences. Of critical importance in classrooms of this kind, is the teacher’s use of judicious questioning and mediation to
support learning. Mediation refers to behaviour on the part of a more experienced other to encourage development and cognitive reorganisation as a result of participation in an activity, rather than internalisation of some external event or technique (Askew, 2013; Rogoff, 1995). From this perspective, the role of the teacher is such that not only should they support the learner to achieve success with the task by modifying the setting in which the learning experience takes place and their questioning. As well, they need to do so in a way that positively impacts on the development of the learner. Cobb et al. (1995) provided an example:

The manner in which Ms Smith communicated to her students what she valued mathematically clarifies one way in which she coped with a tension endemic to teaching. She respected their thinking as she proactively supported their development of important mathematical ideas. In doing so, she avoided the twin dangers of anything-goes romanticism, on the one hand, and of attempting to impose adult concepts on the other. (p. 7)

A key feature of a classroom of this type are the opportunities for discussion about solution strategies and their efficiency. Van den Heuvel-Panhuizen (2001) suggested that “the advantage for students is that sharing and discussing their strategies with each other can function as a lever to raise their understanding. The advantage for teachers is that such problems can provide them with a cross-section of their class’s understanding at any particular moment” (p. 23). Throughout this study, students were encouraged to explain and justify strategies to solve tasks in context and to reflect on these actions in order to foster cognitive reorganisation.

Steffe and Thompson (2000) distinguished between learning and development. Learning is described as occurring in particular interactions in which students modify their current understandings. Development, on the other hand, occurs over a longer period of time and refers to the student’s readjustment of their understanding in their overall cognitive scheme. Steffe and Thompson (2000) believed:
It is important to emphasize that no single observation can be taken as an indication of learning or development. Change is the transition from one point to another and therefore requires at least two observations made at different times. We do not specify any minimal duration between the two times. (p. 299)

As in the first phase of design research, teaching experiments also need to be considered from a sociocultural perspective. The class in question is the point of reference and the particular situation is an emergent phenomenon. Teachers/researchers and students are all learning as the teaching sequence unfolds, and it is important to acknowledge that each classroom community has different dynamics. As previously described in Section 3.4, the interacting systems at play in a classroom community are referred to as a “learning ecology”:

Elements of a learning ecology typically include the tasks or problems that students are asked to solve, the kinds of discourse that are encouraged, the norms of participation that are established, the tools and related material means provided, and the practical means by which classroom teachers can orchestrate relations among these elements. (Cobb et al., 2003, p. 9)

In a classroom community, as in any ecology, the key elements are present, but the exact interplay between these elements will be peculiar to each individual environment and must be accounted for during the revision process.

A teaching experiment is the second phase of design research and is used to trial the initial hypothetical learning trajectory. During the course of the teaching experiment, however, the researcher must endeavour to “forget” this anticipatory thought experiment and instead respond to the actions and dialogue of the students (Steffe & Thompson, 2000). The main goal during teaching should be to promote the greatest development possible in the understanding of all students. Once the teaching experiment is completed, the initial hypotheses can be analysed retrospectively.
3.3.3 Third Phase – Retrospective Analysis

The third and final phase of design research is the retrospective analysis of the translation from the anticipatory thought experiment, to the local instruction theory, to the hypothetical learning trajectory and finally to the realised instructional sequence. The retrospective analysis may well result in refining the local instruction theory and thereby the creation of the need for another teaching experiment, and so the cycle continues.

The purpose of a retrospective analysis is to generate evidence-based claims, to contribute to the theoretical knowledge in the field about the results and to provide direction for the next local instruction theory of a design research based teaching experiment. Barab and Squire (2004) suggested the analysis should “generate evidence-based claims about learning that address contemporary theoretical issues and further the theoretical knowledge of the field” (p. 6). This contribution to the theory of a particular field of knowledge is seen as important by many researchers (Barab & Squire, 2004; Cobb, 2000a; Cobb et al., 2003; Gravemeijer, 1999; Lamberg & Middleton, 2009; Shavelson et al., 2003). The Design-Based Research Collective (2003) proposed that good design-based research exhibits five characteristics, one of which they list as “research on designs must lead to sharable theories that help communicate relevant implications to practitioners and other educational designers” (p. 5). According to Gravemeijer (1999), the aim of the final phase of design research is the reconstruction of an optimal instruction sequence to share with others. However, determining exactly which features of the design experiment were most effective and therefore identifying which should support the development of a “pedagogical domain theory” can be a challenging process (Rittle-Johnson & Koedinger, 2005).

Important in the development of any theory or model as part of a retrospective analysis is the need to keep in mind the perspective from which it was viewed. Steffe and Thompson (2000) cautioned that “the model is viable as long as it remains adequate to
explain students’ independent contributions. But no amount of fit can turn a model into a description of what may be going on” (p. 302).

A positive by-product of participation in the teaching experiment and involvement in the subsequent retrospective analysis is the increase in the teacher’s knowledge and understanding of the mathematics of their students. Simon (1995) described this: “The teacher’s knowledge evolves simultaneously with the growth in the students’ knowledge. As the students are learning mathematics, the teacher is learning about mathematics, learning, teaching, and about the mathematical thinking of his students” (p. 141). This can only be advantageous for any future students of this teacher.

Barab and Squire (2004) referred to one of the benefits of this kind of research as having both “experience-near” and “experience-distant” relevance. Experience-near refers to the specific knowledge gained that is relevant to this classroom community, whilst experience-distant implies that what has been learnt from this particular experiment can be generalised to other classrooms. In attempting to transfer learning from one situation to another, Brown (1992) suggested that

this is intervention research designed to inform practice. For this to be true, we must operate always under the constraint that an effective intervention should be able to migrate from our experimental classroom to average classrooms operated by and for average students and teachers, supported by realistic technological and personal support. (p. 143)

However, whilst transferring learning from one situation to another is important, complete replicability is neither “desirable nor possible” (Cobb, 2000a).

The methodology of design study has come under criticism from proponents of more scientific research. It has been censured for its use of narrative in the retrospective analysis phase, the generalisability of claims made and general unscientific research approaches (The Design-Based Research Collective, 2003). However, three recommendations are made to ensure the scientific rigour of the study and reduce the
likelihood of such criticism: 1) preclude competing conjectures; 2) consider claims with scepticism; and 3) encourage the exploration of rival hypotheses (Shavelson et al., 2003). Similarly, Cobb et al. (2003) recommended that, in order to maintain the rigour of analysis of design research, one should: 1) work systematically through the data in order to validate the claims; and 2) be explicit about the criteria and evidence used to make such claims in order to promote transparency and replicability where appropriate.

The design of a teaching experiment may involve the possibility of micro-analysing thousands of decisions made by the teacher/researcher throughout the teaching experiment. It is therefore important to triangulate data. Triangulation of data is defined as “the idea that looking at something from multiple points of view improves accuracy” (Neuman, 2006, p. 149). The triangulation of data in this study was achieved by considering artifacts (student workbooks), transcripts, field notes and video footage.

Gravemeijer (2004a) summed up the retrospective phase of design research in the following way: “Because of the cumulative interaction between the design of the instructional activities and the assembled empirical data, the intertwining between the two has to be unraveled to pull out the optimal instructional sequence in the end” (p. 112). Cobb et al. (2003) described design research as having two faces, prospective and reflective. The prospective face allows theories to be trialled in the “messiness” of a classroom and the reflexive face allows for these theories to be adapted and amended for use in other classrooms. Of primary importance to practitioners in the classroom, design research offers teachers a pedagogical framework that serves to bridge the gap between researchers and teachers. Supplemented by a collection of targetted instructional activities, it enables teachers to adjust pedagogical approaches and lesson content to the specific needs of their students and their classrooms.
In summary, this chapter provided an overview of the theoretical framework of design research and teaching experiment methodology used in this study, and described its application in the teaching and learning of mathematics. Chapter Four details the research procedure used in conducting this study.
Chapter Four: Research Design

In this chapter, the research procedure used in conducting this study is detailed. The plan of study, the assessments, the class group, the structure of the teaching sequence and each lesson, the role of video recording and the enactment of the teaching cycle are outlined. Finally, the domain of learning with regard to the selection of concrete materials and the use of formal arithmetic notation, is related to the literature review in Chapter Two.

This study was conducted as a teaching experiment following the design research methodology. As described in Chapter Three, a hypothetical learning trajectory (Simon, 1995) was developed, and from this a conjectured local instruction theory (Gravemeijer, 2004a) evolved. This provided the framework for the teaching sequence and the role of researcher and teacher required constant analysis of students’ strategies and behaviours in response to the learning activities. In the role of researcher, the implications of students’ responses in terms of the hypothetical learning trajectory (Simon, 1995) and the importance of these in the development of the environment of the “learning ecology” (Cobb et al., 2003) were always an important consideration. The teaching approach taken in this study aimed at fostering meaningful learning by encouraging students to make sense of new information presented, to organise it into a coherent structure and to integrate it with their current knowledge (Mayer, 2004).

In this teaching experiment, a choice had to be made between acting as researcher only, or teacher and researcher. For the reasons described in Section 3.4.2, acting as both teacher and researcher was more desirable because it allowed a greater level of control over the content and direction of whole class discussions, as well as the flexibility
of being able to seize teachable moments and micro-adjust teaching episodes in order to maximise learning opportunities (Wright, Martland, Stafford, & Stanger, 2006).

4.1 Plan of Study

The participants in this study were a class of students from a Preparatory grade (first year of school) in a Catholic school in a suburb of an Australian city. The study was conducted during the third of four terms in the students’ first year of formal schooling. At the commencement of the study, the students had been attending school for approximately 20 weeks. Twenty-four teaching sessions, each of approximately one hour’s duration, were conducted with the class. The teaching sessions took place over a period of seven weeks. All students were assessed prior to the commencement of the teaching sequence (pre-assessment) and at the conclusion of the teaching sequence (post-assessment). These assessments are described in Section 4.1.1. Throughout the study, the researcher also acted as the teacher. She was employed as the Numeracy Leader of the school and had worked at the school for a period of two years, and, as such, the students and their parents were familiar with her and her work.

4.1.1 Assessments

All assessments were conducted in the same room and each student was individually assessed by the researcher. The same tasks were used in both the pre- and post-assessments. All tasks required oral responses from the student or required the student to manipulate materials to demonstrate knowledge. For example, students were asked to move seven beads on an arithmetic rack, or arrange digit cards to make an equation at the level of formal arithmetic to match the number of red dots and blue dots on a ten-frame. The assessment did not involve the students doing any writing. The first few assessment items were easily mastered by most students; however, this was still considered to be important information as, in conjunction with assessing students’ current knowledge, the assessment
interviews served to inform the starting point for instruction (McClain, 2005) and ensured students had sufficient prior knowledge in order to successfully engage with the planned learning sequence. All assessment and teaching sessions were videotaped for later analysis. The rationale for the use of videotaping is described in Section 4.1.5. Copies of the assessment interview script are provided in Appendix E.

All assessments were carefully constructed so that tasks were presented in a sequence which allowed for the use of strategies at increasing levels of mathematical sophistication both within and across the tasks. For example, Task 3 featured the tasks, 4 + 2 and 9 + 4, presented in the material setting of two screened collections. 9 + 4 was considered to be a more difficult task for students to solve because, with the sum of the two addends greater than 10, this task exceeded the student’s “finger range” (Baroody et al., 2003; Wright, Martland, and Stafford, 2006). Exceeding this finger range meant that the student was unable to represent a collection on each hand, which they may have wanted to do out of convenience or habit rather than necessity.

The second collection was also carefully chosen from a minimum of two and a maximum of four counters for both tasks, in order to limit the number of counts that needed to be made by the student. Wright, Martland, Stafford, and Stanger (2006) suggested that:

> The second number is limited to this narrow range because that is the range in which it is considered useful for students to become facile at keeping track of counting. Generally, the higher the second number, the more difficult the task. (p. 109)

**4.1.2 Class Group**

There were 22 students in the class group, 11 boys and 11 girls. At the commencement of the study, when the pre-assessments were conducted, the average age of the students in the class group was 5 years and 11 months. All 22 students were included in the teaching and assessment components of this study. However, the results of two students (one boy and one girl) were excluded from the whole class group analysis. One student only
began to verbally communicate in full sentences six months after the conclusion of the teaching cycle. Another student was diagnosed with a severe language disorder eight months after the conclusion of the teaching cycle. This diagnosis qualified her to have a teacher’s aide to support her learning in the classroom. Both families opted to have their student repeat the Preparatory grade in the subsequent school year. As both of these students were incapable of being successful participants in the rigours of a mainstream mathematics classroom, their results have been excluded from the whole class group analysis. Thus, as a result of these two exclusions, the group size was 20 students.

### 4.1.3 Organisational Structure of the Teaching Sequence

Table 4.1 shows the schedule of the teaching sequence of 24 lessons, and the pre- and post-assessments across a seven-week period.

<table>
<thead>
<tr>
<th>Week</th>
<th>Monday</th>
<th>Tuesday</th>
<th>Wednesday</th>
<th>Thursday</th>
<th>Friday</th>
</tr>
</thead>
<tbody>
<tr>
<td>Week 1</td>
<td>Pre-assess</td>
<td>Pre-assess</td>
<td>Pre-assess</td>
<td></td>
<td>Pre-assess</td>
</tr>
<tr>
<td>Week 2</td>
<td>Lesson 1</td>
<td>Lesson 2</td>
<td>Lesson 3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Week 3</td>
<td>Lesson 4</td>
<td>Lesson 5</td>
<td>Lesson 6</td>
<td>Lesson 7</td>
<td></td>
</tr>
<tr>
<td>Week 4</td>
<td></td>
<td>Lesson 8</td>
<td></td>
<td>Lesson 9</td>
<td>Lesson 10</td>
</tr>
<tr>
<td>Week 5</td>
<td>Lesson 11</td>
<td>Lesson 12</td>
<td>Lesson 13</td>
<td>Lesson 14</td>
<td></td>
</tr>
<tr>
<td>Week 6</td>
<td>Lesson 15</td>
<td>Lesson 16</td>
<td>Lesson 17</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Week 7</td>
<td>Lesson 18</td>
<td></td>
<td>Lesson 19</td>
<td>Lesson 20</td>
<td></td>
</tr>
<tr>
<td>Week 8</td>
<td>Lesson 21</td>
<td>Lesson 22</td>
<td>Lesson 23</td>
<td>Lesson 24</td>
<td></td>
</tr>
<tr>
<td>Week 9</td>
<td>Post-assess</td>
<td>Post-assess</td>
<td>Post-assess</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

All pre-assessments were conducted prior to the commencement of the teaching sequence, and all post-assessments were conducted after the final lesson of the teaching sequence. Delivering the 24 lessons in an uninterrupted sequence was not possible due to class commitments to specialist lessons, excursions and whole school closure days. The researcher acted as teacher for each of the 24 lessons. The regular class teacher, referred to
as the support teacher, was also present for each of the lessons. Prior to each lesson, the researcher met with the support teacher to explain the learning intention for the session, and the nature and purpose of the learning experiences to be presented to the students. This pedagogical strategy enabled the researcher and the support teacher to circulate among the students during the student activity component of each lesson in order to “gain a sense of the diverse ways in which students [were] interpreting and solving instructional activities” (Cobb et al., 2001, p. 117), and to support their learning in appropriate and consistent ways.

4.1.4 Structure of the Lessons

Each of the 24 lessons of the teaching sequence comprised a similar structure (individual lesson plans are provided in Appendix D). All lessons were of approximately one hour’s duration and were structured using what is referred to as a “whole-small-whole” (Clarke, 2001) approach. In this approach, each lesson is divided into four main parts. These parts are described below.

4.1.4.1 Part 1: Warm up

Each lesson began with a short, fast-paced whole class session designed to engage the students in mathematical thinking. This session was considered to be an introductory warm up which involved some sort of mental mathematical activity and often revisited an activity from a previous session. Typically, this part of the lesson was of approximately 10 minutes’ duration.

4.1.4.2 Part 2: Whole class teaching

The second part of the lesson involved the presentation of a particular addition task, often in a realistic setting using the Realistic Mathematics Education (RME) (Gravemeijer et al., 2000) approach, such as passengers getting on and off a bus. The researcher used the setting, questioning and discussion to elicit from the whole class group
of students appropriate strategies to solve the given addition task. Typically, this part of the lesson was of approximately 15 minutes’ duration.

4.1.4.3 Part 3: Student activity

During the third part of the lesson, students engaged in tasks similar to those presented in the second component. Typically, during this time, students attempted to solve addition tasks in pairs or independently. One sociomathematical norm (Gravemeijer, 2004a; McClain & Cobb, 2001) of the classroom community was that, during this independent learning time, students were expected to discuss their strategies with their partner and record their results in their workbook, in a manner “so that others might understand their reasoning” (Cobb et al., 1995, p. 9). Accordingly, during this part of the lesson, students typically worked in pairs. During this independent learning time, the researcher and the support teacher circulated through the classroom, mediating (Askew, 2013) tasks in the material setting for students through the use of judicious questioning as necessary. Typically, this part of the lesson was of approximately 25 minutes’ duration.

4.1.4.4 Part 4: Plenary session

The fourth part of each lesson was the plenary session. During this time, students were invited to discuss examples of strategy use from the independent activity time, and the researcher indicated that “certain mathematical interpretations and solutions are particularly valued” while simultaneously ensuring that students “[did] not merely attempt to superficially imitate such solutions” (Cobb et al., 1995, p. 5). The discussions were directed in such a way that solutions moved towards higher levels of sophistication (McClain, 2005). “Norms for argumentation” (McClain, 2005), such as what constituted a novel solution and expectations regarding justification, were established. Frequently, during this part of the lesson terminology or expressions came to be “taken-as-shared” (Cobb & Whitenack, 1996; McClain, 2002) by the class group. This was a vital part of the lesson.
during which misconceptions were addressed and explicit connections made to prior and future learning. Typically this part of the lesson was of approximately 10 minutes’ duration.

4.1.5 Video recording of Assessment and Teaching Sessions

All pre- and post-assessments were video recorded, as was each lesson of the teaching sequence. In each lesson, a video camera was used to film Part 1: The initial whole class warm up activity; Part 2: The whole class explicit teaching component; and Part 4: The plenary session. This facilitated analysis of student–student and student–researcher interactions. During Part 3: The student activity part of the lessons, the video camera was repositioned to film a group of students solving tasks independently of the researcher. For all of the lessons, the same group of students was filmed.

Video recording was used for two different purposes to capture two different aspects of the study. These are described below.

4.1.5.1 Video recording of assessments

The pre- and post-assessments were conducted as video recorded, interview-based assessments. This approach was described by Wright (2008):

The interview is videotaped. Subsequently, the teacher reviews in detail the videotaped record of the interview, and in doing so, a detailed written record is generated via a standard pro forma corresponding to the interview schedule. The process of generating the written record allows for (a) the categorizing of responses; (b) the recording of correct and incorrect responses; and (c) the recording of observations of the child’s mathematical behavior and interpretations of that behavior. (p. 235)

In order to inform the enactment of the hypothetical learning trajectory, repeatedly reviewing the videotapes of the strategies used by students to solve addition tasks was very valuable. This enabled the researcher to look for evidence of advancement in the sophistication of strategy use.
4.1.5.2 Video recording of lessons

All lessons were video recorded. The whole class first, second and fourth parts were video recorded in such a way that all students in the class could be seen and heard if they contributed to the whole class discussion. The value of using video recording in teaching experiments was described by Stecher et al. (2006): “Classroom observations (and by extension, video records of classrooms) are the most proximal indicators of instruction, providing rich descriptions of activities that occur in the classroom” (p. 101). However, video recording all students working during Part 3 was beyond the scope of this study. Accordingly, during Part 2 of each lesson eight students were selected for video recording using purposeful sampling (Patton, 2002) and, in particular, intensity sampling. In consultation with the class teacher and through the pre-assessment interviews, these eight students were identified as appropriate samples according to the criterion of being articulate in terms of their ability to express clearly their mathematical strategies, yet also representative of a variety of levels of mathematical knowledge of the class group as a whole. This group of students were “information-rich cases that manifest[ed] the phenomenon of interest intensely” (Patton, 2002, p. 234).

The rationale behind the decision to select eight students on whom to focus was also informed by Cobb and Whitenack (1996):

The children’s small group activity was influenced by their realization that they would be expected to explain their interpretations and solutions in the subsequent whole class discussions. Conversely, their small group activity served as the basis for whole class discussions. Thus the classroom microculture both constrained and was sustained by the small group relationships the children developed. (p. 216)

After the conclusion of the teaching sequence, three students were selected as case studies from this group of eight students. Section 4.2 describes the selection of these students in more detail.
As described earlier, this study was developed through the lens of a hypothetical learning trajectory and enacted as a teaching experiment, with ongoing cyclic review and evaluation of the teaching sequence. Therefore, it was important that each video recording was watched, transcribed and analysed soon after each lesson, and used to inform planning for the next session. In this approach “daily analyses are used to guide the microlevel decisions about appropriate tasks for the following day” (McClain, 2005, p. 92).

4.1.6 Sequence of Lessons

As previously stated, the purpose of this study was to investigate the use of grouping strategies to solve addition tasks involving two addends in the range 1 to 10 by students in their first year of school. As described in Section 3.4.1, this sequence of lessons was developed from a conjectured local instruction theory (Gravemeijer, 2004a). Ellemor-Collins and Wright (2009, pp. 58–59) provided an instructional sequence for a progression of teaching 10 topics in structuring numbers in the range 1 to 20:

1. Patterning and partitioning.
2. Five-wise patterns for numbers 1 to 10.
3. Pair-wise patterns for numbers 1 to 10.
5. Adding two numbers with a sum in the range 1 to 10.
6. Doubles from 6 and 6 to 10 and 10.
7. Ten-wise patterns for numbers 1 to 20.
8. Pair-wise patterns for numbers 1 to 20.
9. Adding two numbers in the range 1 to 10.
10. Adding and subtracting in the range 1 to 20.
The teaching sequence taught in the present study was informed by the progression outlined by Ellemor-Collins and Wright (2009) above, with the exception that in this study only minimal attention was given to their topics 3 and 6 both of which involved the use of doubles. Recall that in Section 2.2.2, Groen and Parkman’s (1972) research found that doubles (e.g., $7 + 7$) or ties are easier to retrieve from memory than the other addition combinations, and all doubles are accessed at the same rate, independent of the magnitude of the addends. Therefore, for this reason, and others described later in Section 6.1, only minimal focus was placed on doubles addition combinations in this study.

For each of the 24 lessons, the learning intention, focus strategy and instructional materials as they were enacted during the teaching experiment are presented (see Appendix D). Whilst a hypothetical learning trajectory similar to that described by Ellemor-Collins and Wright (2009) above was developed in the first phase of the planning, in the manner described by McClain (2005): “the actual activity sheets and the particular number of class sessions for each activity type were not predetermined. These decisions were made on a day-to-day basis, based on ongoing analyses of the students’ current mathematical understandings” (p. 100). All teaching materials are described in detail in Appendix B. Section 4.1.2 explained the reasons for two students in the class group being excluded from the data collection of this study. However, for the 24 lessons of the teaching sequence, these students were always part of the class group in the whole class components of the lesson. As these students were unable to work at the same cognitive level as their peers during Part 3 (the student activity) of the lesson, the researcher differentiated the tasks completed by the whole class in order to cater for the needs of these two students.

Similarly, whilst this teaching sequence was designed with the whole class group in mind, there were times during the teaching sequence when at least two sub-groups of the whole class appeared to be consolidating different concepts, skills and levels of knowledge.
As well as catering to the needs of the two students described earlier, at this point in the teaching sequence, the researcher deemed it appropriate to provide two different tasks for students to complete during Part 3 (the student activity). The two activities were designed to more explicitly cater to the needs of each group of students. The lessons in which two different activities were provided during Part 3 of the lesson are indicated by the shaded area in Appendix D.

At the same time as a hypothetical learning trajectory is developed for the class as a whole, as in any classroom, individual student needs must also be considered. Cobb et al. (2001) stated that one of their criteria for an appropriate analytical approach to the design research cycle is: “It should enable us to document the developing mathematical reasoning of individual students as they participate in the practices of the classroom community” (p. 116). As argued by Cobb et al., considering a learning trajectory for every individual student is a questionable idealisation. They propose that the collective mathematical development of the class group be considered when developing conjectures, as “descriptions of planned instructional approaches written so as to imply that all students will reorganize their thinking in particular ways at particular points in an instructional sequence involve, at best, questionable idealizations” (Cobb et al., 2001, p. 117).

The diversity in student knowledge and behaviour, which became evident as the students attempted tasks independently of the support teacher and researcher, highlighted the critical importance of the role of the teacher during Part 4 of each lesson. According to Cobb et al. (2001), these parts of the lesson routinely focus on the qualitative difference in students’ reasoning in order to develop conjectures about mathematically significant issues that may, with the teacher’s proactive guidance, emerge as topics of conversation. The intent is to capitalize on the diversity in the students’ reasoning by identifying interpretations and solutions that, when compared and contrasted, may lead to substantive mathematical discussions … this diversity is, in the hands of a skillful teacher, a
primary motor of the collective mathematical learning of the classroom community. (p. 117)

Thus, Part 4 of each lesson, within the context of the hypothetical learning trajectory, often informed the focus for the following lesson.

4.1.7 Relating the Domain of Learning With the Literature

The domain of learning relevant to this study is students’ strategies to solve simple addition tasks. Chapter Two reviewed the literature in terms of the role of concrete materials, arithmetic notation and the use of grouping strategies to solve addition tasks while Sections 3.3 and 3.4 outlined the theoretical framework which informed this study. In the following sections, distinctive features of the teaching sequence enacted during this study will be linked to the relevant literature.

4.1.7.1 Establishing a classroom culture

The teaching approach adopted in this study featured a strong focus on the use of quinary-based strategies alongside the use of base-ten strategies. At all times, students were encouraged to articulate the strategies they used to solve addition tasks during whole class discussions. However, through judicious use of questioning, the teacher made students aware that grouping strategies were more highly valued and a more efficient means to solve tasks than count-by-ones strategies. A similar attitude to the sharing of strategies was described by Gravemeijer et al. (2000): “Ms Smith and the students negotiated that grouping solutions were particularly valued” (p. 254).

This judicious use of questioning was an important aspect of the teacher’s role. Finding the balance between opportunities for discovery learning and instructional teaching was important. As Hiebert et al. (1996) commented:

Clearly students can benefit from having access to relevant information; they would make very slow progress if they were asked to rediscover all of the information available to the teacher. On the other hand, too much information imposed with a heavy hand undermines students’ inquiries. Our position is that the
teacher is free, and obligated to share relevant information with students as long as it does not prevent students from problematizing the subject. (p. 16)

4.1.7.2 Material settings used in this study

Conclusions drawn from the literature review in Chapter Two are that sustained use of simple, but well-considered, materials is best (McNeil & Jarvin, 2007); however, in order to be effective, this must be supported by regular reflection and discussion about the key mathematical concepts involved. Strategic withdrawal of the materials (distancing the material setting) and the use of formal symbolic recording were also recommended to support student learning. Each of these approaches was used throughout this teaching experiment.

As described by McClain (2005): “The design research cycle involves the interplay of (1) the development of instructional materials; and (2) the ongoing research of the effectiveness of those materials in the context of the classroom” (p. 92). The materials used in this study were thoughtfully selected according to characteristics designed to engender the use of grouping strategies to solve addition tasks. Strategic use of colour occurred in the selection of materials. Counters were used in conjunction with canonical dot patterns, unifix cubes, five-frames, ten-frames, double ten-frames and an arithmetic rack (see Appendix B).

When counters were used during this study, each addend was represented by a different colour. This deliberate use of colour encouraged students to see each addend as a group, as well as seeing the individual counters. This was thought to support the use of at least a count-on-by-ones strategy in preference to a count-by-ones from one strategy. In the setting of two screened collections, the use of a different colour for each addend supported students’ use of visual imagery (Wright, Martland, Stafford, & Stanger, 2006). The use of colour was also an important aspect of the structure of unifix towers. In constructing unifix
towers, unifix cubes were grouped in one of two ways: they were arranged in towers of ten, either with each group of five cubes represented by a different colour, or with all cubes in the same colour. Unifix towers in two groups of five cubes were used to represent five-plus addition combinations, partitions of 10 and doubles. Unifix towers in one group of 10 cubes were used to represent ten-plus combinations.

The Australian curriculum document for the third year of school (seven-year-olds) dictates the standard: “They [students] perform simple addition and subtraction calculations using a range of strategies” (ACARA, 2011). In the statement which elaborates this standard, ten-frames are suggested as an appropriate material setting. This is the first reference to ten-frames in the Australian curriculum document. In this study, ten-frames were used in the first year of school as a key material setting for five-year-old students. The ten-frames were used to: (a) ascribe number to, and model, one-digit numbers; (b) support the development of knowledge of the partitions of 10; and (c) model and support the use of a build-through-ten strategy to solve one-digit addition tasks.

Section 2.6.3 described the structure of quinary- and ten-based materials as supporting the use of grouping strategies to solve addition tasks (Murata, 2004; Van den Heuvel-Panhuizen, 2008; Verschaffel et al., 2007; Wright, Martland, Stafford, & Stanger, 2006). In the early stages of this study, concrete materials of a five-frame and sets of five unifix cubes were used to scaffold the child’s learning of the partitions of five (two numbers that add together to make five) and the part-part-whole combinations of the numbers in the range 1 to 5 (e.g., partitions of five include five and zero, four and one, and three and two). However, in the assessments, the researcher made the decision not to include the partition of 0 and 5. The partitions presented to the students were 1 and 4, 2 and 3, 3 and 2, and 4 and 1. Reading, writing and counting with numerals in the range 1 to 10 is the beginning point for most formal mathematics curricula. The natural or counting
numbers 1–10 are often children’s first experience of counting, and introducing the concept of zero too early can encourage children to incorrectly include zero as a counting number. R. J. Wright (personal communication, June 27, 2011) recommends delaying the formal introduction of zero in an arithmetical sense, that is, as a number that can be added to and subtracted from the counting numbers (1, 2, 3 …), notwithstanding that students are likely to encounter zero in several forms, for example, as the number after “one” when counting down (“four, three, two, one, zero!”) and as a digit on the right in decuples (10, 20, 30 …). Thus, zero in an arithmetical sense should not be introduced until students are facile with grouping strategies to solve addition tasks.

Once these partitions of five were mastered, the focus moved to developing students’ reasoning about numbers in the range 6 to 10 as five-plus combinations. A ten-frame or two coloured sets of five unifix cubes were used to support this thinking. Once this knowledge was mastered, a ten-frame or the unifix cubes were used to support students’ reasoning about the partitions of 10. A twenty-frame or arithmetic rack was used to support students in reasoning about teen numbers as ten-plus combinations; and once the students demonstrated facile knowledge of the partitions of 10, and ten-plus combinations, these were then used to support reasoning about addition tasks involving two one-digit numbers, through the use of a build-through-ten strategy.

All of the material settings used in this study were explicitly chosen for their capacity to foster facile knowledge of key mathematical concepts. The fundamental elements of these settings were their simplicity (Clements & Sarama, 2009), the judicious use of colour and the repetitive element of five and 10 as key numbers.

4.1.7.3 The role of arithmetic notation

In this study, arithmetic notation was used as a shorthand way to communicate actions or relationships that were represented using concrete materials. The use of symbols
in this way is aligned with the approach used in RME (see Section 3.1) which Gravemeijer (1999) described as “a process of gradual growth in which formal mathematics comes to the fore as a natural extension of the student’s experiential reality” (p. 156). Whatever the strategy used to solve a task (and regardless of whether or not the strategy required manipulation of concrete materials), intuitive strategies (rather than standardised, formal procedures), combined with reflection on strategy use, were considered preferable to the rote manipulation of written symbols or concrete materials. The approach adopted was guided by the tenets of RME as described by Cobb et al. (1997):

Instructional sequences … typically involve establishing nonstandard means of symbolizing that are designed both to fit with students' informal activity and to support their transition to more sophisticated forms of mathematical activity. These means are, in effect, offered to students as resources that they might use as they solve problems and communicate their thinking. (p. 162)

An example of the progressive use of formal arithmetic notation across a series of whole class episodes during this study was performed as follows. A total of seven counters were represented on a ten-frame as five counters on the upper row and two counters on the lower row. When the upper row of a ten-frame is filled before counters are placed on the lower row, this is described as a “five-wise” ten-frame. Figure 4.1 shows the type of graphical representation that was depicted in the early stages of the teaching sequence. The researcher proposed notation of this type to the class as one possibility to represent the combination at a symbolic level. The class community of learners then determined if this accurately represented their understanding of seven as being made of five and two. Once introduced, the notation then became a reference point to be revisited as a tool to support mathematical reasoning (McClain, 2005). Facilitating discussion in this way allowed students to be active participants in the learning process, as described by Van den Heuvel-Panhuizen (2001) with respect to the RME approach: “The students, instead of being the receivers of ready-made mathematics, are considered to be active participants in
the teaching-learning process, in which they develop mathematical tools and insights” (p. 4). This notation was particularly chosen to highlight five and two as one representation of the part-part-whole combination of seven.

![Figure 4.1 Informal Recording to Symbolise Five Counters and Two Counters Making Seven Counters in All, on a Five-wise Ten-frame](image)

After determining that this means of symbolising the relationship between five counters, two counters and seven counters had come to be taken-as-shared (Cobb & Whitenack, 1996; Cobb, 2000a) by the classroom community, the researcher proposed the inclusion of the formal symbol for addition (see Figure 4.2).

![Figure 4.2 Formal Recording to Symbolise Five Counters and Two Counters Making Seven Counters in All, on a Five-wise Ten-frame](image)

Finally, this arrangement of seven counters on a five-wise ten-frame was symbolically recorded using formal arithmetic notation as $5 + 2 = 7$. The students were encouraged to focus on the connection between the setting and the arithmetic notation; that
is, on the relationship between the five counters and the symbol 5, combined with two
counters and the symbol 2, resulting in a total of seven counters, and the symbol 7.

This approach to the introduction of symbolic notation is similar to that
described by Cobb et al. (1995):

> It is important to stress that, from the students’ perspective, 
> Ms Smith appeared to introduce notation almost in passing. 
> They were not obliged to use these forms of notation, and 
> indeed, never practised how to write number sentences 
> correctly. Further, the notation Ms Smith used was not 
> limited to conventional notation but was, instead, designed to 
> fit with the students’ thinking. (p. 9)

In the manner described by Gray and Tall (1994) in Section 2.5.3, discussion
regarding the use of formal arithmetic notation in this study encouraged students to think
proceptually about $5 + 2$ in three ways: (1) as the result of two added to five; (2) as the
combination of five and two; and (3) as a total of seven. Ginsburg et al. (1998) described
the use of symbolic notation in a similar way:

> The child may realize that the numeral 2 corresponds to two cubes, 
> that the numeral 3 corresponds to three cubes, that the $+$ refers to 
> combining the cubes, and that the numeral 5 corresponds to the result 
> obtained. The child may also realize that the 2 cubes are just like 2 
> fingers, and that combining 2 and 3 cubes gives the same result as 
> combining 2 and 3 fingers. In this respect, the cubes and fingers serve 
> as a bridge between informal knowledge (counting, addition 
> concepts) and the written symbols, concepts and procedures of formal 
> mathematics (the numerals 2 and 3, the symbol $+$, the concept of 
> commutativity). The bridge allows the formal mathematics to be 
> assimilated into the informal knowledge. (p. 422)

In describing the RME approach (see Section 3.1) to the teaching and learning of
mathematics, the term “realistic” is used. The Dutch use this term to describe the “emphasis
that RME puts on offering the students problem situations which they can imagine” (Van
den Heuvel-Panhuizen, 2001, p. 3), in the sense that the context is imaginable for students.
Maclellan (2001) suggested that, as well as the context being imaginable for students “if
children are to appreciate that external symbolic representation supports thinking and
enables communication about mathematical ideas, they need to understand that the
formalisms of mathematics do indeed have real world meaning” (p. 12).

In this study, “realistic” settings such as passengers on a bus were used to support
numerical reasoning. The use of horizontal and vertical mathematisation occurred when,
within the context of passengers on a bus, addition tasks were posed and described using
increasingly formal arithmetic notation. The “model of” passengers on a bus became a
“model for” addition tasks in the manner described by Van den Heuvel-Panhuizen and
Wijers (2005):

Models serve as an important device for bridging this gap between
informal, context-related mathematics and more formal mathematics
… In order to fulfill the bridging function between the informal and
formal levels, models have to shift from a “model of” a particular
situation to a “model for” all kinds of other, but equivalent,
situations. (p. 289)

As the teaching sequence progressed, the context of passengers on a bus was
revisited as a springboard for extending students’ mathematical thinking and deepening the
complexity of their additive reasoning: “The situations that serve as starting points should
continue to function as paradigm cases that involve rich imagery and thus anchor students’
increasingly abstract mathematical activity” (McClain, 2005, p. 97).

4.1.7.4 The role of visualisation

Spatial visualisation abilities were described by Clements and Sarama (2009) as
“processes involved in generating and manipulating mental images of two- and three-
dimensional objects, including moving, matching, and combining them” (p. 110). In this
study, the approach of distancing the material setting (see Section 2.4.6) was used to
develop visualisation strategies. In this approach, a setting such as a ten-frame or the
arithmetic rack was briefly shown to students and then screened. Students’ were then
encouraged to reason additively about the images they saw. Similarly, visualisation
strategies were used to create mental images of canonical dot patterns to develop students’
subitising ability. In this study, the approach taken was that the combination of carefully chosen material settings and visualisation strategies, supported by the sagacious use of questioning and reflection, served to develop robust knowledge of the part-part-whole structure of numbers. The mental strategy of using part-part-whole knowledge of numbers to support addition strategies is referred to as reasoning additively. Yackel and Wheatley (1990) described student responses when a strong emphasis was placed on visualisation strategies in the classroom: “We observed that, as the pupils listened to each other’s explanations during the discussions, they began to think about the visually presented images in more than one way and to elaborate on and extend their own ideas” (p. 54).

The structure of each material setting, the use of visualisation strategies, the incremental mastery of knowledge, the use of symbolic recording to describe student reasoning and the language used by the teacher were all specifically developed to engender the use of grouping strategies in preference to count-by-ones strategies to solve addition tasks.

4.2 Case Study Design

In Chapter Seven, case studies are presented to show the advancement of three students in the use of grouping strategies during the teaching sequence. Intensity sampling was used to determine these three case studies, each of which exhibited “sufficient intensity to illuminate the nature of success or failure, but not at the extreme” (Patton, 2002, p. 234). The case study research design adopted was based on a collective approach, “to understand a theory or problem by combining information from smaller cases” (Hancock & Algozzine, 2006, p. 33). The three students, Jack, Bridget and Tracey (pseudonyms), chosen for the case studies were representative of differing levels of mathematical knowledge (McClain, 2005) and the range of growth in mathematical knowledge from the pre- to the post-assessments. These students were selected in order to obtain a broad picture of students’ learning progressions throughout the teaching sequence. Jack was selected as a
representative of students from the low range, Bridget from the mid-range and Tracey from the high range. The researcher conducted an analysis of the students’ use of strategies to solve addition tasks in response to the teaching activities presented. Video footage of the participation of the three students in the teaching sequence as well as anecdotal notes and artefacts (such as the students’ workbooks) were triangulated for evidence of learning progressions and cognitive reorganisation (Steffe & Cobb, 1988).

In summary, Chapter Four detailed the research procedure used in conducting this study. The specific details of the plan of study, the assessments, the class group, and the structure of the teaching sequence were outlined, and the domain of learning was related to the literature review in Chapter Two, with regard to the selection of concrete materials and the use of formal arithmetic notation throughout this study. Chapter Five documents students’ results from the 12 task groups in the pre- and post-assessments and compares results from key tasks within and across task groups.
Chapter Five: Results – Students’ Progression on the Twelve Assessment Task Groups

This chapter describes the students’ results in the pre- and post-assessments conducted as part of this study. The primary aim of each assessment task group was to ascertain the students’ knowledge through targeted questioning and observation of the most efficient strategy use. The researcher probed the students for further clarification of the strategy used and their knowledge of this aspect of early number until she was satisfied that she had gained a clear insight into their thinking.

Analysis of the students’ strategy use during the assessments allows for comparisons to be made between the different strategies used by students to solve particular tasks as well as allowing comparisons to be made between the strategies used by particular students across a range of tasks. Strategies were arranged according to their level of mathematical sophistication. As students become more facile (Wright, Martland, & Stafford, 2006) in solving addition tasks involving two addends in the range 1 to 10, they use increasingly sophisticated numerical strategies (Bobis et al., 2009). Students did not necessarily use all of the strategies listed as they progressed from the lowest to the highest level of sophistication.

In Task Groups 5, 8, 9, 10, 11 and 12 students were presented with tasks in bare number format. In each of these task groups, the term “bare number” means that formal arithmetic notation is used to pose the addition task. At times during the assessments, students spontaneously self-corrected; that is, they stated an incorrect response, but then immediately and spontaneously altered their answer to the correct response without any
comment or prompting on the part of the researcher. In all of the assessments conducted during this study, if a student spontaneously self-corrected, their response was regarded as correct.

5.1 Pre- and Post-assessment Results For Each Task Group

In Sections 5.1.1 to 5.1.12, each of the 12 task groups is described in terms of its presentation, purpose and setting. The frequencies of correct responses for each of the tasks and the frequencies of strategy use by the students for most of the tasks are reported. Results for each task are presented in graphical format. Column graphs are used to display frequencies of correct responses and frequencies of strategy use in each task group. Most column graphs facilitate comparisons of responses to each task on the pre- and post-assessments. Twenty students completed both the pre- and post-assessments. This chapter reports on both the frequency of correct responses to tasks posed by the researcher and the solution attempts categorised into response types.

5.1.1 Task Group 1 – Forward Number Word Sequence (FNWS)

Task Group 1 consisted of four tasks. Each student was instructed to “count by ones” from a beginning number. When the student reached the pre-determined end point of the sequence, the researcher instructed them to stop counting. The purpose of Task Group 1 was to assess the student’s facility with the FNWS in the range 1 to 35. Students were asked to count forwards from four different starting points: from 1 to 20, 5 to 16, 13 to 24, and 17 to 35. These ranges of numbers were chosen as all addition tasks posed throughout the assessments and the teaching cycle were within the range 1 to 35, and each sequence assessed the students’ ability to say a sequence which spanned one or two decuples. The ability to recite the FNWS in this range was considered to be a prerequisite skill in order for students to access the hypothetical learning trajectory (Cobb et al., 2001; Gravemeijer, 2004a; Simon, 1995) developed as part of this study. All students correctly recited the
FNWS from 1 to 20, and from 5 to 16 in both the pre- and post-assessments. Although all students responded correctly to this aspect of the assessment, collecting this data was important to inform the starting point for the teaching sequence (McClain, 2005).

Figure 5.1 Frequencies of Correct Responses on FNWS Tasks

As depicted in Figure 5.1, the number of students who recited correctly the FNWS in the range 13 to 24 increased from 17 in the pre-assessment to 20 in the post-assessment. The number of students who recited correctly the FNWS in the range 17 to 35 increased from 12 in the pre-assessment to 18 in the post-assessment.

5.1.2 Task Group 2 – Numeral Identification

Task Group 2 consisted of 20 tasks. The student was asked to say the name of the numeral displayed on an individual card. The 20 cards were organised into two groups. The first group consisted of the numerals 1 to 10 presented in non-counting order. The second group consisted of the numerals 11 to 20 presented in non-counting order. The purpose of Task Group 2 was to determine each student’s ability to identify numerals in the range 1 to 20.

In the first part of the task, the numerals 1 to 10 were presented in the following order: 3, 7, 1, 5, 4, 6, 8, 2, 9 and 10. All students were able to identify all numerals in the
range 1 to 10 in both the pre- and post-assessments. In the second part of the task, the numerals 11 to 20 were presented in the following order: 13, 17, 15, 14, 16, 18, 11, 19, 12 and 20. All students correctly identified the numeral “11” in both the pre- and post-assessments.

![Figure 5.2 Frequencies of Correct Responses in Identifying Numerals in the Range 12 to 20](image)

Figure 5.2 shows that the frequencies of students who identified correctly the numerals 13, 17 and 19 remained the same from the pre-assessment to the post-assessment. There was a decrease of one in the frequencies of students who identified correctly the numerals 14, 15 and 18, and a decrease of two in the frequencies of students who identified correctly the numeral 16 from the pre-assessment to the post-assessment. The disaggregated results show that one student was not able to correctly identify the numerals 12, 13, 17, 15, 14 and 16 in the post-assessment, yet had correctly identified eight numerals (13, 17, 15, 14, 16, 18, 11, 20) in the pre-assessment.

### 5.1.3 Task Group 3 – Addition Tasks Involving Two Screened Collections

Task Group 3 consisted of two addition tasks presented in the setting of two screened collections. The purpose of Task Group 3 was to ascertain the most advanced strategy used by students to determine how many counters in all when both collections were screened. The two collections were briefly displayed and then screened by an opaque
piece of card. The researcher said, “Under this card I have four red counters and two yellow counters, how many counters altogether?”

In planning the first addition task involving screened collections, the researcher deliberately chose to use two collections which, when combined, remained in the range 1 to 10. If the student was incorrect in solving the task with both collections screened (4 + 2), the second collection was unscreened (displayed) and the student was then asked how many counters there were in all. If the student was correct in solving the task 4 + 2, a more advanced task with both collections screened was presented: 9 + 4. The purpose of this task was to give the student the opportunity to demonstrate use of a more advanced strategy than count-by-ones from one.

![Bar chart]

Figure 5.3 Frequencies of Correct Responses to Solve Screened Collections Tasks 4 + 2 and 9 + 4

Figure 5.3 depicts an increase in the number of students who correctly solved 4 + 2 and 9 + 4 from the pre-assessment to the post-assessment. Of the four students who were unsuccessful in solving the task 4 + 2 presented with both collections screened in the pre-assessment, three were successful when the second addend was unscreened. The one student who was unsuccessful in solving the task 4 + 2 presented with both collections screened in the post-assessment, was successful when the second addend was unscreened.
Students’ responses to Task Group 3 were reviewed and grouped into seven categories. The responses are categorised from lowest to highest in terms of mathematical sophistication of the strategies observed. These categories are described in the table below.

Table 5.1
Description of Response Categories to Solve Addition Tasks Presented as Two Screened Collections

<table>
<thead>
<tr>
<th>Response Categories</th>
<th>Description of Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. Did not pose task (DNP)</td>
<td>The task is not presented to the student for one of two reasons:</td>
</tr>
<tr>
<td></td>
<td>1. The student is correct on the previous tasks in this task group and responds with certitude. The researcher is satisfied the most sophisticated strategy has been determined (Wright, Martland, Stafford, &amp; Stanger, 2006).</td>
</tr>
<tr>
<td></td>
<td>2. The student is repeatedly incorrect on the previous tasks and/or the researcher determines they are unable to succeed at this task as they have not exhibited the necessary prerequisite knowledge (e.g., knowing the FNWS in this range).</td>
</tr>
<tr>
<td>b. Incorrect response or unknown strategy (Incorrect)</td>
<td>The student does not use an identifiable strategy or the student gives an incorrect response, regardless of the strategy observed.</td>
</tr>
<tr>
<td>c. Counts-by-ones from one, building perceptual replacements (Cb1sfrom1wPR)</td>
<td>The student first builds perceptual replacements for both collections and then counts-by-ones from one (e.g., 1, 2, 3, 4, 5, 6).</td>
</tr>
<tr>
<td>d. Counts-by-ones from one, without building perceptual replacements (Cb1sfrom1woPR)</td>
<td>The student counts-by-ones from one, once, to solve the task, but does not make either addend on their fingers (e.g., 1, 2, 3, 4, 5, 6).</td>
</tr>
<tr>
<td>e. Counts-on-by-ones (Con b1s)</td>
<td>The student counts-on-by-ones from the first collection (e.g., 4, pause, 5, 6), either with or without a physical or vocal means of keeping track of the counts.</td>
</tr>
<tr>
<td>f. Reasons with reference to another addition combination (Reasons)</td>
<td>The student solves the task by referring to another addition combination they know (e.g., 4 + 2 = 6 because 5 + 2 = 7).</td>
</tr>
<tr>
<td>g. Responds immediately with certitude (Immediate)</td>
<td>The student solves the task immediately and with certitude (e.g., 4 + 2 = 6).</td>
</tr>
</tbody>
</table>
Each response category was given an abbreviated label which is written in brackets after the Response Category title in Table 5.1. These abbreviated labels are used in Figures 5.4 and 5.5 to facilitate ease of reference.

Figure 5.4 Frequencies of Categorised Responses Used to Solve the Task 4 + 2 Presented as Two Screened Collections

Figure 5.4 shows that, in the post-assessment, to solve the task 4 + 2 presented as two screened collections, eight students used the higher level strategies (f and g) which did not involve counting-by-ones. The post-assessment showed that 19 out of 20 students in the class were at least using a strategy involving counting-on-by-ones from the first collection, with four students reasoning with reference to another known addition combination, for example: 4 + 1 → 5 + 1 → 6. Four students were able to find the sum of 4 and 2 immediately and with certitude. The range of strategy use was more varied for solving the addition task 4 + 2 presented as two screened collections in the pre-assessment than in the post-assessment.
As shown in Figure 5.5, in the pre-assessment, the task 9 + 4 was not presented to the one student who could not correctly solve 4 + 2 with the second addend unscreened. In the post-assessment, seven students did not correctly solve 9 + 4, and 13 students used one of the higher level strategies (e and f). In the pre- and post-assessments, no student responded immediately and with certitude to solve this task. The strategies used by students to solve the task 9 + 4 as two screened collections were polarised, with most students either unable to correctly solve the task or using at least a count-on-by-ones strategy. Strategy use was more varied for solving the addition task 9 + 4 presented as two screened collections in the pre-assessment than in the post-assessment.

The results in Figures 5.4 and 5.5 show that the students were more successful in solving the task 4 + 2 than the task 9 + 4 when presented as two screened collections.

5.1.3.1 Alternative Analysis of Screened Collections Tasks

Below is an alternative analysis of the strategies used by the students to solve the addition tasks presented as two screened collections. The SEAL model (see Table 2.1) developed by Wright, Martland, and Stafford (2006) is a progression of the counting
strategies that students use in increasingly sophisticated ways to solve addition and subtraction tasks. It is not appropriate to apply the SEAL directly to the pre- and post-assessment tasks used in this study as the data collected are not adequate to make an informed judgment about a student’s stage on the SEAL. However, the SEAL model was used to inform the construction of four bands which describe the range of strategies that students use to solve addition tasks. In order to aid proximal alignment of these bands with the SEAL, the bands are arranged from lowest to highest in terms of mathematical sophistication and are described in Table 5.2.

Table 5.2
Description of Response Bands and the SEAL to Solve Addition Tasks Presented as Two Screened Collections

<table>
<thead>
<tr>
<th>Response Band</th>
<th>SEAL</th>
<th>Description of Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>Band 1</td>
<td>Stages 0 and 1</td>
<td>The student is unable to count a screened collection. Student needs to build perceptual replacements to solve the task. The student gives an incorrect response, regardless of the strategy used.</td>
</tr>
<tr>
<td>Band 2</td>
<td>Stage 2</td>
<td>The student counts-by-ones from one to determine the total number of counters in two screened collections.</td>
</tr>
<tr>
<td>Band 3</td>
<td>Stages 3 and 4</td>
<td>The student counts-on from the first collection to determine the total number of counters in two screened collections.</td>
</tr>
<tr>
<td>Band 4</td>
<td>Stage 5</td>
<td>The student solves the task immediately and with certitude; reasons with reference to another addition combination or uses a grouping strategy such as build-through-five/ten or use of doubles/near doubles.</td>
</tr>
</tbody>
</table>

Frequencies of students’ responses to solve the addition tasks 4 + 2 and 9 + 4, presented as two screened collections, categorised into bands according to strategy use, are presented in Figure 5.6.
This graph depicts that strategies used to solve both addition tasks $4 + 2$ and $9 + 4$ were more varied in the pre-assessments than in the post-assessments. In the pre-assessment, 11 students used the higher level strategies of Bands 3 and 4 to solve $4 + 2$ as two screened collections, and in the post-assessment this increased to all students using the higher level strategies of Bands 3 and 4.

The high representation of Band 1 strategies in the pre-assessment is indicative of the 13 students who gave an incorrect solution to the task $9 + 4$. The number of students using Band 1 strategies was almost halved in the post-assessment, to seven students. In the pre-assessment, four students used the higher level strategies of Bands 3 and 4 to solve $9 + 4$ as two screened collections, and in the post-assessment this increased to 13 students.

5.1.4 Task Group 4 – Ascribing Number to Dot Patterns on a Five-frame

Task Group 4 consisted of five tasks. The student was presented with a five-frame pattern card (see Appendix B) and asked to orally state how many dots they could see (e.g., a two dot five-frame has two black dots in the far left hand squares, and three blank squares). The purpose of Task Group 4 was to ascertain the most advanced strategy each
student used to determine the number of dots on a five-frame. The five-frame was flashed at
the student for one or two seconds. Regardless of their response on the previous task, the
student was presented with each of the frames for the numbers one to five, presented in the
following order: two dots, four dots, three dots, one dot and five dots.

In the pre-assessment, the five-frame was presented to the student and left
unscreened as the student stated the number of dots they could see. At the time of the pre-
assessment, students had not been introduced to five-frame pattern cards. For this reason,
the frames were displayed rather than flashed. Thus, in the pre-assessment, students were
able to use perceptual (Wright, Martland, & Stafford, 2006, p. 22) count-by-ones strategies
to determine the number of dots. In the post-assessment, the frames were flashed.

In both the pre- and post-assessments all students were able to correctly ascribe
numbers to dot patterns on a five-frame. As well, in both the pre- and post-assessments all
students ascribed number to two, three, one and five dots on the five-frame immediately
and with certitude. One student reasoned in both assessments, with reference to the one
empty square, to explain that there were four dots on the five-frame.

Table 5.3 depicts the number of students who immediately and spontaneously
corrected their response (self-corrected) without any prompting or questioning from the
researcher. Self-correction was a behaviour that was more apparent in this task group than
in most other task groups in the assessments.

Table 5.3
Frequencies of Self-correction in Ascribing Number to Dot Patterns on a Five-frame

<table>
<thead>
<tr>
<th>Number of Dots on the Five-frame</th>
<th>Pre-assessment</th>
<th>Post-assessment</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
Two students self-corrected during the pre-assessment, and four students self-corrected during the post-assessment. The one student, who self-corrected when ascribing number to the dot pattern for five on the five-frame, changed their response from 10 dots to five dots. Of the five students who self-corrected when ascribing number to the dot pattern for four, three changed their response from five dots to four dots, and two changed their response from three dots to four dots. These results suggest that, on the tasks of ascribing number to five-frame patterns, students have most difficulty with the pattern for four.

5.1.5 Task Group 5 – Partitions of Five on a Five-frame

Task Group 5 consisted of four tasks. A five-frame partition card (see Appendix B) was shown to the student, and they were asked how many red dots and how many blue dots they could see. The student selected two numeral cards from a set of four numeral cards (1, 2, 3, 4) to complete the bare number equation \( \square + \square = 5 \). The student was instructed to place the numeral for the number of red dots as the first addend, and the numeral for the number of blue dots as the second addend. As described in Section 4.1.7.2, the partition of zero and five was not included, as zero in an arithmetical sense was not introduced until students were facile with grouping strategies to solve addition tasks. (i.e., strategies within Band 4 as described in Task Group 3). At the time of the pre-assessment, very few students were using strategies within Band 4 to solve addition tasks and therefore the partitions of five and 10 involving zero were not included.

The purpose of Task Group 5 was to ascertain the most advanced strategy each student used to determine the partitions of five. Regardless of their level of success on the previous task, the student was presented with each of the partitions of five in the following order: \( 4 + 1, 2 + 3, 3 + 2, 1 + 4 \). In the pre-assessment, the five-frame was presented to the student and left unscreened as the student constructed the number sentence. At the time of the pre-assessment, students had not been introduced to the idea of partitions of five. For
this reason, the frames were displayed rather than flashed. Thus, in the pre-assessment, students were able to use count-by-ones to determine the number of red dots and the number of blue dots. In the post-assessment, the frames were flashed.

In some instances, the student used the correct partition of five to complete the bare number equation, but placed the addends in the reverse order. For example, if there were three red dots and two blue dots on the five-frame, the student constructed the bare number equation as $2 + 3 = 5$. As the purpose of this task group was to determine the strategy the student used to identify partitions of five, this reversal of the order was not considered important, and the student’s response was judged to be correct.

All 20 students constructed all of the number equations for the partitions of five correctly in the pre-assessment. In the post-assessment, all students correctly constructed the number equations for $2 + 3$ and for $1 + 4$. Nineteen students correctly constructed the number equation for $4 + 1$ and 18 students correctly constructed the number equation for $3 + 2$.

Frequencies of students’ responses to represent the partitions of five as bare number equations from the material setting of a five-frame were categorised into four types. The responses that constitute each type are described in Table 5.4. The responses are arranged from lowest (Type I) to highest (Type IV) in terms of the mathematical sophistication of the strategy used.
Table 5.4
Description of Response Categories Used to Represent the Partitions of Five as Bare Number Equations from the Setting of a Five-frame

<table>
<thead>
<tr>
<th>Response Categories</th>
<th>Description of Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type I</td>
<td>The student does not use an identifiable strategy, the task is not presented to the student or the student gives an incorrect response, regardless of the strategy used.</td>
</tr>
<tr>
<td>Type II</td>
<td>To determine the first addend, in the pre-assessment, the student counts-by-ones from one, or in the post-assessment they build perceptual replacements. In order to determine the second addend, they either read the dot pattern (in the pre-assessment), count-by-ones from one, or count up from the first addend to five.</td>
</tr>
<tr>
<td>Type III</td>
<td>The student states the first addend as a known dot pattern. In the pre-assessment, they then read the dot pattern of the second addend. The student does not have to count-by-ones in the pre-assessment to determine the first addend, nor build perceptual replacements for it in the post-assessment. The student pauses long enough before constructing the bare number equation that the partition of five cannot be considered to have been immediately recalled.</td>
</tr>
<tr>
<td>Type IV</td>
<td>The student either reasons with reference to another addition combination or states immediately and with certitude both the first and second addend of the partition of five. They do not use any count-by-ones strategy.</td>
</tr>
</tbody>
</table>

As the frame was unscreened in the pre-assessment, the researcher cannot be certain that students assigned a Type IV response have acquired the partitions of five as known addition combinations. They may have ascribed numbers to the dots in each addend with just a very quick glance before they turned their attention to completing the bare number equation.
Figure 5.7 depicts an increase in the frequency of Type IV responses in the post-assessment and a decrease in the frequency of Type II responses. Recall that the five-frame was left unscreened in the pre-assessment and flashed in the post-assessment. This increase in complexity between the pre- and post-assessments may account for the increase in the frequency of Type I or incorrect responses in the post-assessment. Disaggregated data indicate that the student who responded incorrectly to the numeral identification tasks in Task Group 2, gave three of the four Type I responses in this task group.

5.1.6 Task Group 6 – Ascribing Number to Five-wise Patterns on a Ten-frame

Task Group 6 consisted of 10 tasks. The student was presented with a ten-frame pattern card (see Appendix B) and asked how many dots they could see. A ten-frame pattern card has dots placed in the boxes in a five-wise arrangement, that is, one row is filled with dots from left to right before any dots are placed in the second row. Seven, for example, is represented as five plus two (see Figure B4 in Appendix B). Frames for numbers less than five have an incomplete row of dots on the upper row and an empty
lower row. There is one ten-frame pattern card for each number in the range 1 to 10. The purpose of Task Group 6 was to ascertain the most advanced strategy each student used to determine the number of dots in a five-wise arrangement on a ten-frame pattern card. All of the numbers from 1 to 10 were presented, not in counting order, regardless of responses on the earlier arrangements. Task Group 6 was divided into two sub-groups – the ten-frame pattern cards for 1 to 5 and the ten-frame pattern cards for 6 to 10.

As in Task Group 5, in the pre-assessment, the ten-frame pattern card was presented to the student and left unscreened. Therefore, in the pre-assessment, students were able to use count-by-ones to determine the number of dots. In the post-assessment, the frames were flashed. In the first part of the task, the dot patterns for numbers 1 to 5 were presented in the following order: 3, 1, 4, 2 and 5. All students were able to identify the dot patterns for 3, 1 and 2 in the pre-assessment. One student incorrectly named the dot patterns for 4 and 5. In the post-assessments, all students correctly identified all dot patterns in the range 1 to 5.

In the pre-assessment, all students named the patterns for 3, 1 and 2 dots immediately and with certitude. Nineteen students and 18 students respectively named the patterns for 4 dots and 5 dots with certitude. In the post-assessment, all students named the patterns for 1 to 5 dots immediately and with certitude, with the exception of one student who reasoned to explain how they knew there were 4 dots on the ten-frame. As students may sometimes “describe a strategy that differs from their original strategy” (Wright, Martland, & Stafford, 2006, p. 45), neither a reasoning strategy nor responding immediately is considered more advanced than the other because:

1. The student coded as using a form of reasoning may immediately and correctly ascribe number to the dot pattern on the ten-frame, but nevertheless explain their response by reasoning with reference to another dot pattern; and
2. The student may reason with reference to another dot pattern to ascribe number but, as they respond almost immediately and with certitude, the researcher assumes that this is a known addition combination for the student.

In the second part of Task Group 6, the dot patterns for numbers 6 to 10 were presented in the following order: 6, 10, 9, 8 and 7. The frequencies of correct responses to ascribe number to dots in the range 6 to 10 on a five-wise ten-frame are shown in Figure 5.8.

![Figure 5.8 Frequencies of Correct Responses to Ascribe Number to Dots in the Range 6 to 10 on a Five-wise Ten-frame](image)

Students’ responses to Task Group 6 were reviewed and grouped into five categories. These categories are described in Table 5.5. Each response category has been given an abbreviated label which is written in brackets after the Response Category title in Table 5.5. The responses are categorised from lowest to highest in terms of the mathematical sophistication of the strategies used.
Table 5.5
Description of Response Categories Used to Ascribe Number to Dots in a Five-wise Arrangement on a Ten-frame Pattern Card

<table>
<thead>
<tr>
<th>Response Categories</th>
<th>Description of Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. Incorrect response or unknown strategy</td>
<td>The student does not use an identifiable strategy or the student gives an incorrect response, regardless of the strategy used.</td>
</tr>
<tr>
<td>Incorrect (Incorrect)</td>
<td></td>
</tr>
<tr>
<td>b. Counts-by-ones from one</td>
<td>The student counts the dots by-ones from one once, to solve the task (e.g., 1, 2, 3, 4, 5, 6).</td>
</tr>
<tr>
<td>Cby1sfrom1</td>
<td></td>
</tr>
<tr>
<td>c. Counts-on by-ones from five</td>
<td>The student counts the dots by-ones from five (e.g., 5, pause, 6, 7).</td>
</tr>
<tr>
<td>Conby1sfrom5</td>
<td></td>
</tr>
<tr>
<td>d. Reasons with reference to another dot pattern</td>
<td>The student states the number of dots by referring to another arrangement of dots they know (e.g., I know there are six dots because there are five dots on the upper row and one dot on the lower row).</td>
</tr>
<tr>
<td>Reasons (Reasons)</td>
<td></td>
</tr>
<tr>
<td>e. Responds immediately with certitude (Immediate)</td>
<td>The student solves the task immediately and with certitude (e.g., four dots).</td>
</tr>
</tbody>
</table>

The frequencies with which each strategy was used to ascribe number to dots in the range 6 to 10 in a five-wise arrangement on a ten-frame pattern card are presented in Figure 5.9. The abbreviated labels from Table 5.5 are used to facilitate ease of reference.

![Figure 5.9 Frequencies of Student Response Types to Ascribe Number to Dots in the Range 6 to 10 on a Five-wise Ten-frame-in the pre- and post-assessments](image)
There were 200 responses to the task of ascribing number to dot patterns on a ten-frame in the range 6 to 10, that is, 20 students responded to five combinations in the pre- and post-assessments. Seven of the 200 responses used a count-on-by-ones from five strategy and 16 of the 200 responses were categorised as an unknown strategy or an incorrect response. Only two responses involved a count-by-ones from one strategy, once each for six dots and eight dots respectively. All other solution attempts (175 responses) involved ascribing number to the dot patterns immediately and with certitude or by reasoning with reference to another dot pattern. Strategy use was more varied for ascribing number to dots in the range 6 to 10 in the pre-assessment than it was in the post-assessment. These results show an increase in the use of strategies at a higher level of mathematical sophistication to ascribe number to all ten-frame pattern cards in the range 6 to 10.

On Task Group 6 – ascribing number to dot patterns in the range 1 to 10 – in both the pre- and post-assessments, strategy use was more varied for ascribing number to dot patterns in the range 6 to 10 than it was for ascribing number to dot patterns in the range 1 to 5.

5.1.7 Task Group 7 – Moving Beads in the Range 1 to 10 on the Arithmetic Rack

Task Group 7 consisted of 10 tasks. The student was shown an arithmetic rack (see Appendix B) and its structure was explained. In these tasks, students moved beads on the upper row only. This task group involved the researcher stating a number in the range 1 to 10 and the student moving that number of beads from the right hand side of the arithmetic rack to the left. All numbers from 1 to 10 were presented, regardless of responses on previous combinations. Task Group 7 was divided into two groups, numbers in the range 1 to 5 and numbers in the range 6 to 10. The purpose of Task Group 7 was to ascertain the most advanced strategy each student used to move the beads. In the first part of the task, students were asked to move a number of beads in the following order: 3, 4, 1, 5 and 2.
In the pre-assessment, all students correctly moved 3, 1 and 2 beads. Eighteen and 19 students were correct in their attempts to move 4 and 5 beads respectively. In the post-assessment, all students correctly moved beads for all numbers in the range 1 to 5.

Students’ responses to Task Group 7 were reviewed and grouped into five categories. These categories are described in Table 5.6. Each response category was given an abbreviated label which is written in brackets after the Response Category title in Table 5.6. The responses are categorised from lowest to highest in terms of the mathematical sophistication of the strategies used.

Table 5.6
Description of Response Categories Used to Move Beads on the Arithmetic Rack

<table>
<thead>
<tr>
<th>Response Categories</th>
<th>Description of Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. Incorrect response or unknown strategy (Incorrect)</td>
<td>The student does not use an identifiable strategy or the student gives an incorrect response, regardless of the strategy used.</td>
</tr>
<tr>
<td>b. Moves beads by ones from one (Mby1sfrom1)</td>
<td>The student moves each bead sequentially in coordination with counting from one.</td>
</tr>
<tr>
<td>c. Moves beads as a group of five-plus other beads (M5plus1s)</td>
<td>The student moves or counts five beads simultaneously then the remainder sequentially while counting on from five.</td>
</tr>
<tr>
<td>d. Reasons with reference to another addition combination (Reasons)</td>
<td>The student solves the task by referring to another addition combination that they know (e.g., to move eight beads, they move ten across and then move two back).</td>
</tr>
<tr>
<td>e. Responds immediately with certitude (Immediate)</td>
<td>The student solves the task immediately and with certitude (e.g., moves nine beads in one movement with no apparent need to use a count-by-ones strategy).</td>
</tr>
</tbody>
</table>

Response Category (c) describes students moving beads as a group of five, and then the remainder by ones. When moving less than five beads, this is not an available strategy. As described previously in Section 5.1.6 neither of the two Response Categories (d) or (e) in Table 5.6 is considered more advanced than the other.
The frequencies with which each response category was used for moving a number of beads in the range 1 to 5 on the arithmetic rack are presented in Figure 5.10. The abbreviated labels from Table 5.6 are used to facilitate ease of reference.

![Figure 5.10 Frequencies of Student Response Types for Moving a Number of Beads in the range 1 to 5 on the Arithmetic Rack](image)

Variation in strategy use on the arithmetic rack in the range 1 to 5 was greatest for moving four beads. The total number of students’ responses which involved moving the beads immediately and with certitude was 87 (from a possible 100) in the pre-assessment, and 93 in the post-assessment.

In the second part of Task Group 7, students were asked to move a number of beads in the following order: 6, 9, 7, 10 and 8. In the pre-assessment, 19 students correctly moved 6, 10 and 8 beads. Eighteen and 17 students were correct in their attempts to move 9 and 7 beads respectively. In the post-assessment, all students correctly moved beads for all numbers in the range 6 to 10.
As with moving beads in the range 1 to 5, students’ solution strategies to move beads in the range 6 to 10 were reviewed and grouped into five categories. These were described in Table 5.6. The frequencies with which each response category was used for moving a number of beads in the range 6 to 10 on the arithmetic rack are presented in Figure 5.11. The abbreviated labels from Table 5.6 are used to facilitate ease of reference.

![Figure 5.11 Frequencies of Student Response Types for Moving a Number of Beads in the Range 6 to 10 on the Arithmetic Rack](image)

Variation in strategy use on the arithmetic rack in the range 6 to 10 was greatest for moving 7, 8 and 9 beads. Sixteen and 17 students moved 8 and 7 beads respectively, immediately and with certitude. This may suggest that these combinations were more difficult for students to visualise as five-plus some more beads. The total number of students’ responses which involved moving the beads immediately and with certitude increased from 74 (from a possible 100) in the pre-assessment, to 92 in the post-assessment.

5.1.7.1 Range 1 to 10

There was a total of 400 responses to the task of moving beads in the range 6 to 10 on the arithmetic rack, that is, 20 students responded to five combinations in the pre- and
post-assessments. Eight students moved beads-by-ones from five (2% of total responses), 20 students moved beads-by-ones from one (5% of total responses) and 14 students were observed using an unknown strategy or gave an incorrect result (3.5% of total responses). Moving beads immediately and with certitude or by reasoning with reference to another group of beads occurred 358 times (89.5% of total responses).

In the pre-assessment, students moved an incorrect number of beads three times (0.75% of total responses) in the range 1 to 5, and eight times (2% of total responses) in the range 6 to 10. In the post-assessment, all students correctly moved the stated number of beads in the range 1 to 10.

As described in Table 5.6, Response Categories (a), (b) and (c) are the least advanced in terms of mathematical sophistication. Students were observed using less advanced strategies more often to move beads in the range 6 to 10 than in the range 1 to 5 in the pre-assessment. The least sophisticated strategy of moving beads-by-ones from one was observed 14 times (3.5% of total responses) in the pre-assessment, and five times (1.25% of total responses) in the post-assessment. Of the 14 times the least sophisticated strategy of moving beads-by-ones from one was observed in the pre-assessment, nine (or 64.2%) were in the range 6 to 10. Of the five times the least sophisticated strategy of moving beads-by-ones from one was observed in the post-assessment, one (or 20%) was in the range 6 to 10.

5.1.8 Task Group 8 – Partitions of 10 on a Ten-frame

Task Group 8 consisted of nine tasks. A ten-frame partition card (see Appendix B) was shown to the student, and they were asked how many red dots and how many blue dots they could see. The following convention is adopted for describing these tasks: for the partition 6 and 4, for example, expressed verbally or written, we refer to the 6 as the first addend and 4 as the second addend. Red and blue dots were used to show the first and
second addends respectively. The red dots filled the ten-frame in a five-wise arrangement from the left hand side of the upper row and the blue dots filled the remainder of the empty boxes. After seeing the ten-frame, the student selected two numeral cards from a set of ten numeral cards (1, 2, 3, 4, 5, 6, 7, 8, 9) to complete the bare number equation \( \square + \square = 10 \). The student was instructed to place the numeral for the number of red dots as the first addend, and the numeral for the number of blue dots as the second addend.

The purpose of Task Group 8 was to ascertain the most advanced strategy each student used to identify the partitions of 10. Regardless of responses on the previous tasks, the student was presented with each of the partitions of 10 in a non-sequential order. Task Group 8 was divided into two sub groups – the ten-frame partition cards where the number of red dots was greater than or equal to the number of blue dots (i.e., 9 + 1, 8 + 2, 7 + 3, 6 + 4, and 5 + 5) and the frames where the number of red dots was less than the number of blue dots (i.e., 1 + 9, 2 + 8, 3 + 7, and 4 + 6). These sub-groups are described as the partitions of 10 with first addend in the range 5 to 9, and the partitions of 10 with first addend in the range 1 to 4. The partitions presented to the students were 9 and 1, 8 and 2, 7 and 3, 6 and 4, and 5 and 5, and their inverses. The researcher made the decision not to include the partition of 0 and 10 in the assessment. The reasons for this exclusion are described in Section 4.1.7.2.

In the pre-assessment, the ten-frame was presented to the student and left unscreened as the student constructed the number sentence. At the time of the pre-assessment, students had not been introduced to the idea of partitions of 10. For this reason, the frames were displayed rather than flashed. Thus, in the pre-assessment, students were able to use a count-by-ones strategy to determine the number of red dots and the number of blue dots. In the post-assessment, the frames were flashed.
In some instances, the student used the correct partitions of 10 to complete the bare number equation, but placed the addends in the reverse order. For example, if there were 8 red dots and 2 blue dots on the ten-frame, the student may have constructed the bare number equation as \(2 + 8 = 10\). As the purpose of this task group was to determine the strategy the student used to identify partitions of 10, this reversal of the order was not considered to be important, and the student’s response was judged to be correct.

Figure 5.12 Frequencies of Correct Responses to Represent Partitions of 10 with First Addend in the Range 5 to 9 as Bare Number Equations From the Material Setting of a Ten-frame

Figure 5.12 depicts that, in comparing the pre- and post-assessments, the number of students who correctly represented the partitions of \(6 + 4\) and \(5 + 5\) as bare number equations increased, and the number of students who correctly represented the partitions of \(9 + 1\), \(8 + 2\), and \(7 + 3\) as bare number equations decreased. This can be explained by the fact that, in the pre-assessment, the ten-frame was left unscreened in front of the student and they were able to use this visual aid to determine the partition of 10. In the post-assessment, the ten-frame was only briefly flashed at the student and then removed from sight. Thus, students relied on visualisation strategies to determine the partition of 10, and counting the dots by ones was not an available strategy. It can be seen that, even though the
complexity of the task was increased in the post-assessment, students achieved a similar level of success.

To facilitate analysis of the strategies used to represent the partitions of 10 as bare number equations from the setting of a ten-frame, frequencies of students’ responses were categorised into four types. The responses that constitute each type are described in Table 5.7. The responses are arranged from lowest (Type I) to highest (Type IV) in terms of the mathematical sophistication of the strategy used.

Table 5.7
Description of Response Categories Used to Represent Partitions of 10 with First Addend in the Range 5 to 9 as Bare Number Equations from the Material Setting of a Ten-frame

<table>
<thead>
<tr>
<th>Response Categories</th>
<th>Description of Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type I</td>
<td>The student does not use an identifiable strategy, the task is not presented to the student, or the student gives an incorrect response, regardless of the strategy used.</td>
</tr>
<tr>
<td>Type II</td>
<td>To determine the first addend, in the pre-assessment, the student counts-by-ones from one or in the post-assessment they build perceptual replacements. In order to determine the second addend they either read the dot pattern (in the pre-assessment), count-by-ones from one, or count up from the first addend to 10.</td>
</tr>
<tr>
<td>Type III</td>
<td>The student states the first addend either as a known dot pattern or they count-by-ones from five. They then count-by-ones or reason with reference to another dot pattern to determine the second addend, or in the pre-assessment read the dot pattern of the second addend. The student does not have to count-by-ones in the pre-assessment to determine the first addend, nor build perceptual replacements for it in the post-assessment.</td>
</tr>
<tr>
<td>Type IV</td>
<td>The student either reasons with reference to another dot pattern or states immediately and with certitude both the first and second addend of the partition of 10. They do not use any count-by-ones strategy.</td>
</tr>
</tbody>
</table>
Figure 5.13 depicts that, in the post-assessment, the frequency of Type IV responses was higher for each of the partitions 6 + 4, 5 + 5, and 9 + 1 than it was in the pre-assessment. For the partition 8 + 2, the same number of students gave Type IV responses in the pre- and post-assessments. In the post-assessment, the frequency of Type IV responses was lower for the partition 7 + 3 than it was in the pre-assessment. The frequency of Type IV responses increased from 50 responses (from a possible 100) in the pre-assessment to 63 in the post-assessment. Recalling that, in the post-assessment, the ten-frame was only briefly flashed at the student and then removed from sight (in contrast to being left unscreened in the pre-assessment), this increase is noteworthy.

The graph below depicts the frequencies of students’ correct responses to the task of representing partitions of 10 with first addend in the range 1 to 4 as bare number equations from the material setting of a ten-frame.
Fewer students correctly represented $4 + 6$ and $2 + 8$ as bare number equations in the post-assessment than in the pre-assessment. Similarly, the number of correct responses decreased from 69 responses (from a possible 80) in the pre-assessment to 57 in the post-assessment. This decrease in the overall frequencies of correct responses for the partitions of 10 with first addends in the range 1 to 4 between the pre- and post-assessments, can be explained by the fact that in the pre-assessment the ten-frame was left unscreened in front of the student and they were able to use this visual aid to determine the partition of 10. In the post-assessment, the ten-frame was only briefly flashed at the student and then removed from sight. This meant that the students had to rely on visualisation strategies to determine the partition of 10, and counting the dots by ones was not an available strategy.

To facilitate analysis of the strategies used to represent partitions of 10 with first addend in the range 1 to 4 as bare number equations from the material setting of a ten-frame, the four category types described in Table 5.7 were used again.
Figure 5.15 depicts a slight increase in the frequency of Type IV responses from 32 responses (from a possible 80) in the pre-assessment to 33 in the post-assessment. Recalling that, in the post-assessment, the ten-frame was only briefly flashed at the student and then removed from sight (in contrast to being left unscreened in the pre-assessment), this increase is noteworthy.

Figures 5.13 and 5.15 depict the strategies used by students to represent a partition of 10 as a bare number equation from the setting of a ten-frame. The equations are of the form \(a + b = 10\), and the results show that the strategies are more varied when \(a < b\) than when \(a \geq b\). There was a total of 360 responses to the task of representing a partition of 10 as a bare number equation, that is, 20 students responded to nine combinations in the pre- and post-assessments. The frequency of Type IV responses was greater when \(a \geq b\). Sixty-five responses were categorised as Type IV (18.06% of total responses) when \(a < b\) and 117 responses were categorised as Type IV (32.5% of responses) when \(a \geq b\). This
indicates that students used higher level strategies more often to represent a partition of 10 as a bare number equation from the setting of a ten-frame in the form a + b =10 when a≥b than when a<b.

5.1.9 Task Group 9 – Small Doubles Presented in Bare Number Format

Task Group 9 consisted of five tasks. The small doubles, 1 + 1, 2 + 2, 3 + 3, 4 + 4, and 5 + 5, were presented in bare number format. Each student was asked to read aloud what was written on the card. If the student did not appear to spontaneously begin to calculate the sum, they were asked if they had a way to work out “what that equals”. The purpose of Task Group 9 was to ascertain the most advanced strategy each student used to determine the sum when the two addends were the same and in the range 1 to 5. The small doubles were presented in the following order: 3 + 3, 5 + 5, 1 + 1, 4 + 4, and 2 + 2.

Figure 5.16 Frequencies of Correct Responses to Find the Sums of the Small Doubles Presented in Bare Number Format

Figure 5.16 depicts an overall increase in the number of correct responses to the small doubles presented in bare number format from 66 (out of a possible 100) correct responses in the pre-assessment, to 94 correct responses in the post-assessment. The
number of correct responses to each item also increased, with all students correctly responding to the tasks $5 + 5$ and $2 + 2$ in the post-assessment.

Below is an analysis of the strategies used by the students to determine the sums of the small doubles presented in bare number format. As with Task Group 3, students’ strategies were categorised into four bands. A detailed explanation of the construction of these bands can be found in Section 5.1.3.1. The bands are described in Table 5.8.

<table>
<thead>
<tr>
<th>Response Bands</th>
<th>SEAL</th>
<th>Description of Bands</th>
</tr>
</thead>
<tbody>
<tr>
<td>Band 1</td>
<td>Stages 0 and 1</td>
<td>The student needs to build perceptual replacements for one or both of the addends to solve the task; the student gives an incorrect response, regardless of the strategy used; or the task was not presented as the student had given incorrect responses to previous tasks.</td>
</tr>
<tr>
<td>Band 2</td>
<td>Stage 2</td>
<td>The student counts-by-ones from one to determine the sum of the double. They may or may not use fingers to keep track of the counts.</td>
</tr>
<tr>
<td>Band 3</td>
<td>Stages 3 and 4</td>
<td>The student counts-on from the first addend to determine the sum of the double.</td>
</tr>
<tr>
<td>Band 4</td>
<td>Stage 5</td>
<td>The student solves the task immediately and with certitude or reasons with reference to another addition combination or uses a grouping strategy such as build-through-five.</td>
</tr>
</tbody>
</table>

Frequencies of students’ responses to find the sums of the small doubles presented in bare number format, categorised into bands according to strategy use, are presented in Figure 5.17.
As described in Table 5.8, Band 4 responses are the most advanced in terms of the mathematical sophistication of the strategies used and Band 1 the least advanced. Figure 5.17 depicts an increase from a frequency of 60 (from a possible 100) Band 4 responses in the pre-assessment to a frequency of 90 Band 4 responses in the post-assessment to find the sums of the small doubles presented in bare number format. Similarly, there was a decrease in the frequency of less sophisticated Band 1 responses from the pre-assessment to the post-assessment for each of the small doubles except 3 + 3. In the pre-assessment, the greatest variety of responses was used to solve 3 + 3. This was the only task in which students’ responses were from all four bands. This result could possibly be accounted for by the fact that this was the first task in the assessment which was presented in bare number format.

These results for finding the sums in the small doubles task group indicate that, in the post-assessment, frequency of responses was generally polarised into Band 1 or Band 4 responses. In the post-assessment, a group of four students used a strategy with a low level of mathematical sophistication from Band 1 to solve the tasks 3 + 3 and 4 + 4,
whereas the rest of the students typically used a high level, grouping strategy from Band 4. No students gave a Band 2 or Band 3 response. This was also true in the pre-assessment with the exception of the task 3 + 3, which was the first task presented.

5.1.10 Task Group 10 – Ten-plus Addition Combinations Presented in Bare Number Format

Task Group 10 consisted of 10 tasks. The ten-plus addition combinations (i.e., 10 + 1, 10 + 2, through 10 + 10) were presented in bare number format. Each student was asked to read aloud what was written on the card. If the student did not appear to spontaneously begin to calculate the sum, they were asked if they had a way to work out “what that equals”. The purpose of Task Group 10 was to ascertain the most advanced strategy each student used to determine the sum when a number in the range 1 to 10 is added to 10.

![Figure 5.18](image)

**Figure 5.18 Frequencies of Correct Responses to Find the Sums of Ten-plus Addition Combinations Presented in Bare Number Format**

Figure 5.18 depicts increases in students’ correct responses from the pre-assessment to the post-assessment for all of the ten-plus addition combinations. The
The greatest increase in the number of correct responses was for $10 + 2$, with 11 and 19 students responding correctly to this in the pre- and post-assessments respectively. The smallest increase in the number of correct responses was for $10 + 4$, with 14 and 18 students responding correctly to this in the pre- and post-assessments respectively.

In the pre-assessment, the two ten-plus addition combinations with the least number of correct responses were $10 + 2$ with 11 students giving correct responses, and $10 + 1$ with 13 giving correct responses. By way of contrast, the low frequency of correct responses was not evident in the post-assessment, as 19 students correctly solved both of these ten-plus addition combinations.

The disaggregated data show that, in the post-assessment, the same student was unable to solve any of the ten-plus combinations. The first four tasks in Task Group 10 were presented to her but, when she was incorrect in her attempts to solve these, the remaining tasks were not presented. Tasks not presented were classified as incorrect, as the researcher decided not to present the task because the students had been incorrect on prior tasks. Task Group 10 was not presented to six students in the pre-assessment as they had given incorrect responses to prior task groups in bare number format.

Figure 5.18 depicts an overall increase in the number of correct responses to the ten-plus combinations presented in bare number format from 136 (out of a possible 200) in the pre-assessment to 189 in the post-assessment.

Below is an analysis of the strategies used by the students to determine the sums of the ten-plus addition combinations presented in bare number format. As with Task Groups 3 and 9, students’ strategies were categorised into four bands. A detailed explanation of the construction of these bands can be found in Section 5.1.3.1. The bands are described in Table 5.9.
Table 5.9
Description of Response Bands and the SEAL to Find the Sums of the Ten-plus Addition Combinations Presented in Bare Number Format

<table>
<thead>
<tr>
<th>Response Bands</th>
<th>SEAL</th>
<th>Description of Responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>Band 1</td>
<td>Stages 0 and 1</td>
<td>The student needs to build perceptual replacements for one or both of the addends to solve the task; the student gives an incorrect response, regardless of the strategy used; or the task was not presented as the student had given incorrect responses to previous tasks.</td>
</tr>
<tr>
<td>Band 2</td>
<td>Stage 2</td>
<td>The student counts-by-ones from one to determine the sum of the ten-plus addition combination. They may or may not use fingers to keep track of the counts.</td>
</tr>
<tr>
<td>Band 3</td>
<td>Stages 3 and 4</td>
<td>The student counts-on from 10 to determine the sum of the ten-plus addition combination. The student may or may not use fingers to keep track of the counts.</td>
</tr>
<tr>
<td>Band 4</td>
<td>Stage 5</td>
<td>The student uses a grouping strategy or solves the task immediately and with certitude. The student may reason with reference to another addition combination such as 10 + 4 = 14 so therefore 10 + 5 must be 15.</td>
</tr>
</tbody>
</table>

Frequencies of students’ responses to find the sums of the ten-plus addition combinations presented in bare number format, categorised into bands, are presented below.
As described in Table 5.9, Band 4 responses are the most advanced in terms of the mathematical sophistication of strategies used and Band 1 the least advanced. Figure 5.19 depicts that, in the pre-assessment, the highest frequencies of Band 1 and 2 responses were given by students to solve the tasks 10 + 4, 10 + 10, 10 + 2, and 10 + 1. In the post-assessment, Band 1 and 2 responses were given by no more than two students to find the sum of any of the ten-plus addition combinations in the task group.

In the pre-assessment, responses to the task 10 + 4 were most varied, and the two highest frequencies of a count-on Band 3 strategy occurred for the first two ten-plus addition combinations presented, 10 + 4 and 10 + 5, with seven and three students respectively using these strategies. In the post-assessment, 19 students used at least a Band 3 count-on strategy to solve 10 + 5, 10 + 3, 10 + 9, 10 + 7, 10 + 6, 10 + 8, 10 + 2, and 10 + 1. Eighteen students used at least a count-on strategy to solve 10 + 4 and 10 + 10.
Comparing the post-assessment to the pre-assessment, the three highest increases in students giving a Band 4 response occurred for the ten-plus addition combinations 10 + 4, 10 + 2, and 10 + 1. Figure 5.19 depicts an increase from 111 Band 4 responses (from a possible 200) in the pre-assessment, to 165 in the post-assessment to find the sums of the ten-plus addition combinations presented in bare number format.

5.1.11 Task Group 11 – Large Doubles Presented in Bare Number Format

Task Group 11 consisted of five tasks. The large doubles, 6 + 6, 7 + 7, 8 + 8, 9 + 9, and 10 + 10, were presented (not in this order) in bare number format and each student was asked to read aloud what was written on the card. If the student did not appear to spontaneously begin to calculate the sum, they were asked if they had a way to work out “what that equals”. The purpose of this assessment task group was to ascertain the most advanced strategy each student used to determine the sum when the two addends were the same and in the range 6 to 10.
Figure 5.20 depicts increases in the number of students who responded correctly to all of the large doubles from the pre-assessment to the post-assessment. However, excluding the double 10 + 10, there were only three (out of a possible 80) correct responses in the pre-assessment and 12 correct responses in the post-assessment when students were asked to find the sums of the large doubles presented in a bare number format. These results show that, in this task group, the percentages of correct responses were significantly smaller in both the pre- and post-assessments than they were for any other task group.

Similar to Task Group 9, students’ responses were categorised into bands. However, for this task a fifth band, Band 0, has been included in order to distinguish between the students who were not presented with the large doubles tasks and the students who attempted the large doubles tasks but were incorrect. The bands are described in Table 5.10 which corresponds to Table 5.8, with the inclusion of Band 0.

<table>
<thead>
<tr>
<th>Response Bands</th>
<th>SEAL</th>
<th>Description of Bands</th>
</tr>
</thead>
<tbody>
<tr>
<td>Band 0</td>
<td></td>
<td>The task is not presented as the student has given incorrect responses to previous tasks.</td>
</tr>
<tr>
<td>Band 1</td>
<td>Stages 0 and 1</td>
<td>The student needs to build perceptual replacements for one or both of the addends to solve the task; the student gives an incorrect response, regardless of the strategy used; or the task was not presented as the student had given incorrect responses to previous tasks.</td>
</tr>
<tr>
<td>Band 2</td>
<td>Stage 2</td>
<td>The student counts-by-ones from one to determine the sum of the double. They may or may not use fingers to keep track of the counts.</td>
</tr>
<tr>
<td>Band 3</td>
<td>Stages 3 and 4</td>
<td>The student counts-on from the first addend to determine the sum of the double.</td>
</tr>
<tr>
<td>Band 4</td>
<td>Stage 5</td>
<td>The student solves the task immediately and with certitude or reasons with reference to another addition combination or uses a grouping strategy such as build-through-five.</td>
</tr>
</tbody>
</table>
Figure 5.21 represents frequencies of students’ responses to find the sums of large doubles presented in bare number format, categorised into bands according to strategy use.

![Figure 5.21 Frequencies of Response Bands to Find the Sums of the Large Doubles Presented in Bare Number Format](image)

This graph depicts Band 0 responses with the highest frequency in the pre-assessment. This can be accounted for by the fact that many of the students had responded incorrectly to previous tasks such as the small doubles, and therefore the task was not presented. In the post-assessment, 19 students were presented with all of the large double tasks. The frequency of Band 0 responses in the pre-assessment increased according to the chronological order of presentation of large doubles tasks in the assessment. This could be because, when students made errors or repeatedly used a strategy such as count-on-by-ones from the first addend on one or more tasks, the researcher felt there was nothing to be
gained in presenting subsequent tasks as the student had demonstrated their most advanced strategy.

The number of students giving Band 1 responses increased from the pre-assessment to the post-assessment for each of the tasks 8 + 8, 6 + 6, 9 + 9, and 7 + 7. These increases can be accounted for in part by the fact that all tasks were presented to 19 students in the post-assessment, and therefore none of those students’ responses was categorised as Band 0. In the post-assessment, for each of the tasks 8 + 8, 6 + 6, and 7 + 7, 19 students who gave a Band 1 response were incorrect in their attempt to solve the large doubles when they were presented as a bare number task. For the task, 9 + 9, 15 students who gave a Band 1 response were incorrect in their attempt.

As described earlier, these results and those depicted in Figure 5.21 show that (excluding 10 + 10) at least 15 students gave incorrect responses to the large doubles presented in bare number format in the pre-assessment and the post-assessment. Recall that, for reasons described in Section 4.1.6, doubles were not a strong focus in the teaching sequence in this study. This may account for the comparatively low success rate of this task group in the post-assessment. Due to this low success rate, the researcher decided to scaffold the large doubles task for the students by introducing the arithmetic rack. The results are described in the following section.

5.1.11.1 Task Group 11(a) – Large Doubles Presented on the Arithmetic Rack

Task Group 11(a) consisted of four tasks. The large doubles, 6 + 6, 7 + 7, 8 + 8, and 9 + 9, were presented (not in order of smallest to largest) in the setting of an arithmetic rack. Each student was asked to state the total number of beads. In the first task, 8 + 8, the researcher made the two addends by pushing eight beads from the right to the left hand side on the upper and lower rows of the arithmetic rack. In subsequent tasks, the student generally made the two addends on the arithmetic rack. When large doubles are represented
on the arithmetic rack there are 10 beads of one colour (five on the upper row and five on the lower row) and some beads of another colour (half on the upper row and half on the lower row) (see Figure 5.22 for the example of 8 + 8 modelled on an arithmetic rack). By asking students to point to the arrangements of beads on the arithmetic rack and questioning them about the colours and the groups of the beads, the researcher made explicit to the students the significance of both of these features. The students were encouraged to use these features, and their knowledge of ten-plus addition combinations and small doubles, to assist them in finding the sum of the large doubles.

![Figure 5.22 8 + 8 in the Setting of the Arithmetic Rack](image)

The task 10 + 10 was omitted from this task group as 19 students were able to state this correctly from the setting of a bare number task in the post-assessment in Task Group 11. The purpose of this assessment task group was to ascertain the most advanced strategy each student used to determine the sum when the two addends were the same in the range 6 to 10 and presented as beads on an arithmetic rack.

The class group for this task group was reduced to 18 students. One student had responded correctly to all of the large doubles presented as bare number tasks. Another student had given incorrect responses to previous task groups and her assessment had been terminated after Task Group 10. Below, the frequencies of students’ correct responses to large doubles tasks presented as bare number tasks are contrasted with those presented in the setting of the arithmetic rack.
Figure 5.23 depicts an increase in the frequencies of students’ correct responses to finding the sum of all the large doubles when using the arithmetic rack compared with the bare number tasks. One student responded correctly to the task $7 + 7$ presented in bare number format, whereas 17 students responded correctly when it was presented in the setting of an arithmetic rack. If the task $9 + 9$ is disregarded, there was an increase in the frequencies of students correctly finding the sums of the large doubles in the setting of the arithmetic rack, which parallels the chronological order in which the tasks were presented. This was possibly due to students becoming more familiar with the structure of the arithmetic rack as each task was presented.

5.1.12 Task Group 12 – One-digit Additions, Bridging 10, Presented in Bare Number Format

Task Group 12 consisted of nine addition tasks with both addends in the range 1 to 10 presented in bare number format. Each student was asked to read aloud what was written on the card. If the student did not appear to spontaneously begin to calculate the
sum, they were asked if they had a way to work out “what that equals”. The purpose of Task Group 12 was to ascertain the most advanced strategy each student used to determine the sum of the two addends, when both addends were less than 10 and the sum exceeded 10. Algebraic notation was used in this task group to provide a generic description of bare number tasks in a given task group. Thus, Task Group 12 consisted of addition tasks of the form $9 \geq a \geq 7, \ 7 \geq b \geq 3, \ a+b>10$, where $a$ and $b$ are the two addends. The intention of this task group was to present students with addition tasks with sums greater than 10 in order to move them beyond their “finger range” (see Section 4.1.1 for further explanation). This was an attempt to elicit more advanced strategies than count-by-ones to solve addition tasks in bare number format, such as build-through-ten.

Task Group 12 was not presented to any of the 20 students in the pre-assessment, as none of the students had been able to consistently and correctly solve the bare number tasks presented in Task Groups 9, 10 and 11. Therefore, Figure 5.24 depicts the frequencies of students’ correct responses in the post-assessment only. In the post-assessment, not all tasks were presented to all students because of incorrect responses in previous tasks. In terms of frequencies of correct responses, tasks not presented were classified as incorrect. Figure 5.24 depicts the frequencies of students’ correct responses to solve addition tasks of the form $9 \geq a \geq 7, \ 7 \geq b \geq 3, \ a+b>10$. 
As the purpose of this assessment task group was to ascertain the most advanced strategy each student used to determine the sum when it exceeded 10, if the strategy used by the student to solve the first task was not a Band 4 strategy, the researcher asked if using a ten-plus addition combination might help. Students were familiar with the phrase “ten-plus” from the teaching sequence.

Below is an analysis of the strategies used by students to solve addition tasks presented in bare number format, with addends less than 10, and sums greater than 10.

Similar to Task Group 11, students’ responses were categorised into bands, and Band 0 was included in order to distinguish between the students who were not presented with the tasks, and the students who attempted to find the sums of the addition tasks but were incorrect. The bands are described in Table 5.11.
Table 5.11
Description of Response Bands and the SEAL to Solve Tasks of the Form $9 \geq a \geq 7$, $7 \geq b \geq 3$, $a+b>10$

<table>
<thead>
<tr>
<th>Response Bands</th>
<th>SEAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>Band 0</td>
<td></td>
</tr>
<tr>
<td>Band 1</td>
<td>Stages 0 and 1</td>
</tr>
<tr>
<td>Band 2</td>
<td>Stage 2</td>
</tr>
<tr>
<td>Band 3</td>
<td>Stages 3 and 4</td>
</tr>
<tr>
<td>Band 4</td>
<td>Stage 5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Description of Responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>The task is not presented as the student has given incorrect responses to previous tasks.</td>
</tr>
<tr>
<td>The student needs to build perceptual replacements for one or both of the addends to solve the task; or the student gives an incorrect response, regardless of the strategy used.</td>
</tr>
<tr>
<td>The student counts-by-ones from one to determine the sum of the two addends. They may or may not use fingers to keep track of the counts.</td>
</tr>
<tr>
<td>The student counts-on from the first addend to determine the sum of two addends. The student may or may not use fingers to keep track of the counts.</td>
</tr>
<tr>
<td>The student uses a grouping strategy or solves the task immediately and with certitude. The student may reason with reference to another addition combination such as build-through-ten, or knowledge of doubles.</td>
</tr>
</tbody>
</table>

Frequencies of students’ strategies in the post-assessment to find the sums of the form $9 \geq a \geq 7$, $7 \geq b \geq 3$, $a+b>10$, categorised into bands according to strategy use are presented in Figure 5.25.
This graph shows that Band 2 responses, or use of a count-by-ones from one strategy, were used least often by students to find the sums of the form $9 \geq a \geq 7, 7 \geq b \geq 3, a+b>10$. The frequencies of Band 1 responses, which included students who were incorrect, were highest for the task $8 + 6$ with 11 students observed giving this response; second highest were the tasks $7 + 5$, and $8 + 7$ with 10 students.

Band 3 responses were observed most frequently to solve the tasks $8 + 6$, $7 + 5$, and $8 + 3$. As described in Table 5.11, Band 4 responses are considered to use the most advanced strategies in terms of mathematical sophistication. There is a general trend for Band 4 responses to be observed more frequently when students solve nine-plus tasks. A Band 4 response was observed by seven students to solve the task $9 + 4$, by nine students to solve the task $9 + 5$, by 11 students to solve the task $9 + 6$, and by 10 students to solve the task $9 + 3$. 

*Figure 5.25 Frequencies of Response Bands to Solve Addition Tasks of the Form $9 \geq a \geq 7, 7 \geq b \geq 3, a+b>10$ in the Post-assessment*
5.2 Comparisons of Results From Key Tasks Within and Across Task Groups

In the following section, the results of key tasks across the 12 task groups will be compared and contrasted. These are considered key tasks as they were the only assessment tasks presented in bare number format. The absence of a material setting in the presentation of these addition tasks allowed students to use a strategy at their highest level of mathematical sophistication. 9 + 4 presented as two screened collections is also compared with 9 + 4 presented as a bare number task. These results will then be discussed in Chapter Six.

5.2.1 Task Group 12 – Nine-plus Tasks

In the post-assessment, four nine-plus tasks presented as bare number tasks were interspersed through Task Group 12. As shown in Figure 5.24, these four nine-plus tasks had the highest frequency of correct responses from all addition combinations in this task group presented in bare number format. The tasks 9 + 4, 9 + 5, 9 + 6, and 9 + 3 were solved by 12, 16, 13 and 14 students respectively. As previously described, students whose responses were categorised as Band 4 used strategies of the highest level of mathematical sophistication. Figure 5.26 depicts a comparison of the frequencies of correct responses with the frequencies of Band 4 responses to the nine-plus bare number tasks in the post-assessment.
This graph indicates that more than half of the students who correctly responded to a nine-plus task presented as a bare number task gave a Band 4 response. The disaggregated data indicate that the same six students consistently gave Band 4 responses to solve each of the four nine-plus tasks.

5.2.2 Strategies Used to Solve 9 + 4 Presented as Two Screened Collections Compared With 9 + 4 Presented in Bare Number Format

The task 9 + 4 was presented in Task Group 3 as two screened collections, and in Task Group 12 as a bare number task. Students’ responses to solve the task 9 + 4 presented as two screened collections (Task Group 3) and as a bare number task (Task Group 12), in the post-assessment, categorised into bands are presented in the graph below.
Figure 5.27 depicts Band 3 responses, which described the use of a count-on strategy, as more common to solve $9 + 4$ presented as two screened collections than $9 + 4$ presented in bare number format. Band 2 responses, which described the use of count-by-ones from one strategy, were not observed to solve either task.

Figure 5.28 depicts the disaggregated results of the advancement of students’ responses categorised according to bands to solve the task $9 + 4$ presented as two screened collections and as a bare number task, from the pre- to the post-assessment.
As shown graphically, of the seven students who solved $9 + 4$ presented as a bare number task by using a Band 4 strategy, only three of them (Students 6, 10 and 13) used a Band 4 strategy to solve $9 + 4$ presented as two screened collections. The other four...
students (Students 3, 5, 7 and 19) counted-on to solve $9 + 4$ presented as two screened collections. Only one student (Student 16) responded with a Band 4 response to $9 + 4$ presented as screened collections, and this student was unable to correctly solve $9 + 4$ presented in bare number format.

5.2.3 Comparison of Frequencies of Correct Responses Across Four Tasks Presented in Bare Number Format

Four task groups in the assessments consisted of bare number tasks only. These were Task Groups 9, 10, 11 and 12. The frequencies of correct responses for these tasks in the pre- and post-assessments are presented in Table 5.12.

Table 5.12
Summary of Frequencies of Correct Responses in the Pre- and Post-assessments for Bare Number Tasks Presented in Task Groups 9, 10, 11 and 12

<table>
<thead>
<tr>
<th>Task Group</th>
<th>Pre</th>
<th>Post</th>
</tr>
</thead>
<tbody>
<tr>
<td>TG 9 Small Doubles</td>
<td>66</td>
<td>94</td>
</tr>
<tr>
<td>TG 10 Ten-plus Combinations</td>
<td>136</td>
<td>189</td>
</tr>
<tr>
<td>TG 11 Large Doubles</td>
<td>15</td>
<td>31</td>
</tr>
<tr>
<td>TG 12 One-digit Additions</td>
<td>0</td>
<td>99</td>
</tr>
</tbody>
</table>

As indicated, there was an increase from the pre- to the post-assessment in the frequency of correct responses for all task groups. The greatest increase was for Task Group 12 as this task group was not presented to any students in the pre-assessment (for reasons described in Section 5.1.12). The lowest frequency of correct responses to tasks presented to all students was in response to the large doubles task, in both the pre- and post-assessments. In order to determine whether students were reasoning additively, it is also useful to summarise the frequencies of Band 4 responses which indicated the use of grouping strategies, in the pre- and post-assessments for each of Task Groups 9, 10, 11 and 12.
Table 5.13
Summary of Frequencies of Use of Band 4 Grouping Strategies in the Pre- and Post-assessments for Bare Number Tasks Presented in Task Groups 9, 10, 11 and 12

<table>
<thead>
<tr>
<th>Task Group</th>
<th>Pre</th>
<th>Post</th>
</tr>
</thead>
<tbody>
<tr>
<td>TG 9 Small Doubles</td>
<td>60</td>
<td>90</td>
</tr>
<tr>
<td>TG 10 Ten-plus Combinations</td>
<td>111</td>
<td>165</td>
</tr>
<tr>
<td>TG 11 Large Doubles</td>
<td>12</td>
<td>25</td>
</tr>
<tr>
<td>TG 12 One-digit Additions</td>
<td>0</td>
<td>45</td>
</tr>
</tbody>
</table>

Considering Tables 5.12 and 5.13 together, these results indicate that both the frequencies of correct responses and the use of grouping strategies increased from the pre- to the post-assessment for addition problems presented as bare number tasks.

5.2.4 Comparison of Frequencies of Counting-based Strategies with Grouping-based Strategies Across Five Tasks

Tables 5.14 and 5.15 display the frequencies of response types across Task Groups 3, 9, 10, 11 and 12 in the pre- and post-assessments respectively. The bands categorise the strategies used by the students to solve each task. A dash indicates that, in the case of Task Groups 3, 9 and 10, Band 0 was not included as a response option. In the case of Task Group 12, the dash indicates that this task was not presented to any students in the pre-assessment (for reasons described in Section 5.1.12). Only the task 9 + 4 (Task Group 3) is presented as two screened collections; the other four task groups all describe simple additions presented in bare number format.

Table 5.14
Frequencies of Response Bands for Five Task Groups in the Pre-assessment

<table>
<thead>
<tr>
<th>Task Group</th>
<th>Band 0</th>
<th>Band 1</th>
<th>Band 2</th>
<th>Band 3</th>
<th>Band 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>TG3 9+4 as Screened Collections</td>
<td>–</td>
<td>14</td>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>TG9 Small Doubles</td>
<td>–</td>
<td>35</td>
<td>3</td>
<td>2</td>
<td>60</td>
</tr>
<tr>
<td>TG10 Ten-plus Combinations</td>
<td>–</td>
<td>67</td>
<td>1</td>
<td>21</td>
<td>111</td>
</tr>
<tr>
<td>TG11 Large Doubles</td>
<td>64</td>
<td>19</td>
<td>5</td>
<td>0</td>
<td>12</td>
</tr>
<tr>
<td>TG12 One-digit Additions</td>
<td>180</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td><strong>TOTAL FREQUENCY</strong></td>
<td><strong>224</strong></td>
<td><strong>135</strong></td>
<td><strong>11</strong></td>
<td><strong>26</strong></td>
<td><strong>184</strong></td>
</tr>
</tbody>
</table>
Tables 5.14 and 5.15 show the comparatively low frequency of Band 2 responses to these five tasks. Recall that Band 1 responses describe students who answered incorrectly or had to build perceptual replacements to solve the task, and Band 2 responses described students’ counting-by-ones from one to solve the task. Band 3 responses described students’ counting-on-by-ones to solve the task, and Band 4 responses accounted for those students who used grouping strategies. These results suggest that, although some students in this study were either unable to solve the task or used their fingers to support them in doing so, most students used a counting-on or grouping strategy to solve addition tasks. These results are discussed in Section 6.1.1.

These results show a general increase in the use of grouping strategies by many students across a range of tasks from the pre- to the post-assessment. In Chapter Six, the results from these 12 task groups are discussed in terms of students’ advancements in the frequency of correct results and mathematical sophistication of strategy use from the pre- to the post-assessment, and the aspects of the teaching sequence which contributed to this development.
Chapter Six: Discussion – Students’ Progression on the 12 Assessment Task Groups

The aim of the teaching sequence in this study was to encourage the use of grouping strategies in preference to count-by-ones strategies to solve addition tasks involving two addends in the range 1 to 10. In this chapter, results of students’ pre- and post-assessments as a class group are described, and an analysis of these results for evidence of changes in the frequencies of correct responses to solve addition tasks is presented. Episodes from the teaching experiment reveal examples of student or researcher behaviour that provide insight into the learning progressions of the whole class, a group of students or individual students. Evidence of cognitive reorganisation manifested in the advancement of the mathematical sophistication of strategies used by students to solve addition tasks is sought through consideration of vicissitudes in the frequencies of correct responses between the pre- and post-assessments.

Wright, Martland, and Stafford (2006) describe this cognitive reorganisation as “advancements in which children reorganise their numerical thinking and construct novel strategies that, in a mathematical sense, are more sophisticated than their previous strategies” (p. 52). Recall that this study seeks to answer five research questions re-stated below:

1. What levels of student knowledge are prerequisite to the efficient use of grouping strategies to solve addition tasks involving two addends in the range 1 to 10?

2. How does a teaching focus on grouping strategies influence students’ methods of solving simple addition tasks?
3. To what extent is it appropriate and useful to introduce formal arithmetic notation to students in the first year of school who are reasoning additively in the material settings of quinary- and ten-based materials?

4. To what extent does knowledge of the part-part-whole structure of numbers support the use of grouping strategies to solve addition tasks involving two addends in the range 1 to 10?

5. What role does a teaching focus on visualisation have in supporting the use of grouping strategies to solve simple addition tasks?

In this chapter, the results from Chapter Five are described within the context of the literature review, taking account of these five research questions. Because the results of some assessment task groups provide more pertinent data to particular research questions than others, not all task groups are discussed in response to each research question.

6.1 What Levels of Student Knowledge Are Prerequisite to the Efficient Use of Grouping Strategies to Solve Addition Tasks Involving Two Addends in the Range 1 to 10?

The results from this study indicate that some prerequisite knowledge and skills may assist students to efficiently use grouping strategies to solve simple addition tasks. The first two task groups of the assessments conducted in this study measured students’ knowledge of the FNWS in the range 1 to 35 and students’ ability to identify the numerals 1 to 20. Task Group 1 assessed students’ ability to recite the FNWS to 35 from a variety of starting numbers. As described in Section 5.1.1, in the pre-assessment, all 20 students could correctly recite the sequence to 16; 17 students could correctly recite the sequence to 24; and 12 students could correctly recite the sequence to 35. In the post-assessment, all 20 students could recite the sequence correctly to 24, and 18 students could recite the sequence correctly to 35.
Task Group 2 assessed students’ ability to identify numerals in the range 1 to 20. As shown in Section 5.1.2, in the pre- and post-assessments, all students correctly identified all numerals in the range 1 to 10. In each assessment, there was a total of 200 responses identifying numerals in the range 11 to 20. Figure 5.2 indicates that there was a total of 190 correct responses given in the pre-assessment and a total of 188 correct responses given in the post-assessment. As described in Section 5.1.2, the disaggregated results show that one student was not able to correctly identify the numerals 12, 13, 17, 15, 14 and 16 in the post-assessment, yet had correctly identified eight numerals (13, 17, 15, 14, 16, 18, 11, 20) in the pre-assessment. Numeral identification was not an explicit focus in the teaching sequence; however, whilst engaging in tasks, students had many opportunities to read and write numerals in the range 1 to 20. Interestingly, this was the only task group in which the frequency of correct results decreased from the pre- to the post-assessment, and one of only two task groups which did not have an explicit focus in the teaching sequence. Whilst acknowledging a decrease in the frequency from the pre- to the post-assessment, it is still reasonable to conclude that, at the commencement of this study, most students identified most numerals in the range 11 to 20.

Task Groups 4 and 5 assessed students’ ability to ascribe number to dot patterns on a five-frame, and to identify the partitions of five on a five-frame partition card. Section 5.1.4 shows that, in both the pre- and post-assessments, all students correctly ascribed number with certitude to all dot patterns on a five-frame, with the exception of one student for the pattern of four dots. Similarly, Section 5.1.5 shows that all students correctly represented the partitions of five from an unscreened five-frame in the pre-assessment. In the post-assessment, 18 students correctly represented the partitions of five from a flashed five-frame.
Based on previous pertinent literature (Carpenter & Moser, 1984; Groen & Parkman, 1972; Steffe & Cobb, 1988; Wright, 1998) it was clear that Task Groups 1, 2, 4 and 5 had been shown as necessary preliminary knowledge to successfully complete other aspects of the assessments and participate productively in the teaching sequence. Results from these Task Groups in Chapter Five agree with this. Therefore, it is reasonable to assume that knowledge of the FNWS to 20, the ability to identify numerals in the range 1 to 20, and the ability to visualise numbers in the material setting of a five-frame were advantageous when using grouping strategies to solve simple addition tasks presented in this study.

Task Groups 9 and 11 required students to solve small (1 + 1 to 5 + 5) and large (6 + 6 to 10 + 10) doubles tasks presented in bare number format. These tasks were included in the pre-assessment to assess the students’ knowledge of doubles as a grouping strategy to solve addition tasks. As indicated by Figure 5.16, in the pre-assessment, many students were incorrect in their attempts to solve the small doubles tasks, and students typically either knew the answer immediately or were unable to solve the task. In the post-assessment, no student used count-by-ones from one (Band 2) or count-on-by-ones (Band 3) strategies to solve a small doubles task.

Recall that Figure 5.20 shows that, in the pre-assessment, excluding the double 10 + 10, when students were asked to find the sums of the large doubles presented in a bare number format, there were only three (out of a possible 80) correct responses, and in the post-assessment, there were 12 correct responses.

When the assessments were first developed and implemented, the teaching sequence had only been constructed in the form of a hypothetical learning trajectory (Gravemeijer, 2004a; Simon, 1995). Prior to the commencement of the teaching cycle, the hypothetical learning trajectory included the study of doubles and small doubles. As the 24
lesson teaching sequence was enacted, it became apparent that, whilst the use of doubles is a valid grouping strategy to solve addition tasks, students were responding more positively to a focus on quinary- and ten-based strategies (Murata, 2004; Verschaffel et al., 2007; Wright, Martland, Stafford, & Stanger, 2006).

Accordingly, the use of doubles did not feature prominently in the teaching sequence and only three of the 24 lessons involved students working with doubles. As a group, the students had only limited knowledge of small and large doubles at the beginning and conclusion of the teaching cycle, and therefore knowledge of doubles is not considered prerequisite knowledge for students’ use of quinary- and ten-based grouping strategies to solve simple addition tasks.

On face value, the findings from this study appear to be in contrast with the findings of Groen and Parkman (1972) and Ashcraft (1997) (described in Section 2.2.2), which found that doubles or ties are easier to retrieve from memory than the other addition combinations. However, this contrast can be explained as Groen and Parkman’s (1972) research focused on the use of reproductive rather than reconstructive processes to solve these addition combinations.

In summary, the results from Task Groups 1 and 2 described above indicate that knowledge of the FNWS and identification of numerals in the range 1 to 20 are skills which provide a solid base for learning about simple addition tasks. However, as indicated by the pre-assessment results discussed above, 17 out of 20 students had these skills in place prior to the commencement of the teaching sequence, yet not all of these students progressed to using grouping strategies to solve simple addition tasks. Therefore, whilst these skills may be prerequisite for solving addition tasks, they cannot be regarded as an indicator of readiness to do so.
6.1.1 Counting-based Strategies as Prerequisites for Using Grouping Strategies to Solve Addition Tasks

Section 2.3.2 in the literature review described five theoretical frameworks which account for student progression through the counting stages as they advance towards the use of more flexible and efficient strategies to solve simple addition tasks. Each of these frameworks outlines the progression of students from counting all items by ones to counting-on by ones. In the frameworks described by Carpenter and Moser (1984), Steffe and Cobb (1988) and Wright (1998), the counting stages have the most sophisticated level for using efficient non-counting-based strategies to solve simple addition tasks, whether presented using concrete materials or at the level of formal arithmetic.

The progression of skills described in these frameworks implies that students need to have mastered the use of count-by-ones from one and count-on-by-ones strategies to solve addition tasks in order to successfully use grouping strategies to solve simple addition tasks. However, the low frequencies of Band 2 responses in the post-assessment for Task Groups 3, 9, 10, 11 and 12 presented in Table 5.15 indicate that this may not necessarily be the case. Absence of the use of count-by-ones from one strategies to solve 9 + 4 presented as two screened collections, small doubles and ten-plus presented as bare number tasks in the post-assessment, and very low frequencies of this strategy to solve large doubles and other bare number tasks, suggest that students may not need to count-by-ones from one as a prerequisite skill to using efficient grouping strategies to solve simple addition tasks.

The suggestion that students may not need to progress through each of the counting stages to develop facile addition skills leads to the consideration of an alternative sequence of learning to that described in the three counting-based theoretical frameworks. Sections 6.6 and 6.7 below describe an alternative path to developing efficient addition
strategies. The following section describes episodes from the teaching sequence which highlight a focus on grouping strategies in preference to counting-based strategies. These examples are indicative of the teaching that occurred throughout the sequence and may help to explain the higher frequencies of grouping strategies compared with counting-based strategies used by students in the post-assessment.

The material setting of screened collections of counters does not have an inherent grouping structure, and is therefore assumed more likely to engender the use of count-by-ones strategies. For this reason, the material setting of screened collections of counters was only ever used during Part 1 (the warm up) of lessons. This allowed judicious use of questioning by the researcher to guide discussion which encouraged the use of grouping strategies to solve tasks. Whole class exploration of strategies to find the sum of addition tasks presented in the setting of two screened collections occurred three times during the teaching cycle, in Lessons 10, 17 and 23. When addition tasks were presented in the setting of screened collections, the second addend was always within the range 1 to 4. This range is considered to be appropriate “because that is the range in which it is considered useful for children to become facile at keeping track of counting” (Wright, Martland, Stafford, & Stanger, 2006, p. 109). If the second addend is greater than four and a count-by-ones strategy is used, the necessity of long counts increases the likelihood of errors. Similarly, if the second addend is greater than four, a grouping strategy such as build-through-ten can be used.

In Lesson 10, the explicit focus of Part 1 of the lesson was on fostering the use of a count-on strategy to determine the total number of counters in two collections. In previous lessons, a significant proportion of the class group was observed using a count-by-ones from one strategy to solve addition tasks. As an interim step to using a more mathematically advanced grouping strategy, it was considered desirable to have the
students at least begin to use a count-on-by-ones strategy. To encourage this, the two addends were presented as collections of counters; the first collection was flashed and then screened, and the second addend was left unscreened. The first task presented was $5 + 2$. One student stated this as a five-plus number combination (i.e., $5 + 2 = 7$) immediately, correctly and with certitude. The next three tasks presented were $8 + 2$, $12 + 3$, and $15 + 2$. These three tasks were solved by three different students, but for each task the solution process was explained to the class via a count-on-by-ones strategy. The task $5 + 2$ was re-presented at the end of the session. In this instance, a different student counted-on-by-ones from five.

In whole class discussions, students were encouraged to use count-on-by-ones as a preferred strategy if the sum of the two addends was not a number combination for which they could recall the result immediately and with certitude. In the protocols described in this chapter, the text in italics is a direct record of an utterance made by the researcher or the student. The text in square parentheses describes a physical action on the part of the researcher or the student. In the following protocol from Lesson 10, the whole class focus on solving the tasks $5 + 2$ and $8 + 2$ presented as two screened collections is described.

Protocol 1 (Lesson 10):

R: *I have five counters* [flashes screened collection] *and I’ve got two more here*

[indicates two unscreened counters]. *How many have I got altogether?*

St: Seven.

R: *How do you know?*

St: *Because five and two make seven.*

R: *They do. We’ve got five* [lifts screen to display five counters] *and we know there’s five, and we can think six, seven* [touches each counter in displayed collection in coordination with count] *if we need to. I wonder if this works another way? Let’s try this* [screens eight counters as the first addend]. *I’ve got eight counters hiding under there* [indicates screen] *and I’ve got two more here. If I’ve got eight counters here*[indicates screen] *and I’ve got two more, how many altogether?* [Pause for five seconds] *Mary?*
St: Ten!

R: Why do you think it is 10?

St: Because there’s eight under there [indicates screen] and one more makes nine and then another one makes 10.

R: Fantastic!

The pedagogical approach described above involves using dialogue between the researcher and the whole class to build upon natural strategies articulated by the students, in order to emphasise the use of particular strategies for solving tasks. Cobb et al. (2001) write about this approach for solving tasks:

> When we take the local classroom community rather than the discipline as our point of reference, a practice is seen to be an emergent phenomenon rather than an already-established way of reasoning and communicating into which students are to be inducted. (p. 120)

In Protocol 1, the emergent phenomenon is the use of a count-on-by-ones strategy to solve a screened collections task. The first student described in the protocol had automatic knowledge of the five-plus number combination, and this was recognised. However, the researcher was aware that this was not the case for many of the students in the class and therefore also chose to have a student demonstrate the use of a count-on-by-ones strategy, as described, as an interim method of solving the task. An emergent phenomenon (Cobb et al., 2001) of this classroom community, articulated in many whole class discussions, was that the use of count-on-by-ones was considered to be an appropriate strategy to solve addition tasks in the early stages of the teaching sequence. However, as the sequence progressed and students became more familiar with grouping strategies to solve addition tasks, this strategy type came to be highly valued and regarded as the most mathematically advanced strategy for solving addition tasks.

Addition tasks in the setting of screened collections were presented on two other occasions in the teaching sequence – both during Part 1 of Lessons 17 and 23. These activities were of five minutes’ duration and involved the use of flashed canonical dice
patterns rather than collections of counters. Students were asked to state the total number of dots in the two collections. In Lesson 17, the first addend, which was flashed, was always the dice pattern for five, with the second addend a dice pattern for another number in the range 1 to 4. The addition tasks solved were $5 + 2$, $5 + 4$, $5 + 3$, and $5 + 1$. Most students who volunteered responses indicated that they “just knew” the total number of dots because they knew their five-plus number combinations. The presumption is that this was due to previous five-plus activities completed in the setting of a ten-frame, and this knowledge had come to be “taken-as-shared” (Cobb & Whitenack, 1996; McClain, 2002) by the classroom community. One student described how he added five and two, “because I remembered that there was five and then I counted-on by five and I went two, and I went plus two equals seven”. Although he describes his strategy as count-on-by-ones, the rest of his explanation indicates that it is more likely that he knew this result as a five-plus number combination.

Addition tasks in the setting of flashed canonical dice patterns were used again in Lesson 23, but the first addend was not always five. The tasks solved by individual students as part of a whole class discussion were $4 + 1$, $5 + 2$, $4 + 2$, $4 + 3$, $6 + 2$, and $5 + 3$. One student described using the partitions of five to solve $4 + 1$. Students described using a count-on-by-ones strategy to solve the tasks $5 + 2$, $4 + 3$, and $6 + 2$. Other students described using five-plus number combinations to solve the tasks $5 + 2$ and $5 + 3$. In Protocol 2 below, Rupert’s use of a grouping strategy to solve $4 + 3$ is described.

Protocol 2 (Lesson 23):

R: What about this one? Are you ready? [Simultaneously flashes dice pattern for four and dice pattern for three] What do you think Rupert?
Ru: Seven!
R: Why do you think it is seven?
Ru: ’Cos I remembered that three, but then there’s [pause for one second] … ’cos there’s four and one more equal five, and then two more equal seven.
R: Great explaining Rupert, thank you.
In the following paragraphs, Rupert’s strategy is examined in more detail. To solve this task, he partitioned the number three into one and two, used the one to build-to-five and then used his knowledge of the five-plus number combination, to determine that \( 5 + 2 = 7 \). Rupert does this in the setting of screened canonical dice patterns, which do not have an inherent five structure. By this stage of the teaching sequence, Rupert was very familiar with five-plus tasks in the setting of a ten-frame. Thus, Rupert was able to apply his knowledge of five-plus number combinations to assist him to solve this task, in the setting of screened canonical dice patterns.

A key feature of this teaching sequence was a strong focus on the use of grouping strategies to solve addition tasks in material settings with an inherent five and ten structure. Throughout the 24 lessons (each of approximately one hour’s duration), there were only three five-minute teaching episodes that focused on the use of count-on-by-ones in the setting of screened collections and canonical dice patterns. As mentioned previously, greater importance is placed on the change in strategy use from the pre- to the post-assessment to solve the task \( 9 + 4 \) than \( 4 + 2 \). With this in mind, it seems reasonable to assert, from the pre- and post-assessment results described above, that this strong focus on the use of grouping strategies in the teaching sequence resulted in an advancement of the strategies used to solve addition tasks involving two screened collections.

Episodes from Lessons 10, 17 and 23 of the teaching sequence that were designed to encourage the use of a count-on-by-ones strategy to solve addition tasks in the setting of a screened collection have been described above. The episode in Lesson 17 featured an increase in the complexity of the tasks compared with Lesson 10, and the episode in Lesson 23 featured an increase in complexity of the tasks compared with Lesson 17. The first episode used the material setting of individual counters in a screened collection, with the second addend unscreened. The second episode used the setting of
flashed canonical dice patterns printed on cards. The dice patterns were a more formal setting for the addends because they were no longer represented by individual counters which could be manipulated. Thus, it seems likely that the dots printed in the canonical dice pattern of five encouraged students to recognise them as a group of five from which they could count-on to find the total of the two addends rather than five individual items.

Again, in the third episode in Lesson 23, both addends were represented by canonical dice patterns on cards and were flashed. However, in this episode, the first addend was not always five. Again, it seems likely that, as in Lesson 17, presenting the addends as canonical dice patterns on cards encouraged students to “trust the count” (Willis, 2002) and to regard this first addend as an abstract composite unit, rather than a numerical composite (Steffe & Cobb, 1988) made up of iterable units (Olive, 2001). This consideration of the first addend as a group is commented on by Willis (2002): “Learning to ‘see’ numbers as groups, enables children to come to ‘trust’ a number as signifying a quantity that does not change as a result of counting differently, or rearranging parts, or indeed rewriting in a different form” (p. 123).

The example from Protocol 2 of Rupert using a grouping strategy to solve $4 + 3$ occurred in Lesson 23 of the 24 lesson sequence. In the pre-assessment, to solve $4 + 2$ involving two screened collections, Rupert counted-by-ones from one, building perceptual replacements to facilitate his count, and he was incorrect in his attempt to solve $9 + 4$. In the post-assessment, Rupert counted-on-by-ones to solve correctly both addition tasks $4 + 2$ and $9 + 4$ (Task Group 3) in the setting of two screened collections. This count-on-by-ones strategy is not considered to be at as high a level of mathematical sophistication as Rupert previously demonstrated in Lesson 23. Therefore, it seems reasonable to conclude that he may have used this count-by-ones strategy out of convenience. In summary, Rupert advanced in the sophistication of his strategy use to solve $9 + 4$ presented as two screened
collections, from a Band 0 strategy in the pre-assessment to a Band 3 strategy in the post-assessment. He also advanced in the sophistication of his strategy use to solve 4 + 2 presented as two screened collections, from a Band 0 strategy in the pre-assessment to a Band 3 strategy in the post-assessment. Rupert is one example of a student who did not use a Band 2 count-by-ones from one strategy in either assessment and was not observed using this strategy throughout the teaching cycle. However, although he was not observed offering a Band 2 type response at any stage, it cannot be assumed that he never used this type of strategy throughout the teaching experiment.

As well as being presented in the material setting of two screened collections, the task 9 + 4 was also presented in bare number format (Task Group 12) in the post-assessment. To solve this task, Rupert used a count-on-by-ones strategy again, and used his fingers to keep track of the four counts. When asked if the use of ten-plus number combinations might assist him to solve this task, he was able to correctly solve the task using this strategy, but he did not use this strategy spontaneously. Rupert’s change in strategy use from the pre-assessment to the post-assessment was indicative of a trend in the results of the whole class, as shown by the low frequencies of count-by-ones from one, Band 2 responses presented in Table 5.15.

The results presented in Table 5.15, supported by the examples from the teaching sequence, indicate that being able to count-by-ones from one to solve addition tasks is not a prerequisite to solving simple addition tasks using grouping strategies. This contrasts with the progression of learning through the counting stages described by the three theoretical frameworks of Carpenter and Moser (1984), Steffe and Cobb (1988) and Wright (1998).
6.2 How Does a Teaching Focus on Grouping Strategies Influence Students’ Methods of Solving Simple Addition Tasks?

In this section, discussion focuses on the influence of learning activities designed to promote the use of grouping strategies on students’ solution strategies. This influence is considered with regard to the pre- and post-assessment results reported in Chapter Five, and episodes from the teaching sequence. As described in Sections 2.3.3 and 2.3.4 of the literature review, for students in their first year of school, the mathematics curriculum typically has a focus on the use of counting-based strategies to solve addition tasks. Appendix D describes the sequence of activities enacted in the implementation of the teaching sequence. The materials used in most lessons had an inherent five or ten structure, which when used judiciously reduce the need to use counting-based strategies to solve addition tasks (Labinowicz, 1985).

6.2.1 Progression in Students’ Use of Grouping Strategies

Conducted in a framework of design research (Bannan-Ritland, 2003; Collins et al., 2004; Gravemeijer et al., 2000), this teaching experiment began with a hypothetical learning trajectory (Gravemeijer, 2004a; Simon, 1995). This trajectory was then modified and adapted in response to conceptual analysis (Von Glasersfeld, 1995) of the cognitive reorganisations made by students as the teaching sequence was enacted. Therefore, this constituted a conjectured local instruction theory (Gravemeijer, 2004a). For instance, doubles featured as an important part of the hypothetical trajectory for using grouping strategies, but the focus on doubles was greatly reduced in the enactment of the teaching sequence for the reasons outlined in Section 6.1.

A key feature of this teaching sequence was the progressive development of students’ knowledge. As the students mastered one topic of conceptual knowledge, it was used as a building block for the next area of knowledge. For example, as described above,
five-plus knowledge was used to support students’ reasoning about the partitions of 10. As the teaching sequence progressed, it was no longer necessary for students or the researcher to describe the pattern for eight dots on a ten-frame as being comprised of five dots and three dots. Rather, the class accepted this as a known fact, it had become “taken-as-shared” (Cobb & Whitenack, 1996; McClain, 2002). According to McClain (2005): “An indication that a classroom practice has been established is that explanations about that practice are no longer necessary; it is beyond justification” (p. 95).

The conjectured local instruction theory (Gravemeijer, 2004a) described the first stage of the teaching sequence as student exploration and guided reinvention (Freudenthal, 1991) around the partitions of five. Once students started to consolidate this knowledge, the numbers 6, 7, 8, 9 and 10 were considered as five-plus number combinations. For example, eight was considered as five plus three. Next, the focus was on partitions of 10, incorporating five-plus knowledge and the use of visualisation strategies (see Section 6.5 for discussion).

In the next stage of the teaching sequence, focus shifted to working with 10 as a base number, similar to the prior focus on five as a base number. Once partitions of 10 were established, the focus shifted to teen numbers as ten-plus number combinations. Finally, the focus shifted to the strategy of using known number combinations (partitions of five, five-plus, partitions of 10, ten-plus) to add through 10 (i.e., solving tasks such as 9 + 5 by adding through ten). This involved using material settings of ten- and twenty-frames. As described above, only three lessons focused on the small doubles (1 + 1 to 5 + 5) and the large doubles (6 + 6 to 10 + 10). Whilst the hypothetical learning trajectory and the enacted learning sequence were developed for the classroom community as a whole, student exploration through the process of guided reinvention (Freudenthal, 1991) was mediated (Askew, 2013). Therefore, whilst the local instruction theory was designed with the whole
class group in mind, there were times during the teaching sequence when at least two sub-
groups of the whole class were consolidating different concepts, skills and levels of
knowledge, and the tasks were adapted accordingly.

The material settings used to support the use of grouping strategies were quinary- or
ten-based (Murata, 2004; Verschaffel et al., 2007, Wright, Martland, Stafford, & Stanger,
2006). These included five-frames, ten-frames, bead strings with 10 beads arranged in two
sets of five beads of the same colour, twenty-frames, the arithmetic rack and unifix cubes
arranged in two sets of five cubes of the same colour (each of these material settings is
described in full in Appendix B).

In the assessments, the material setting of a ten-frame pattern card was used in
Task Group 6 and a ten-frame partition card was used in Task Group 8. As described in
Chapter Five, the purpose of these task groups was for students to ascribe number to a dot
pattern on a ten-frame and to make an equation for the partition of 10 represented by the
dots on the ten-frame respectively. In the pre-assessment, the ten-frames were displayed for
the students to see; in the post-assessment, they were flashed.

As described in Section 5.1.6, all students could ascribe number to dot patterns in
the range 1 to 5 on the pre- and post-assessments. In the pre-assessment, at least 15 students
ascribed number to dot patterns in the range 6 to 10, and in the post-assessment at least 17
students were correct. The only exception was for the five-wise pattern of eight dots, which
was correctly ascribed by only 14 students.

The high frequencies of correct responses to the task of representing the
partitions of 10 as bare number equations with the first addend in the range 6 to 10 were
reflective of the high frequencies of students who correctly ascribed number to ten-frame
pattern cards in this range in Task Group 6. In the post-assessment, at least 17 students
were correct in representing the partitions 6 + 4, 5 + 5, 9 + 1, 8 + 2, and 7 + 3 as equations.
However, the results were different for partitions of 10 with the first addend in the range 2 to 4. Eighteen students correctly represented the partition 1 + 9 as an equation, and 13, 11 and 15 students correctly represented the partitions 4 + 6, 2 + 8, and 3 + 7 respectively. Interestingly, in Task Group 6, eight was the number of dots students were least likely to correctly ascribe number to, and 2 + 8 was the partition that the least number of students correctly represented as a bare number equation in Task Group 8.

These results indicate that students have more difficulty recalling and representing the partitions of 10 as bare number equations when the first addend is smaller than the second addend. This accords with the finding that “addition-task combinations … of the smaller-addend-first variety … are more difficult than the larger-addend-first variety for children who use abstract (verbal) counting procedures but who do not yet disregard addend order” ((Baroody et al., 2003, p. 132). Whilst Task Group 8 was not posed in the sense of a standard, bare number addition task, the students were expected to represent the relationship between the partitions at the level of formal arithmetic and, consistent with Baroody et al.’s findings, they did find these tasks more difficult when the first partition was smaller than the second partition.

A possible explanation for why students were more successful finding the partitions of 10 with the first addend in the range 5 to 10 is that they were able to ascribe number to this first addend quickly, because it was represented in the material setting of a ten-frame. They may then have used their knowledge of partitions of 10 to complete the bare number equations. Many times throughout the teaching sequence, students were encouraged to use their familiarity with the five and 10 structure of the material setting to support a reasoning strategy to find “how many more to make 10?” This is illustrated in the following protocol from Part 1 of Lesson 9 in the teaching sequence in which the task was posed in the setting of an arithmetic rack.
Protocol 3 (Lesson 9):
R: I’m going to make a number on my rack and I want to know how many beads you can see. Are you ready? [Researcher shows arithmetic rack with 7 beads (5 blue and 2 yellow) at the left hand end of the upper row]
St: Seven!
R: Danielle, how do you know?
St: Because 5 and 2 make 7.
R: Magnificent! If that’s 7, how many are left down this end here? [Researcher indicates the remaining beads at the right hand end of the upper row]
St: Ten!
R: How many are left down this end? You’re right, there are 10 altogether, how many yellow beads can you see?
St: Three!
R: Three, you’re right! We’ve got 7 [indicates 7 beads] and we’ve got 3 [indicates 3 beads], we can write 7 and 3, we can say that makes 10, because I’ve got 10 beads altogether, made up of 7 and 3 [ten as made up of 7 and 3 is recorded informally as described in Figure 4.1]

The protocol above illustrates that students used their knowledge of five-plus combinations to determine the first addend, and then reasoned in the setting of the arithmetic rack to determine how many more to make 10. In this example from Lesson 9 the beads of the second addend were visible, however, in later lessons when the second addend was screened, students could reason that if there were two yellow beads visible as part of the seven, there must be three yellow beads remaining, as two and three are a partition of five. Recalling that the average age of these students during the teaching sequence was just over six years, it is encouraging to note that, for Task Group 8 in the post-assessment, at least five students gave a Type IV response, which involved the use of a grouping strategy to represent the more difficult smaller-addend first partitions of 10, 3 + 7, 1 + 9, 4 + 6, and 2 + 8 as bare number equations. Similarly, at least eight students gave a Type IV response, which involved the use of a grouping strategy to represent the larger-addend first partitions of 10, 6 + 4, 9 + 1, 8 + 2, and 7 + 3.
Task Group 10 required students to state the sum of ten-plus number combinations when they were presented in bare number format. The post-assessment results for this task group provide another example of students’ use of grouping strategies to solve simple addition tasks. In the post-assessment, 18 students correctly stated the sum of all ten-plus number combinations. At least 15 of these students used a grouping strategy to solve all of the ten-plus tasks. In a similar manner to the protocol described for Lesson 9 above, the arithmetic rack was used in conjunction with expression cards from 10 + 1 to 10 + 10 in Part 1 of Lessons 12, 13 and 14. In Lesson 12, the 10 beads on the upper row and the beads on the lower row were unscreened for the students, and the researcher posed the task as follows: “I have 10 beads on the top, and 3 beads on the bottom, how many beads do I have altogether?” The phrase “ten-plus” came to be taken-as-shared (Cobb & Whitenack, 1996; McClain, 2002) within the classroom community to describe these number combinations. During Part 1 of Lesson 13, the researcher wrote a ten-plus combination on the board as a bare number task (e.g., 10 + 3). Students were asked if they knew the answer to this, and then to prove it by sliding beads on the arithmetic rack in a ten-plus formation. In the whole class setting, most students were able to respond immediately with the ten-plus combination and check it by arranging the beads with 10 on the upper row, and the remaining number of beads on the lower row. After all ten-plus combinations had been made and connected with a bare number expression, the researcher used an arithmetic rack to pose the task 3 + 10.

Protocol 4 (Lesson 13):
R: *If I have 3 on the top* [slides 3 beads to the left hand end of the upper row of the arithmetic rack], *and 10 on the bottom* [slides 10 beads to the left hand end of the lower row of the arithmetic rack], *how many do I have altogether?*

St: Ah, 13!

R: *Is it still 13? How come?*
St: *Because, you’ve still got the 10 but it’s not up there* [indicates upper row of arithmetic rack] *but you’ve still got the 3, so you will still have 13!*

R: *Instead of having 10* [slides 10 beads to the left hand end of the upper row of the arithmetic rack] *and 3* [slides 3 beads to the left hand end of the lower row of the arithmetic rack], *you have 3* [slides 3 beads to the left hand end of the upper row of the arithmetic rack], *and 10* [slides 10 beads to the left hand end of the lower row of the arithmetic rack].

St: *It’s a turnaround!*

This protocol illustrates that after two lessons the students related ten-plus expression cards with the arrangement of beads on the arithmetic rack in both the form of 10 + ? and ? + 10. Students stated the sum of two addends, when either addend was 10, without using counting strategies. In Lesson 14, the students were shown an expression card of the form 10 + ? or ? + 10 and asked to state the sum. They then checked their responses by moving beads on the arithmetic rack. As described above, the results from Task Group 10 indicate that instruction in the use of grouping strategies can reduce students’ need to use counting to solve addition tasks, and encourage them to perceive addition tasks as binary operations (Baroody et al., 2003).

A teaching focus on the use of grouping strategies influenced the ways students solved simple addition tasks, as the inherent 5 and 10 structure of the material settings supported the use of grouping strategies but still allowed for the use of counting strategies. In Part 2 of Lesson 22, the tasks 9 + 5, 9 + 7, and 9 + 3 had been presented previously to the whole class. Representing 9 with counters on the top ten-frame and 5 with counters on the bottom ten-frame was accepted practice by the students; there was no need for discussion, this was taken-as-shared (Cobb & Whitenack, 1996; McClain, 2002). As shown in Figure 6.1 below, the dots represent counters. In accordance with the notion of guided reinvention (Freudenthal, 1991), students proposed moving one counter from the bottom frame to the top frame to make 10. This was then identified as 10 + 4 and solved via knowledge of the
ten-plus fact. The tasks $9 + 7$ and $9 + 3$ were solved by the whole class in a similar way.

The fourth task presented in the setting of two ten-frames was $9 + 6$ (see Figure 6.1).

![Figure 6.1 9 + 6 Presented in the Material Setting of Two Ten-frames](image)

Also available to the students was a set of expression cards ranging from $9 + 1$ to $9 + 10$, and a set of expression cards ranging from $10 + 1$ to $10 + 10$. Protocol 5 is taken from Part 2 of Lesson 22.

Protocol 5 (Lesson 22):
R: *How many on the top frame?*
St: *Nine.*
R: *How many on the bottom frame?*
St: *Six.* [Unprompted, the student collects expression card that reads $9 + 6$ and places it next to the two ten-frames]
R: *Could you have a guess at how many counters there are altogether on both frames?*
St: [Looks at card and counters] *Hmmm, 15?*
R: *How are you going to check?*
St: *Put one up the top.*
R: *Yes, you do that.*
St: [Moves one counter from bottom frame to fill top frame, as indicated by the arrow on Figure 6.1] *R: How many dots now altogether?*
St: *15.*
R: *Why is it 15?*
St: *Because I know that’s 5 [student sweeps hand across upper row of top frame] and now that’s 10 [student indicates lower row of top frame with upper row of top frame] and that’s 5 [indicates upper row of bottom frame] and if you have 10 and you put 5 there it makes 15.*
R: Good thinking. You made 10 on the top frame. Can you find the card for us?

St: [Collects $10 + 5$ expression card]

R: So we know $10 + 5$ is 15, so what is $9 + 6$?

St: 15!

As illustrated in the above protocol, this student used the structure of the ten-frame to re-arrange $9 + 6$ into $10 + 5$. As will be discussed in Section 6.5, the student was encouraged to visualise and predict what the solution might be before checking to see if he was correct by manipulating the counters on the ten-frames. Self-checking by manipulation of the ten-frames was used repeatedly throughout this study.

The student described in the protocol above used a grouping strategy to solve the task described. However, one student in the class group was observed using a count-all strategy and others were observed using a counting-on strategy to solve this task. Thus, although the material setting is designed to encourage the use of a grouping strategy, some students used a counting strategy.

These results suggest that students who were cognitively ready to learn, and had the prerequisite skills described in Section 6.1, can be influenced by a teaching focus on grouping strategies to solve addition tasks in the material setting of two ten-frames, and at the level of formal arithmetic. The strong emphasis on the use of five and 10 as a base number (Van den Heuvel-Panhuizen, 2008; Verschaffel et al., 2007; Wright et al., 2012) during the teaching cycle may provide an explanation for this success. The role of five as a base number and the judicious selection of material settings for this study are described in the following section.

6.2.2 The Role of Material Settings in Effecting Strategy Use

In this section, the effect of material settings in influencing students’ use of grouping strategies to solve simple addition tasks is discussed, and the phenomenon of individual students using different strategies to solve tasks presented in different material
settings is considered (Cowan et al., 2011). For example, some students successfully solve addition tasks using a grouping strategy in one material setting but do not use this same strategy in another material setting.

In the pre- and post-assessments of this study, the task 9 + 4 was presented in two ways – as two screened collections and as a bare number task. Figure 5.3 in the results chapter shows that the frequencies of students who correctly solved the task 9 + 4 presented as two screened collections increased from seven students in the pre-assessment to 13 students in the post-assessment. Figure 5.5 shows that nine students used a count-on-by-ones strategy, and four students used a grouping strategy. These results indicate that, for students who were cognitively ready, strategy use increased in terms of mathematical sophistication. In contrast, seven students were still unable to correctly solve the task 9 + 4 in the post-assessment. When considering this in terms of the SEAL (see Table 2.1) (Wright, Martland, & Stafford, 2006), none of the students in the cognitively ready group was observed using Stage 2 – Figurative Counting strategies – to solve addition tasks involving two screened collections. Thus, in the context of the curriculum documents analysed in Section 2.3.3, the strong emphasis on the use of grouping strategies to solve addition tasks in a range of settings in this study has resulted in students progressing relatively quickly to count-on-by-ones strategies from Stage 0 strategies. This approach has encouraged them to see the first addend as a numerical composite (Steffe, 1992), and thus use a count-on-by-ones rather than a count-from-one-by-ones strategy.

Figure 5.28 shows that, of the seven students who gave a Band 4 response and used a grouping strategy to solve 9 + 4 presented as a bare number task, only three gave a Band 4 response and used a grouping strategy to solve 9 + 4 when it was presented as two screened collections. The other four students used a Band 3 count-on-by-ones strategy. Therefore, to solve the same addition task presented in different settings, four students used
a high level strategy, in terms of mathematical sophistication, and a low level strategy within the same assessment interview. This suggests that the material setting of screened collections is more likely to engender the use of a lower level count-by-ones strategy than the setting of formal arithmetic. Therefore, it seems reasonable to assume that these students who had been taught using a grouping strategy to solve addition tasks were more likely to use a grouping strategy to solve a bare number task than the same task presented in the setting of two screened collections.

Also interesting to note is that, in the post-assessment, three of the four students who gave a Band 4 response by using a grouping strategy to solve $9 + 4$ presented as two screened collections also used Band 4 strategies to solve $9 + 4$ at the level of formal arithmetic. Therefore, it seems reasonable to assume that students who use a grouping strategy to solve tasks presented as screened collections will generally also use a grouping strategy approach to solve addition tasks at the level of formal arithmetic, but the reverse is not true. In other words, it is not possible to presume that the use of a grouping strategy to solve addition tasks at the level of formal arithmetic indicates that the same student would use a grouping strategy to solve a screened collections task. This indicates that some students solve addition tasks using a grouping strategy in one material setting but do not use this same strategy in another material setting.

6.3 To What Extent is it Appropriate and Useful to Introduce Formal Arithmetic Notation to Students in the First Year of School Who Are Reasoning Additively in the Material Settings of Quinary- and Ten-based Materials?

In this section, this research question is discussed, whilst considering the pre- and post-assessment results reported in Chapter Five, episodes from the teaching sequence and reference to the literature review. Bare number tasks, that is, tasks presented in bare number format notation, were the only mode of presentation for Task Groups 9, 10, 11 and
12 of the assessments. The reader is referred back to the summary of the frequencies of
correct responses for Task Groups 9, 10, 11 and 12 presented in Table 5.12. Task Group 12
was not presented to any students in the pre-assessment, as no students had been
sufficiently successful on previous tasks. Each of these results from Table 5.12 shows an
increase in the frequencies of correct responses from the pre- to the post-assessments,
suggesting that it was appropriate for students in their first year of school to be introduced
to formal arithmetic notation for addition tasks. In order to determine whether students are
reasoning additively, it is also useful to summarise the frequencies of Band 4 responses
which indicated the use of grouping strategies for each of the Task Groups 9, 10, 11 and 12
in the pre- and post-assessments (see Table 5.13 for these results). Considering Tables 5.12
and 5.13 together, many of the students in the first year of school who participated in this
study are reasoning additively to solve simple addition tasks when presented as bare
number tasks, and therefore the introduction of formal arithmetic notation appears to be
both appropriate and useful.

Section 3.4.1 described the development of an anticipatory thought experiment
(Gravemeijer, 2004a) from which a local instructional theory and hypothetical learning
trajectory evolved. This process was also followed with regard to the introduction of formal
arithmetic notation in this teaching experiment. In the following paragraphs, specific
episodes from the teaching sequence are analysed with reference to the enactment of this
planned trajectory.

Protocol 3 above, from Lesson 9, described the use of informal notation similar
to that shown in Figure 4.1 as a means of recording seven and three as partitions of 10.
Protocol 3 also described the use of pre-made expression cards in Lesson 22 to connect the
physical model of ten-plus number combinations in the material setting of the arithmetic
rack with the formal arithmetic notation of 10 + ? This approach was also used in the
context of partitions of five, five-plus number combinations, partitions of 10, small doubles and large doubles.

Throughout the first 12 lessons of the teaching sequence, formal and informal arithmetic notation was used interchangeably to encourage students to regard addition tasks as binary rather than unary operations (Baroody et al., 2003). For instance, students regarding $5 + 3$ as five and three combining to make eight, rather than regarding it as three added onto five. Von Glasersfeld (1982) described his use of the term “and” rather than the formal addition symbol in order to emphasise a “figural joining” and not the numerical operation of addition. The interchangeable use of informal and formal numerical symbols for addition served to support students in using grouping strategies to solve addition tasks presented in bare number format in this teaching sequence. For example, the formal written symbol for addition was used to record the arrangement of dots on a five-frame described by the students in arithmetic notation. This was an emergent phenomenon (Cobb et al., 2001) that arose naturally from the language students used to articulate what they saw when the five-frame was unscreened, in the first instance, and then visualised after it had been flashed. The protocol below describes the way formal arithmetic was used to notate a relationship presented in a material setting, from the beginning of the teaching sequence. In this episode from Lesson 5, formal mathematical notation is used to describe a partition of five. A five-frame with one dot and four empty spaces was shown to the class.

Protocol 6 (Lesson 5):
R: *Yesterday we matched these frames with sums, today we want to write them ourselves. How could I write like a mathematician for this one?*
St: *Um, 1 plus 4 makes 5.*
R: *[Records $1 + 4 = 5$] Good thinking, 1 plus 4 makes 5. Who could tell me what would the frame look like for this sum?* [Records the sum $3 + 2 = 5$ on the board]
St: *Um, that would have 3 dots and 2 spaces.*
R: *What does the 5 mean?*
St: There are 5 dots and spaces altogether.

This formal recording did not appear to be a source of perturbation for any students and when asked to read a bare number task such as $5 + 3$ they used the terms “five plus three” and “five and three” interchangeably.

6.4 To WhatExtent Does Knowledge of the Part-part-whole Structure of Numbers Support the Use of Grouping Strategies to Solve Addition Tasks Involving Two Addends in the Range 1 to 10?

Historically, counting strategies have been taught to students as the preferred means to solve addition tasks in the early stages of their arithmetical learning since “many mathematics educators see counting as the first step towards more advanced mathematical understanding” (Young-Loveridge, 2002, p. 36) and “counting is the means through which the child penetrates addition” (Maclellan, 2010, p. 76). In recent years there has been discussion about the merits of grouping strategies as an alternative to counting strategies to solve addition tasks. Willis (2002) notes that, in Western Australia, there are students of indigenous background and non-indigenous children of graziers who are reported by teachers as not yet having learnt to count. She describes their numerical knowledge as being considerably helped by activities which focus on simply “seeing” the three and the four in seven (and of course the two and the five in seven and the one and the six) … We now understand that, while counting is important, it is insufficient to enable children to develop a sense of number as a representation of quantity. (p. 123)

Whilst acknowledging counting as an important skill, Bobis (1996) also supports the idea of using a grouping, part-whole approach to the teaching of number:

Counting a set of say five objects by ones does not help children recognise that the same set of objects can be decomposed into three and two or four and one. Interpreting number in terms of part-whole relationships makes it possible for children to think about number as compositions of other numbers. (p. 20)

In a similar way, results from this study suggest it is advantageous for students to consider numbers in terms of their part-whole components as a support for the use of
sophisticated, non-counting strategies to solve addition tasks: “Part-whole reasoning supports children’s mathematical learning through the elementary school and beyond” (Hunting, 2003, p. 231). As part of the local instruction theory developed prior to the commencement of the teaching sequence, informal recording as shown in Figure 4.1 was expected to emphasise the part-part-whole structure of number. As described in Section 6.3, interchanging informal recording and formal arithmetic notation throughout the teaching sequence was expected to support students to regard bare number tasks as involving binary rather than unary operations (Baroody et al., 2003).

Throughout this study, the use of five-wise arrangements of dots on ten-frames provided a rich quinary-based setting in which students developed their part-whole knowledge of numbers in the range 1 to 10. Hatano (1982) described how Japanese students mastered additive strategies via an alternative path to the counting stages. This belief is further supported by Sarama and Clements (2009a):

Some studies have reported that Japanese children move from early, more primitive counting strategies, to more sophisticated strategies such as composition and decomposition of numbers without progressing through a long use of counting strategies … This may have to do with the language and a variety of cultural and instructional supports for using five as an intermediate anchor. (p. 110)

Five is described above as an intermediate “anchor” (Ontario Ministry of Education, 2005; Sarama & Clements, 2009a), and this resonates with the importance attributed to five as a key “base” number in this teaching sequence. Once the visual arrangements from one to five on a ten-frame came to be taken-as-shared (Cobb & Whitenack, 1996; McClain, 2002), five was used as a base to represent the quinary structure of the numbers from 6 to 10. Similarly, Warren et al. (2009) report that non-random patterns on ten-frames assist students to quickly ascertain the number of dots. Numbers such as eight were presented in terms of their relationship to five, that is, as an
arrangement of five dots and three dots. As described by Von Glasersfeld (1982), the use of material settings such as ten-frames “makes the fact that a ‘six’ is a ‘five’ and a ‘one’, a ‘seven’ is a ‘five’ and a ‘two’, etc. perceptually accessible [emphasis in original]” (p. 207).

During this teaching sequence, the term “turnaround” came to be taken-as-shared (Cobb & Whitenack, 1996; McClain, 2002) by the classroom community of learners. This term was used to describe the commuted representation of an addition task. In Protocol 4 above, students were able to articulate that 10 + 3 and 3 + 10 resulted in the same sum, even though they were presented in both the material setting of the arithmetic rack and as bare number tasks. However, very few students moved from identifying turnarounds in this way to using this knowledge to assist them in solving addition tasks. Baroody et al. (2003) highlight students’ potential difficulties in using commutativity to solve simple addition tasks when they state with reference to the order-irrelevance principle: “There is no guarantee that children would immediately recognize its applicability to the moderately new (and somewhat different) task of addition” (p. 147).

In Task Group 8, although the tasks were presented using ten-frame partition cards, this lack of ability to use commutativity was evident, when 3 + 7 was presented immediately after 7 + 3. Two students stated that 3 + 7 was the turnaround of 7 + 3 but did not give a Type IV response which indicated the use of a grouping strategy to solve the task. One of these students (Rupert) used a count-on-by-ones strategy, and the other made the equation incorrectly as 3 + 8 = 10. Rupert’s strategy in this instance could be described as Level 0 and Level 1 in Baroody et al.’s (2003) Model of Commutativity Development. Even though he identified 3 + 7 as the turnaround of 7 + 3, he did not use this knowledge to assist him to represent the partition as a bare number equation. According to Baroody et al., at Level 0, students “may interpret, for example, 5 + 3 as five and three more and 3 + 5 as three and five more – as different situations with different outcomes [emphasis in original]”
At Level 1, “children intuitively disregard addend order because it minimizes computational effort and seems to work” (p. 152). Rupert described the task as a “turnaround” (Level 1) but then interpreted it as a different task (Level 0) to the partition 7 + 3 as he used a count-on-by-ones strategy to solve it. Thus, students operating at Level 0 and Level 1 use counting-based strategies to solve addition tasks. At Level 2 of Baroody et al.’s Model of Commutativity Development, students have “unary conception” and “pseudocommutativity” (p. 152). They are aware that tasks such as 5 + 3 and 3 + 5 describe different situations, even though they result in the same sum.

Weaver (1982) cautions that whilst expressions such as 7 + 2 = 2 + 7 appear to exemplify commutativity, they actually do not. He states that in the example of unary operations above “+2” and “+7” are not the same operation. “This ‘pseudocommutativity’, as I choose to term it, is a valid property, but not about an operation” (p. 62). Therefore, students’ ability to represent the partitions of 10: 7 + 3 and 3 + 7, as bare number equations is only indicative of their knowledge of pseudocommutativity. As described by Sarama and Clements (2009a): “They recognize that both outcomes are the same but do not necessarily transform the unary to a binary conception” (p. 114). It is only when students are aware of the part-part-whole structure of number, and have a binary conception of addition tasks, that commutativity is fully understood: “The parts three and five are interchangeable in forming the whole eight” (Baroody et al., 2003, p.152). A review of the literature regarding addition as a binary or unary operation can be found in Section 2.5.3.

Students who correctly solved tasks at the level of formal arithmetic by using grouping strategies could be described as using part-whole operations. Steffe and Cobb (1988) refer to these students as using “operative strategies”: “We use the term ‘operative’ to indicate that the children coordinated arithmetic symbols without involving actual or represented counting” (p. 272).
Steffe and Cobb (1988) describe part-whole operations as those operations made possible by disembedding an abstract composite unit from a containing composite unit. They include comparing the disembedded unit to the containing unit, exhaustively disembedding two abstract composite units from a containing unit while remaining aware of the containing unit, and combining the two disembedded abstract composite units to form the containing composite unit. (p. 339)

In this study, one example of disembedding an abstract composite unit from a containing composite unit arises in finding the partitions of five. In Lesson One, students explored all partitions of five by randomly dropping five, double-sided counters onto the table and arranging counters with the upperface of the same colour to form partitions on five-frames. Once the double-sided counters were placed on the five-frames, students were able to ascribe number to determine the frequency of each colour without having to count-by-ones. The students then recorded their findings. In the plenary session of the lesson, the researcher collated the students’ findings and recorded them using the informal notation depicted in Figure 6.2.

![Figure 6.2 Informal Recording of the Partitions of Five](image)

As shown above, the researcher recorded the partitions in a systematic way and, once the students noticed that some partitions included the same addends in the reverse order (indicated by the horizontal blue arrows), the term “turnaround” was introduced. As described in this episode from Lesson 1, the concept of the part-part-whole structure of
numbers was developed in a context that, from the beginning of the teaching sequence, did not require students to use counting strategies.

6.5 What Role Does a Teaching Focus on Visualisation Have in Supporting the Use of Grouping Strategies to Solve Simple Addition Tasks?

Section 2.3.2 in Chapter 2 described five theoretical frameworks which inform the advancement in complexity of students’ use of counting strategies to solve addition tasks. Two of these frameworks were informed by the work of Les Steffe: Stages in the Construction of the Number Sequence (Steffe & Cobb, 1988), and Stages of Early Arithmetical Learning (SEAL) (Wright, 1998). In both frameworks, Stage 3 describes students as using count-on strategies to solve addition tasks. For students at Stage 3, it is no longer necessary for them to count-by-ones from one to establish the numerosity of the first addend. Section 6.1.1 proposed the view that when students are supported to use grouping strategies by working in material settings with an inherent grouped structure, such as five- and ten-frames, they might not use count-by-ones from one as a strategy to solve simple addition tasks.

Olive (2001) summarises Steffe’s description of the cognitive reorganisation that takes place as students’ strategies move from Stage 1 to Stage 3 of the construction of the number sequence to solve an addition task presented as two screened collections. He describes re-presentation as the cognitive process of mentally repeating counting-by-ones from one in order to determine the total number of items in two collections, until this process becomes redundant and students no longer need to count-from-one (count all), and will count-on from the first addend.

Children who need to count all can be encouraged to mentally re-present their counting acts by covering the prior counted collection when more objects are added. These children will often attempt to visualise the objects under the cover, pointing at the cover while recounting the covered objects. Such activities will help them to at
first *internalize* (make mental representations of) their counting acts, and eventually *interiorize* the results of those counting acts: the result of counting a collection is not only “out there” but is also symbolized mentally by the *interiorized* number word, that now carries with it the records of the experience of counting. The child has constructed an abstract number sequence in the sense that each element of the child’s number word sequence can now stand for the sub-sequence of counting acts that results in that number word [emphasis in original]. (Olive, 2001, p. 6)

Similarly, Steffe and Cobb (1988) use the term empirical abstraction to describe a student’s progress to count-on-by-ones via re-presentation of the counting act:

Empirical abstraction concerns a perceptual (sensory) experience and results in a template that serves to recognize further experiences as similar or equivalent to the past one; eventually the abstracted template turns into a concept and can be re-presented as an internalized item without the presence of the sensory material in actual perception. (p. 333)

R. J. Wright (personal communication, November 14, 2011) also describes the term “re-presentation” as a student’s use of mental imaging to “play through a movie in their mind” of the counting action they had previously completed.

The relatively low frequencies of Band 2 responses in the pre- and post-assessments of this study were evident in Tables 5.14 and 5.15. Recall that Band 2 strategies describe the use of count-by-ones from one. These low frequencies have led to the consideration of an alternative description of the cognitive reorganisation of students performing these tasks. Instead of students coming to construct an abstract composite unit (Steffe & Cobb, 1988) through repeated acts of counting items by ones, a teaching approach such as the one used in this study encourages students to construct a number such as six as an abstract composite unit through repeated experiences with canonical dice patterns and/or five-plus arrangements of dots on a ten-frame.

In the context of using a grouping strategy to solve addition tasks, the act of visualisation corresponds with the act of re-presentation (Olive, 2001) or empirical abstraction (Steffe & Cobb, 1988). Students who have been taught via a teaching sequence
with a strong emphasis on grouping and visualisation strategies may establish a template of a visual pattern, rather than a template of a counting procedure from which they internalise the first addend as an abstract composite unit (Steffe & Cobb, 1988).

In the early stages of the teaching sequence, students participated in many activities in the material setting of ten-frames which encouraged visualisation of the five-plus structure of numbers in the range 1 to 10. They used “ten-frames to visualise addition combinations” (Clements, 1999, p. 404). In Lesson 13, the explicit teaching focus was on adding two one-digit numbers which result in a sum greater than 10. As described in Protocol 5 in Section 6.2.1, ten-frames were used to support a strategy of build-through-ten by representing nine on one ten-frame and the second addend on another ten-frame. Initially, many students chose to go through the process of physically moving one counter from the second ten-frame to “make a 10” on the first ten-frame, and then using their ten-plus number combinations to state the total of counters on the two frames. As students became more advanced in their thinking, judicious screening and flashing were used to encourage students to mentally represent the physical action and visualise their strategy to solve the task. Once many students mastered this task type in the material setting of the ten-frames, the task was presented in bare number format and students checked their solution by unscreening the material setting and physically re-creating the strategy. As noted in Section 2.4.6, the phrase “distancing the setting” (Wright et al., 2007) describes this pedagogical strategy, designed to elicit the use of mental strategies to solve tasks. Hunting (2003) highlights the importance of not only visualising images, but also fostering students’ ability to visualise the use of strategy to solve a task: “A major cognitive tool at work … seemed to be an ability to visualize, not just static configurations, but sequences of actions, when outcomes of such actions were hidden from view” (p. 231).
In the same way that Steffe and Cobb (1988) used the phrase “empirical abstraction”, and Wright et al. (2007) used the phrase “distancing the setting”, Olive (2001) used the phrase “interiorisation of activities”:

Children may do things in action first what they are not yet able to do mentally. The interiorization of activity is a process of reflective abstraction (Von Glasersfeld, 1995). The activity is first internalized through mental imagery; the child can mentally re-present the activity. This mental re-presentation still carries with it the contextual details of the activity. The activity becomes interiorized through further abstraction of these internalized re-presentations whereby they are stripped of their contextual details. (p. 4)

Task Group 12 was presented in bare number format in order to assess each student’s ability to solve these tasks in the absence of a material setting. Whilst in the post-assessment a number of students were able to solve tasks such as build-through-ten in the absence of a setting, students were also observed using visualisation to support their solution strategies. One possibility is that these students had not yet interiorised the activity and still needed the support of the contextual details of the setting to solve the task. Frank’s visualisation of ten-frames to assist him in solving the task $9 + 5$ presented in bare number format is described in Protocol 7.

Protocol 7 (Post-assessment):
R: What about that one? [Places bare number task presented on card on the table]
F: 9 plus 5 [pause for 1 second] 14!
R: [Nods] Good, what did you do for that one?
F: Ah, I counted on.
R: No you didn’t!
F: [Smiles] Nah, it was tricky. I, I pretended there was like a [indicates rectangular shape on the table about the size of an A4 sheet] um a page and then there was $9$ here [making circular movements at the top of the imagined page] and then you put one up [indicates moving a counter to the top of the page] a counter up the top.
F: So then it would make 14.
R: You built to ten!
When Frank uses hand movements to indicate an A4 piece of paper and the movement of a counter to the top of the page, it is likely that he is re-creating the task referred to in Protocol 5 in Section 6.2.1, and using visualisation strategies to solve the addition task.

The examples referred to from the teaching sequence and the post-assessment suggest that a focus on visualisation strategies can play an important role in supporting the use of grouping strategies to solve simple addition tasks. A strong focus on visualisation strategies may reduce the need for students to move through all counting levels as they acquire mastery of efficient, non-counting strategies to solve addition tasks. The following section describes an alternative path for students as they learn to master counting-on and grouping strategies to solve simple addition tasks. This alternative path features a strong focus on visualisation strategies.

6.6 A Grouping-based Progression to Part-part-whole Strategies

In this section, a hypothetical learning progression based on grouping strategies is proposed. According to this progression, students’ learning advances from perceptual counting to part-part-whole operative strategies. This grouping-based progression is an alternative to the counting-based progression referred to as the SEAL (Wright, Martland, & Stafford, 2006) which is described in the theoretical frameworks outlined in Section 2.3.2. This alternative progression incorporates a strong focus on the use of visualisation strategies.

As described in Sections 4.1.7.2 and 4.1.7.3, a key feature of this teaching sequence was the explicit linking of ascribing numbers to the part-part-whole configuration of dots on a ten-frame or beads on an arithmetic rack with formal mathematical notation denoting this as an addition task. Therefore, because of this linking, when presented with a
task at the level of formal arithmetic, students visualised the relevant configuration and thus addition was a binary operation (Baroody et al., 2003) for them.

For example, in Lesson 7, students matched five-plus bare number expressions with the corresponding configuration on an arithmetic rack. An example of an expression card and the corresponding configuration appear in Figure 6.3.

![Figure 6.3 Expression Card for 5 + 3 and the Corresponding Configuration on an Arithmetic Rack](image)

Similarly, when presented with a task such as 9 + 6 as a bare number task (which 11 students solved correctly in the post-assessment), as described in Protocol 5, students might have linked the expression 9 + 6 with a mental image of nine dots on one ten-frame and six dots on another ten-frame. Thus visualisation enabled a build-through-ten strategy.

As shown in Tables 5.14 and 5.15, and described in Section 6.5 above, the aggregated results of Task Groups 3, 9, 10, 11 and 12 all show very low frequencies of the use of count-by-ones from one strategies to solve tasks. These results indicate that a focus on the use of grouping strategies during the teaching sequence expedited the students’ use of Stage 3 counting-on strategies, followed by Stage 5 operative strategies (Steffe & Cobb, 1988) to solve addition tasks at the level of formal arithmetic.

With reference to the curriculum documents reviewed in Chapter Two, it is likely that students in this class group progressed more quickly to counting-on due to the strong focus on grouping strategies, continual distancing of the setting, a strong focus on visualisation encouraged by the researcher and close alignment with formal arithmetic.
notation. Students appear to have followed a different progression to the counting-based model to reach Stage 3 where students count-on-by-ones to solve additive tasks.

Treacy and Willis (2003) proposed subitising and part-whole understanding as an alternative progression for students to see numbers as symbolising quantities. Figure 6.4 shows their alternative path.

![Figure 6.4 Model of Early Number Development (Treacy & Willis, 2003)](image)

Treacy and Willis (2003) acknowledged that counting is not the only means by which students interpret numbers as quantities and proposed this model after a meta-analysis of the work of many researchers. Their model proposes an alternative progression for students learning to see numbers as representations of quantities by developing both counting and grouping strategies.
Ginsburg et al. (1998) also proposed two ways that young children quantify a group of objects – one being counting and the other subitising. However, the alternative model being hypothesised in this study serves to advance students further than quantifying a collection of objects. This alternative model is hypothesised to advance students from being able to quantify two collections with numerosities in the range 1 to 10, either through counting or subitising, to being able to add in the setting of two collections without the need to count-by-ones from one. Initially, some students may need to count-on, to determine the sum, but once this is mastered they are ready to progress to using operative strategies (Steffe & Cobb, 1988) to solve addition tasks in the range 1 to 10 and then 1 to 20. A proposal for this alternative model and the settings used to engender students’ grouping strategies are described in Table 6.1.

6.7 Comparison of Counting-based and Grouping-based Progressions to Solve Addition Tasks

In this section, two alternative models are contrasted. These models describe alternative progressions which students might follow in their advancement towards the use of counting-on and operative strategies to solve addition tasks (Steffe & Cobb, 1988). The counting strategies model is based on the SEAL (see Table 2.1) (Steffe, 1992; Wright, 1994; Wright, Martland, & Stafford, 2006) and the grouping strategies model is a progression of PEGS to solve addition tasks. PEGS is a learning progression of grouping strategies to solve addition tasks which evolved out of the enactment of the anticipatory thought experiment and local instruction theory (Gravemeijer, 2004a) developed prior to the teaching sequence in this study. As the pre-planned model for learning was realised in a class of students in their first year of school, it was monitored, assessed and adapted according to the reactions, responses and evidence of cognitive reorganisation demonstrated by the students. This cyclical monitoring and adaptation emerged as the six phases in
PEGS. In their current form, these phases describe the learning progression followed by students in their first year of school, in this classroom community. Thus, as a result of this study, PEGS can be considered a viable progression through which students develop their use of grouping strategies to solve addition tasks, in the first year of school. Table 6.1 shows a comparison of phases in early counting strategies with phases in early grouping strategies, to solve simple addition tasks. The counting strategies model is adapted from the phases of progression through the SEAL (Wright, Martland, Stafford, & Stanger, 2006, p. 19).

Table 6.1
Comparison of Phases in Early Counting Strategies (adapted from Wright, Martland, Stafford, & Stanger, 2006, p. 19) and Phases in Early Grouping Strategies (PEGS) to Solve Addition Tasks

<table>
<thead>
<tr>
<th>Phases in Early Counting Strategies to Solve Addition Tasks</th>
<th>Phases in Early Grouping Strategies (PEGS) to Solve Addition Tasks</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Phase 1:</strong> Emergent Counting to Perceptual Counting</td>
<td><strong>Phase 1: Numbers in range 1 to 5</strong></td>
</tr>
<tr>
<td></td>
<td>a. Student ascribes number to canonical patterns on five-frames or dice patterns using a count-by-ones strategy</td>
</tr>
<tr>
<td></td>
<td>b. Student ascribes number to canonical patterns on five-frames or dice patterns using grouping strategies</td>
</tr>
<tr>
<td><strong>Phase 2:</strong> Perceptual Counting to Figurative Counting</td>
<td><strong>Phase 2: Partitions of five</strong></td>
</tr>
<tr>
<td></td>
<td>a. Student states partitions of 5 by using a count-by-ones strategy for either or both partitions in setting of a five-frame</td>
</tr>
<tr>
<td></td>
<td>b. Student states partitions of 5 using grouping strategies in setting of a five-frame</td>
</tr>
<tr>
<td></td>
<td>c. Student solves partitions of 5 tasks at the level of formal arithmetic using grouping strategies</td>
</tr>
<tr>
<td><strong>Phase 3:</strong> Figurative Counting to Counting-on and Counting-back</td>
<td><strong>Phase 3: Five-plus number combinations</strong></td>
</tr>
<tr>
<td></td>
<td>a. Student counts-on from 5 to determine number in range 6 to 10 in setting of five-wise ten-frame</td>
</tr>
<tr>
<td></td>
<td>b. Student ascribes number in range 6 to 10 using grouping strategies in setting of five-wise ten-frame</td>
</tr>
<tr>
<td></td>
<td>c. Student solves five-plus tasks at the level of formal arithmetic using grouping strategies</td>
</tr>
<tr>
<td><strong>Phase 4:</strong> Counting-on and Counting-back to Part-part-whole knowledge</td>
<td><strong>Phase 4: Ten-plus number combinations</strong></td>
</tr>
<tr>
<td></td>
<td>a. Student counts-on from 10 to determine number in range 11 to 20 in setting of five-wise twenty-frame</td>
</tr>
<tr>
<td></td>
<td>b. Student uses grouping strategies to ascribe number in range 11 to 20 in setting of five-wise twenty-frame</td>
</tr>
<tr>
<td></td>
<td>c. Student solves ten-plus tasks at the level of formal arithmetic using grouping strategies</td>
</tr>
<tr>
<td></td>
<td><strong>Phase 5: Partitions of numbers in range 6 to 10</strong></td>
</tr>
</tbody>
</table>
### Phases in Early Counting Strategies to Solve Addition Tasks

<table>
<thead>
<tr>
<th>Phase 5: Add-through-tens</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. Student counts-by-ones to determine either or both addends of partitions of ten in setting of five-wise ten-frame</td>
</tr>
<tr>
<td>b. Student uses grouping strategies to determine the partitions of ten in the setting of five-wise ten-frame</td>
</tr>
<tr>
<td>a + b = 10 a\geq b; then a + b = 10 a&lt;b</td>
</tr>
<tr>
<td>c. Student counts-by-ones to determine the partitions of 6, 7, 8 and 9 in the setting of a ten-frame</td>
</tr>
<tr>
<td>d. Student uses grouping strategies to determine partitions of 6, 7, 8 and 9 in the setting of a ten-frame</td>
</tr>
</tbody>
</table>

**Phase 6: Build-through-ten**

<table>
<thead>
<tr>
<th>Phase 6: Build-through-ten</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. Student uses a count-by-ones strategy to solve addition tasks in setting of twenty-frame</td>
</tr>
<tr>
<td>b. Student uses a grouping strategy to solve addition tasks in setting of twenty-frame</td>
</tr>
<tr>
<td>c. Student uses a count-by-ones strategy to solve addition tasks at level of formal arithmetic</td>
</tr>
<tr>
<td>d. Student uses a grouping strategy to solve addition tasks at level of formal arithmetic</td>
</tr>
</tbody>
</table>

In the progression of learning referred to as PEGS, addition tasks are divided into three sets: sums in the ranges 1 to 5, 1 to 10, and 1 to 20 which are indicated by gradations in shading in Table 6.1. As described in Section 2.3.2, Steffe (1992) attributed Piaget’s notion of cardinality to the student who is able to use a count-on strategy, and Phase 3 of the counting model describes students as progressing from figurative counting to counting-on. Students who are at Phase 3 of PEGS are learning about the quinary structure of numbers in the range 6 to 10. Phase 3 is significant in both models, as learning to count-on is important conceptual knowledge (as described by Steffe), and also supports students’ knowledge of numbers in the range 6 to 10 as five-plus number combinations.

In the following paragraphs, the alternative counting and grouping models for mastery of early addition as described above are compared and contrasted in terms of the following six characteristics:
1. Learning progression: The counting model adapted from SEAL describes a research-based learning progression of counting-based strategies to solve addition tasks, whereas PEGS describes a learning progression of grouping strategies to solve addition tasks.

2. Material settings: The material settings used to present addition tasks in the two models are contrasted.

3. Formal arithmetic notation: The models are contrasted with regard to the emphasis placed on reading and recording addition tasks at the level of formal arithmetic.

4. Addition and subtraction: The counting model adapted from SEAL describes a progression in the advancement of strategies used by students to solve addition and subtraction tasks; PEGS focuses only on addition tasks.

5. Number range: PEGS refers only to addition tasks involving two addends in the range 1 to 10, whereas the counting model does not specify a number range.

6. Phases: Phases in PEGS are not invariant, in contrast with the counting-based progression based on the stages of early arithmetical learning in which each stage is characterised by significant cognitive reorganisation.

These six characteristics are described in more detail.

1. SEAL was described by Wright, Martland, and Stafford (2006) as a progression of learning developed from research involving longitudinal studies in which children were taught several times per week during teaching cycles of up to 20 weeks duration, in their first and/or second years of school. An important focus of this research was the ways in which children’s numerical strategies arose during enquiry-based teaching, and how these strategies developed and changed over the course of one or two years of school. (p. 52)

Therefore, it can be seen that SEAL is a progression in the sophistication of counting strategies used to solve arithmetic tasks, developed from the observation of
student behaviour in teaching, learning and assessment environments. In contrast, PEGS is a learning progression for solving addition tasks by using grouping strategies. PEGS describes students’ progression from each stage of SEAL. Inclusion of count-by-ones strategies as the first step of Phase 1 through to Phase 6 of PEGS (denoted by italics in Table 6.1) is necessary because students may use these as intermediate strategies to solve problematic tasks. Nevertheless, these strategies are not the explicit learning intention of any teaching. This is a similar approach to that taken by Murata (2004): “Although it was communicated and understood by students that they are to group numbers to use the break-apart-to-make-ten (BAMT) method, counting was considered and used as a fallback for the learning process to support steps in the BAMT method” (p. 202).

2. A significant difference between the two learning progressions is in the material settings used to engender the respective strategies. An assessment of a student’s SEAL typically involves addition and subtraction tasks presented as collections of counters, which may be screened or unscreened (Wright, Martland, & Stafford, 2006). In contrast, collections of counters are not a material setting that is used as part of the PEGS model for learning to solve addition tasks. Unscreened or screened collections are made up of individual counters, which are likely to encourage the use of a count-by-ones strategy. PEGS is designed as a learning progression for grouping strategies in preference to counting strategies, therefore, the settings used in this model are specifically chosen to support these strategies. As previously described, addition tasks in the PEGS model are divided into three sets, that is, addition tasks working with numbers in the range 1 to 5, 1 to 10, and 1 to 20. Strategies to solve addition tasks within these number ranges are supported by settings with an inherent grouping structure of five (quinary-based) and ten (ten-based), predominantly using five- and
ten-frames. Whilst students are working in the range 1 to 5, most activities are presented in the material setting of a five-frame. As students consolidate their knowledge of partitions of five, the material setting of a ten-frame is introduced to support grouping strategies in the range 1 to 10. Similarly, as the partitions of 10 are being consolidated, the material setting of a twenty-frame or an arithmetic rack is introduced to support the use of grouping strategies in the range 1 to 20. Whilst in the PEGS model it is acknowledged that students may use count-by-ones strategies to solve addition tasks regardless of the material setting, the learning progression focuses on the use of grouping strategies.

The assessment of a student’s SEAL was described as involving the material setting of unscreened or screened collections of counters. This use of screening is also a key feature of PEGS. Within each phase, the material setting is presented as unscreened, providing an opportunity for student–teacher discussion and manipulation by students. As competence develops, the material setting is flashed to encourage visualisation strategies until finally it is screened from view. This process of “distancing the setting” (Wright et al., 2012) is described in Section 2.4.6.

3. The importance of reading and recording tasks at the level of formal arithmetic throughout each of the PEGS is contrasted with the phases adapted from SEAL model. Connection of the formal written notation with the material setting representing the addition task occurs concurrently during each of the stages (unscreened, flashed and screened) of PEGS. So, when a student describes eight dots on a ten-frame as five dots on the upper row and three dots on the lower row, this is recorded as $5 + 3 = 8$. This ongoing, reflexive relationship between solving addition tasks using a grouping approach and the written recording of the addition tasks appears to facilitate the students’ transition to solving addition tasks at the level of formal arithmetic. By the
conclusion of Phase 6 in PEGS, students use a grouping strategy to solve any addition task involving two addends in the range 1 to 10, presented in bare number format. The use of formal arithmetic notation is not described for any of the Phases of Early Counting Strategies, adapted from the SEAL model (Wright, Martland, Stafford, & Stanger, 2006).

4. Phase 3 of the Early Counting Strategies model adapted from SEAL describes the progression in the advancement of strategies used by students to solve addition and subtraction tasks, while PEGS focuses only on addition tasks. The scope of this study restricted the teaching cycle to explore the learning progression for addition tasks only, and, as a result, PEGS has been devised as a model of students’ use of grouping strategies to solve addition tasks. There is, however, scope for further research to extend PEGS as a sequence which includes grouping strategies to solve subtraction tasks in the range 1 to 20. If tasks are presented in similar material settings, once students are familiar with the use of grouping strategies to solve addition tasks, these strategies could be generalised to support the use of grouping strategies to solve subtraction tasks.

5. Phase 4 of the Early Counting Strategies model adapted from SEAL refers to students using a range of grouping strategies to solve problematic addition and subtraction tasks, without stating a numerical range. Phase 6 in PEGS describes students using a build-through-ten strategy to solve addition tasks involving two addends in the range 1 to 10. This limitation of PEGS to addition tasks in the range 1 to 20 is deliberate. Once students establish facile, robust strategies in this range, with effective teaching and place value knowledge, these strategies are easily generalised to solve addition tasks beyond 20. Students are encouraged to finetune strategies used to solve addition tasking involving two addends in the range 1 to 10 and combine these with their
knowledge of place value to solve addition tasks in the range beyond 20. They do not need to learn new mental computation skills and strategies. Verschaffel et al. (2007) highlighted this notion of generalising a strategy: “Further strong motivation for adopting this [make-a-ten] approach is that it lays a strong conceptual foundation for two-digit addition and subtraction in general, and for the understanding of place value in relation to multidigit addition and subtraction” (p. 612). For example, when solving 8 + 5, students may use the build-through-ten strategy:

\[ 8 + 2 = 10 \rightarrow 10 + 3 = 13. \]

When their strategy is mediated through effective questioning (Askew, 2013) and they draw on their knowledge of the place value system, students can modify a strategy used in the range 1 to 20 for use in the range 1 to 100 to solve a task such as 38 + 5. The student builds to the next decuple (40) by generalising the strategy used to build-through-ten. This is described below.

\[ 38 + 2 \rightarrow 40 + 3 \rightarrow 43 \]

Once students have successfully learnt to modify the build-to-ten strategy for use in the range 1 to 100, the teaching focus shifts to include the addition of tens and ones as the next incremental step in the progression towards facile addition in the range 1 to 100. For example, to solve 38 + 25

\[ 38 + 2 = 40 \rightarrow 40 + 3 = 43 \rightarrow 43 + 20 = 63 \]

or

\[ 38 + 20 = 58 \rightarrow 58 + 2 = 60 \rightarrow 60 + 3 = 63 \]

These examples highlight that, for students to be able to apply efficient mental addition strategies when working with two-digit numbers, it is critical that they develop strategies to solve addition tasks involving two addends in the range 1 to 10
that are facile, generalisable and robust in the early stages of their mathematical development.

6. The two learning models described in Table 6.1 can be contrasted with regard to the way students’ progress through the phases. In the PEGS model, the phases are not necessarily in lock-step order (Wright et al., 2012). Teaching activities presented in the same lesson may address more than one phase of the learning progression, as students move towards facile additive strategies in the range 1 to 20. For instance, Part 1 of a lesson may focus on consolidating student knowledge of partitions of five, whilst Parts 2 and 3 of the lesson focus on guided reinvention (Freudenthal, 1991) of numbers in the range 6 to 10 in the form of five-pluses. However, the first three phases of the counting-based model describe students progressing through each of Stages 1 to 3 of SEAL in order. Wright, Martland, and Stafford (2006) reference the work of Steffe (1983) in defining the term “stage” as used in SEAL. They defined a stage as having four characteristics:

1. A characteristic remains constant for a period of time.
2. The stages form an invariant sequence.
3. Each stage builds on and incorporates the previous stage.
4. Each new stage involves a significant conceptual reorganisation (p. 52).

Therefore, the model for Phases in Early Counting Strategies defines a phase as moving from stage to stage in an invariant sequence whereas, as described above, the phases of the learning progression described in PEGS are more flexible. The approach of students working through phases “as though they understand addition” (Askew, 2013, p. 7) was a feature of PEGS enacted as a learning progression in this study.

Students who use strategies at the highest level of sophistication of both learning progressions could be described as facile and robust at finding solutions to problematic
addition tasks. As noted above, the two models describe alternative paths that students might take in their advancement towards the use of operative strategies (Steffe & Cobb, 1988) to solve addition tasks. The cumulative collection of skills developed through the six phases of early grouping strategies results in students solving tasks using strategies similar to those described in Phase 4 of the counting-based model. However, as described above, the instructional paths to acquiring those strategies are quite different.

In summary, this chapter has considered the role of prerequisite knowledge, formal arithmetic notation and the development of part-part-whole knowledge, supported by a focus on visualisation, to advance the sophistication of strategies used by students in their first year of school to solve simple addition tasks. This learning progression is summarised as the PEGS model: Phases of Early Grouping Strategies.

Chapters Seven and Eight describe the influence of PEGS as a learning progression on the results of three students: Jack, Bridget and Tracey. These students were chosen as case studies as they were representative of differing levels of mathematical knowledge levels (McClain, 2005), and of the range of growth in mathematical knowledge from the pre- to the post-assessments.
Chapter Seven: Results – Case Studies of Three Students

In Section 6.7, the PEGS model described the learning progression of students in their advancement towards the use of grouping strategies to solve addition tasks. In this chapter, case studies describe the advancement of three students in the use of grouping strategies across the duration of the teaching sequence. This chapter documents the students’ advancements from the pre- to the post-assessment in terms of the frequency of correct responses and the use of Band 4 grouping strategies to solve simple addition tasks. In Chapter Eight, the results for these three case studies are discussed with reference to the five research questions and key episodes from the teaching sequence. Through this discussion, the observed changes from the pre- to the post-assessment will be contextualised in the teaching sequence and in the emergence of PEGS as a learning progression.

7.1 Overview of Case Studies

As described in Section 4.2, the three students chosen for the case studies were representative of differing levels of mathematical knowledge within the whole class group. These students were selected in order to obtain a broad picture of students’ learning progressions during the instructional sequence. Jack was selected as a representative of students from the low range, Bridget from the mid-range, and Tracey from the high range.

Observations made during the assessments and the teaching sequence were analysed for frequencies of correct responses and strategy use. Video footage from the teaching sequence, anecdotal notes and artifacts, such as the students’ workbooks, were triangulated for evidence of learning progressions and cognitive reorganisation (Steffe & Cobb, 1988).
As described in Section 4.1.1, all pre-assessments were conducted prior to the commencement of the teaching sequence. At the beginning of the sequence, the average age of the students in the class group was five years and 11 months; Jack was five years and six months, and Bridget and Tracey were both six years and one month. Thus, Jack was more than half a year younger than the other two students and five months younger than the average age of the students in the class. He was the least advanced in terms of the use of mathematically sophisticated strategies. However, one of the other students in the class group was one month younger than Jack and, in the post-assessment, he demonstrated advanced grouping strategies to solve addition tasks. Therefore, Jack’s age on its own may not explain his use of strategies at a lower level of mathematical sophistication relative to his peers. Of the 24 lessons of the instructional sequence, Jack was absent for Lesson 18 while Bridget and Tracey were present for all of the lessons. Therefore, it is unlikely that the results of the three case studies were affected by student absence from the teaching sequence. For each of the assessment task groups discussed in this section, a full description of the presentation of the task, including the purpose and material setting, can be found in Chapter Five. Section 7.2 summarises the three case study students’ responses to key tasks across the assessments. Sections 7.3, 7.4 and 7.5 describe the results of Jack, Bridget and Tracey, respectively.

7.2 Results of Case Study Students from the Pre- and Post-assessments

Section 5.2, compared and contrasted students’ responses to key tasks across the twelve task groups for the whole class. Two comparisons of whole class results were made in Chapter Five and will be re-examined here with a particular focus on the three case study students. These two comparisons were for the task 9 + 4 presented in two forms, and for all additions presented as bare number tasks. The comparisons of these results for the case study students follow in Sections 7.2.1 and 7.2.2.
7.2.1 Strategies Used to Solve 9 + 4 Presented as Two Screened Collections Compared with 9 + 4 Presented in Bare Number Format

Figure 5.28 showed the disaggregated results of the advancement of students’ responses, categorised according to bands, to solve the task 9 + 4 presented as two screened collections in Task Group 3, and at the level of formal arithmetic in Task Group 12. In Figure 5.28, the data pertaining to the three case studies are highlighted by an asterisk next to Students 1, 5 and 7 indicating Jack, Bridget and Tracey, respectively. Table 7.1 below extracts the data of the three case studies from Figure 5.28. Recall that the level of mathematical sophistication associated with the observed strategies increased from Category Band 0 to Band 4.

Table 7.1
Categorised Responses of Three Case Study Students From the Pre- to the Post-assessment to Solve 9 + 4 Presented as Two Screened Collections and as a Bare Number Task

<table>
<thead>
<tr>
<th>Student</th>
<th>Screened Coll’ns Pre</th>
<th>Screened Coll’ns Post</th>
<th>Bare Number Pre</th>
<th>Bare Number Post</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jack</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Bridget</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>Tracey</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>4</td>
</tr>
</tbody>
</table>

This data indicate that the strategies used by Tracey increased the most in terms of the level of mathematical sophistication used to solve both tasks, and the strategies used by Jack increased the least. Key episodes from the teaching sequence will be used to explain the observed changes from the pre- to the post-assessment in Sections 7.3, 7.4 and 7.5.

7.2.2 Comparison of Frequencies of Correct Responses Across Four Tasks Presented in Bare Number Format

As described in Section 5.2, the results for Task Groups 9, 10, 11 and 12 were chosen for specific analysis because these were the only assessment tasks that were presented in bare number format. The absence of a material setting in the presentation of these addition tasks enabled students to use a strategy at their highest level of mathematical
sophistication. It has been observed that students may use a lower level strategy out of convenience. Wright (1988) suggested that this phenomenon is not uncommon: “Children frequently use strategies that are less sophisticated than those of which they are capable” (p. 703). For instance, if individual items are available, a student may choose to use a counting strategy out of convenience, even though they are capable of using a strategy of a higher level of mathematical sophistication. Therefore, tasks presented in bare number format notation are less likely to encourage a particular strategy.

Tables 7.2 and 7.3 below show the disaggregated data of the three case studies from the whole class results presented previously in Tables 5.12 and 5.13. The frequencies of correct responses for the three case studies to the four key tasks from the pre- and post-assessments are summarised in Table 7.2.

Table 7.2
Summary of Frequencies of Correct Responses from the Pre- to the Post-assessment for Bare Number Tasks Presented in Task Groups 9, 10, 11 and 12 for Three Case Study Students

<table>
<thead>
<tr>
<th>Task Group</th>
<th>Student</th>
<th>Pre</th>
<th>Post</th>
</tr>
</thead>
<tbody>
<tr>
<td>TG 9 Small Doubles</td>
<td>Jack</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>(5 tasks)</td>
<td>Bridget</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>Tracey</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>TG 10 Ten-plus Combinations</td>
<td>Jack</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>(10 tasks)</td>
<td>Bridget</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>Tracey</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>TG 11 Large Doubles</td>
<td>Jack</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(5 tasks)</td>
<td>Bridget</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Tracey</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>TG 12 Simple Additions</td>
<td>Jack</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(9 tasks)</td>
<td>Bridget</td>
<td>0</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>Tracey</td>
<td>0</td>
<td>9</td>
</tr>
</tbody>
</table>

These results show that the greatest increase in frequency of correct responses from the pre- to the post-assessment occurred for Jack on Task Group 10: Ten-plus Combinations, and for Bridget and Tracey on Task Group 12: Simple Additions. Table 7.3
summarises the frequencies of Band 4 (grouping strategies) responses observed during the pre- and post-assessments, for each of the three students on the four key task groups.

Table 7.3
Summary of Frequencies of Use of Band 4 Grouping Strategies from the Pre- to the Post-assessment for Bare Number Tasks Presented in Task Groups 9, 10, 11 and 12 for Three Case Study Students

<table>
<thead>
<tr>
<th>Task Group</th>
<th>Student</th>
<th>Pre</th>
<th>Post</th>
</tr>
</thead>
<tbody>
<tr>
<td>TG 9 Small Doubles (5 tasks)</td>
<td>Jack</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>Bridget</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>Tracey</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>TG 10 Ten-plus Combinations (10 tasks)</td>
<td>Jack</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>Bridget</td>
<td>0</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>Tracey</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>TG 11 Large Doubles (5 tasks)</td>
<td>Jack</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Bridget</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Tracey</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>TG 12 Simple Additions (9 tasks)</td>
<td>Jack</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Bridget</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>Tracey</td>
<td>0</td>
<td>6</td>
</tr>
</tbody>
</table>

Tables 7.2 and 7.3 indicate an increase in the frequency of correct responses and in the frequency of use of grouping strategies for all three case study students. These results show the greatest increase in frequency of Band 4 responses from the pre- to the post-assessment occurred for Jack and Bridget on Task Group 10: Ten-plus Combinations; for Jack on Task Group 9: Small Doubles; and for Bridget and Tracey on Task Group 12: Simple Additions. For ease of reference, Table 7.4 summarises the aggregated data of these increases in frequency of correct responses and use of grouping strategies.

Table 7.4
Summary of Frequencies of Correct Responses and Band 4 Grouping Strategies from the Pre- to the Post-assessment for Bare Number Tasks Presented in Task Groups 9, 10, 11 and 12 for Three Case Study Students

<table>
<thead>
<tr>
<th>Student</th>
<th>Correct Pre</th>
<th>Correct Post</th>
<th>Band 4 Pre</th>
<th>Band 4 Post</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jack</td>
<td>0</td>
<td>14</td>
<td>0</td>
<td>14</td>
</tr>
<tr>
<td>Bridget</td>
<td>14</td>
<td>25</td>
<td>2</td>
<td>17</td>
</tr>
<tr>
<td>Tracey</td>
<td>15</td>
<td>26</td>
<td>15</td>
<td>23</td>
</tr>
</tbody>
</table>
As seen in Table 7.4, from the pre- to the post-assessment, the frequency of correct responses and the frequency of Band 4 responses increased for all three students. Jack’s results appear to show the greatest overall increase. Nevertheless, 10 of his 14 correct responses and use of Band 4 strategies are accounted for by his responses to tasks involving the ten-plus combinations.

The purpose of the case studies is to document the progress of three different students and to describe episodes from the teaching sequence which may account for variability in performance from the pre- to the post-assessment. In Chapter Eight, the significance of these episodes will be discussed in terms of their impact on students’ cognitive reorganisations. The assessment results of the three case studies Jack, Bridget and Tracey are described in Sections 7.3, 7.4 and 7.5, respectively. The 12 task groups have been grouped into sub-categories for ease of reporting.

7.3 The Case of Jack

7.3.1 Pre-assessment

7.3.1.1 Task Groups 1, 2 and 3

Jack responded correctly and immediately when asked to state the FNWS to 30, beginning at various starting numbers. Jack correctly identified the numerals in the range 1 to 20 presented on cards but gave an incorrect response when asked to solve the tasks 4 + 2 and 9 + 4 presented as screened collections. This is described below in Protocol 8.

Protocol 8 (Task Group 3):
J: [Sub-vocally counts-by-ones from one, tapping his finger on the table] five
R: How do you know that?
J: Because two and five make five.
R: [Re-poses the task as four and two]
J: Because four and two make five.
R: [Presents the addition task 9 + 4 as two screened collections]
J: [Responds immediately] 14!
R: How do you know that?
J: Because 10 and 4 make 14.
R: They do, but I’ve only got 9 and 4.
J: Oh, 19.
R: Why would it be 19?
J: Because 9 and 4 make 19?
R: Do you think?
J: [Nods]

As seen in Protocol 8, Jack repeatedly stated the number 10 more than the first addend, regardless of the second addend.

7.3.1.2 Task Groups 4, 5, 6 and 7

Jack immediately and correctly ascribed number to the patterns on a five-frame pattern card for all numbers in the range 1 to 5, and on a ten-frame pattern card for all numbers in the range 1 to 10 in a five-wise arrangement. In Task Group 4, with the five-frame partition cards displayed, Jack immediately arranged the numerals to make the equation for the partitions $2 + 3 = 5$ and $3 + 2 = 5$ but, for the partitions $4 + 1$ and $1 + 4$, Jack counted the dots by-ones from one to determine the four dots. Jack moved the beads on an arithmetic rack immediately and without counting-by-ones for all numbers in the range 1 to 10, except four. To move four beads, he counted-by-ones, touching each bead.

7.3.1.3 Task Group 8

In Task Group 8, with the ten-frame partition cards displayed, Jack counted-by-ones from one for each of the first addends for the partitions $6 + 4$ and $9 + 1$, but read the dot pattern for the second addend in order to construct the appropriate equation. He read both addends in $5 + 5$ and $7 + 3$ immediately and without counting-by-ones. To make the equation $8 + 2$, Jack counted-by-ones from five to determine there were eight dots, and counted-by-ones from five to determine the seven dots in the partition $3 + 7$. Jack counted-by-ones from one
to determine the larger addend in each of the partitions 1 + 9 and 2 + 8, and he counted-by-one from one to determine both addends in the partition 4 + 6.

7.3.1.4 Task Group 9

Jack’s response when presented with the small double 3 + 3 presented in bare number format is described in Protocol 9.

Protocol 9 (Task Group 9):
R: Read that to me please.
J: 3 and 3 makes 33.
R: What about 3 plus 3?
J: 33.

Jack responded similarly that 5 + 5 made 55 and 1 + 1 made 11.

7.3.1.5 Task Groups 10, 11 and 12

When presented with 10 + 4 in bare number format, Jack stated, “10 and 4 makes 10” to which the researcher replied, “10 and 4 makes 10?” Jack confirmed, “Hmmm, yes”. Jack was not able to solve any tasks from Task Group 11: Large Doubles and, as described in Section 5.1.12, Task Group 12 was not presented to any students in the pre-assessment. In summary, Jack did not correctly solve addition tasks when presented as screened collections or in bare number format. He read five-wise patterns on the five- and ten-frames, but did not use these to reason about the partitions of five or 10.

7.3.2 Post-assessment

7.3.2.1 Task Groups 1, 2 and 3

Jack correctly recited the FNWS to 30, beginning at various starting numbers, and correctly identified the numerals in the range 1 to 20 presented on cards. He gave an immediate and correct response to the addition task 4 + 2 presented in the material setting of two screened collections. When presented with 9 + 4 as two screened collections, he responded incorrectly, “14”. Protocol 10 describes Jack’s response when asked how he could check his answer.
Protocol 10 (Task Group 3):

J: By counting from 10.
R: Yes, what would you do?
J: 10, 11, 12, 13, 14.
R: How many did you count then?
J: By ones.
R: Yep so we had nine [briefly unscreens collections] and you have four.
J: Hmmm, I think it is 14.

As seen, Jack seems to interpret the researcher’s question, “How many did you count then?” to mean how did you count.

7.3.2.2 Task Groups 4, 5, 6 and 7

Jack correctly ascribed number to the patterns on a flashed five-frame pattern card for all numbers in the range 1 to 5, and on a flashed ten-frame pattern card for all numbers in the range 1 to 10 in a five-wise arrangement. Similarly, Jack moved the beads on an arithmetic rack immediately and without counting-by-ones for all numbers in the range 1 to 10.

However, in Task Group 4, with the five-frame partition cards flashed, Jack immediately arranged the numerals to make the equation for the partitions 4 + 1 = 5, 3 + 2 = 5, and 1 + 4 = 5 but, for the partition 2 + 3, Jack initially made the equation 2 + 4 = 5. When asked how he could check, he raised four fingers simultaneously on one hand and two fingers simultaneously on the other hand and then counted-by-ones from one to determine the total as six. He then corrected the equation to read 2 + 3 = 5 and checked this by sequentially raising fingers and counting-by-ones from one.

7.3.2.3 Task Group 8

In Task Group 8, with the ten-frame partition cards flashed, Jack immediately used the numerals to make the equations 6 + 4, 5 + 5, 9 + 1, 8 + 2, 7 + 3, 3 + 7, and 1 + 9. He incorrectly made the equations 4 + 7 and 2 + 6 for the partitions 4 and 6, and 2 and 8.
7.3.2.4 Task Group 9

When presented with the first small double $3 + 3$ in bare number format, Jack immediately responded with “13” and then, one second later, responded with “33”. The researcher then made $3 + 3$ on the arithmetic rack by sliding 3 beads across on the upper row and 3 beads across on the lower row. Jack counted-by-ones from one to determine a total of six. The researcher removed the arithmetic rack from the desk. When presented with the other small doubles in bare number format, $5 + 5$, $1 + 1$, $4 + 4$, and $2 + 2$, Jack responded immediately with the correct answers.

7.3.2.5 Task Groups 10, 11 and 12

Jack responded correctly to all of the ten-plus addition combinations presented in bare number format. When presented with $8 + 8$ in bare number format, Jack said, “I don’t know”. When each of the large doubles was made by the researcher on the arithmetic rack, Jack counted-by-ones from the first addend to correctly determine the total. The only large double Jack stated correctly when presented as a bare number task was $10 + 10$.

When Jack was presented with the task $9 + 4$ in bare number format, he immediately responded with “19” and then one second later responded, “94”. The researcher re-pose[d] the question as 9 plus 4. Jack thought for about five seconds. The ensuing discussion is described in Protocol 11.

Protocol 11 (Task Group 12):

J: That’s a hard challenge [pause] 16!
R: How come?
J: ’Cos $9 + 4 = 16$.
R: How did you work that out though? What did you do?
J: I was just thinking about 16.
R: Hmmm, is there a way you could check if it was 16?
J: [Raises 9 fingers simultaneously] 9, 9 that’s 9.
R: Hmmm, you’ve got to add on 4 more.
J: Equals 16.
In responding to the final task group of one-digit addition tasks presented in bare number format, Jack repeatedly added 10 to the first addend, that is, $8 + 6 = 18$, $9 + 5 = 19$, and $7 + 5 = 17$. The researcher asked Jack if using ten-plus addition combinations might help him to solve the nine-plus tasks, and Jack replied, “I don’t know about that”.

In summary, Jack used a count-on-by-ones strategy to correctly solve addition tasks presented as two screened collections when the total did not exceed 10. When solving tasks involving the partitions of five and 10, ten-plus addition combinations and small doubles, Jack more frequently used a grouping approach than a counting-by-ones strategy. Jack did not seem to have a strategy for solving one-digit additions presented in bare number format.

7.4 The Case of Bridget

7.4.1 Pre-assessment

7.4.1.1 Task Groups 1, 2 and 3
Bridget correctly responded when asked to state the FNWS to 30, beginning at various numbers, and she correctly identified numerals in the range 1 to 20 presented on cards, except 12 and 20 which she identified as 20 and 12 respectively. She used a count-by-ones from one strategy to solve $4 + 2$ presented in the material settings of two screened collections, raising one finger consecutively for each count from one to six. Bridget also used a count-by-ones from one strategy to solve $9 + 4$ presented as two screened collections. She sub-vocally counted from one to nine and then raised one finger to keep track of each of the four counts to 13.

7.4.1.2 Task Groups 4, 5, 6 and 7
Bridget immediately and correctly ascribed number to the patterns on a five-frame pattern card for all numbers in the range 1 to 5, and on a ten-frame pattern card for all numbers in the range 1 to 10 in a five-wise arrangement. In Task Group 4, with the five-frame partition
cards displayed, Bridget correctly arranged the numerals to make the equation for the partitions $2 + 3 = 5$, $3 + 2 = 5$, and $1 + 4 = 5$. For the partition $4 + 1$, Bridget counted the four dots by-ones from one. Bridget moved the beads on an arithmetic rack as a group and without counting-by-ones for all numbers in the range 1 to 10.

7.4.1.3 Task Group 8

In Task Group 8, with the ten-frame partition cards displayed, Bridget seemed to know there were five dots on the upper row of the ten-frame, as she counted-on-by-ones from five to determine the number of dots in the first addend for each of the partitions $6 + 4$, $9 + 1$, $8 + 2$, and $7 + 3$. To determine the second addend for each of these partitions, she ascribed number to the dot pattern and then constructed the equation to correctly represent the partition of 10.

To determine the partitions $3 + 7$, $1 + 9$, $4 + 6$, and $2 + 8$ from a displayed ten-frame, Bridget ascribed number to the pattern for the first addend and used a count-on-by-ones from five strategy to determine the number of dots in the second addend.

7.4.1.4 Task Group 9

In Task Group 9, to solve $3 + 3$ presented in bare number format, Bridget used a count-by-ones from one strategy to add three and three, raising one finger sequentially for each count from one to six. To solve the task $5 + 5$, she sub-vocally counted from one to five by ones, and then stated “10”. She initially read the task $1 + 1$ presented in bare number format incorrectly, but after she was asked to read it aloud she responded immediately with “two”. Bridget used the same strategy of count-by-ones from one using fingers to keep track to solve the task $4 + 4$ presented in bare number format.

7.4.1.5 Task Groups 10, 11 and 12

In Task Group 10, to solve each of the ten-plus combinations presented as bare number tasks, Bridget counted-on-by-ones from ten, raising one finger for each count from 11.
Bridget attempted to count-by-ones from one, but was unable to solve any of the large doubles tasks, and, as described in Section 5.1.12, Task Group 12 was not presented to any students in the pre-assessment.

In summary, the most advanced strategy that Bridget demonstrated, when solving addition tasks presented as two screened collections and small doubles tasks presented in bare number format, was to count-by-ones from one. Yet she counted-on-by-ones from 10 (once the numbers exceeded her finger range) to solve ten-plus tasks presented in bare number format. Bridget ascribed number to five-wise patterns on the five- and ten-frames and used these to reason about the larger addend in the partitions of 10.

7.4.2 Post-assessment

7.4.2.1 Task Groups 1, 2 and 3
Bridget responded correctly when asked to state the FNWS to 30, beginning at various numbers, and she correctly identified the numerals in the range 1 to 20 presented on cards. She responded correctly when presented with the addition tasks 4 + 2 and 9 + 4 presented as two screened collections. To solve 4 + 2, Bridget raised four fingers simultaneously on one hand, held these in her other hand, sub-vocally stated “four” and then sub-vocally counted on “five, six”. To solve 9 + 4, she raised nine fingers simultaneously, stated “nine”, put down the fingers on her left hand and then raised one finger for each sub-vocal count “10, 11, 12, 13”.

7.4.2.2 Task Groups 4, 5, 6 and 7
Bridget correctly ascribed number to the patterns on a flashed five-frame pattern card for all numbers in the range 1 to 5, and on a flashed ten-frame pattern card for all numbers in the range 1 to 10 in a five-wise arrangement. Similarly, she moved the beads on an arithmetic rack immediately and without counting-by-ones for all numbers in the range 1 to 10. When asked to move seven beads, Bridget immediately moved five beads and then two beads.
When the task was re-posed and she was asked to move the beads all in one slide she was able to group the beads and move them altogether.

In Task Group 4, with the five-frame partition cards flashed, Bridget immediately and with certitude arranged the numerals to make the equation for the partitions \(4 + 1 = 5\), \(3 + 2 = 5\), and \(1 + 4 = 5\) but, for the partition \(2 + 3\), Bridget placed the “two” numeral card onto the equation template and then raised two fingers. She looked at these for a moment and then placed the “three” numeral card to complete the equation \(2 + 3 = 5\).

7.4.2.3 Task Group 8

In Task Group 8, the ten-frame partition cards were flashed. When presented with the first card in the task group, \(6 + 4\), Bridget focused her attention in the distance for two seconds and then placed the “six” numeral card as the first addend in the equation. She then raised six fingers simultaneously, looked at these very briefly and then placed the “four” numeral card to complete the equation \(6 + 4 = 10\). For subsequent tasks, Bridget immediately used the numerals to make the equations \(5 + 5\), \(9 + 1\), \(8 + 2\), \(7 + 3\), \(1 + 9\), and \(4 + 6\).

When presented with the ten-frame partition card for \(3 + 7\), Bridget immediately placed the “three” numeral card as the first addend in the equation. She then raised five fingers simultaneously on one hand and said “five”. She then raised three fingers on the other hand, looked at the two fingers still down on one hand and the five fingers from the other hand. She then placed the “seven” numeral card as the second addend in the equation to complete \(3 + 7 = 10\). Bridget used a similar strategy to complete the equation \(2 + 8 = 10\) in that she raised two fingers on one hand and looked intently at her remaining fingers. She then responded, “Eight!”
7.4.2.4 Task Group 9

To solve the first small doubles task $3 + 3$ presented in bare number format, initially, Bridget attempted to count by twos “two, four, six, eight”. She then raised three fingers and sub-vocally counted-by-ones to determine the total of six. When solving the other small doubles tasks presented in bare number format ($5 + 5$, $1 + 1$, and $2 + 2$), Bridget responded immediately with the correct answers. She commented that these were all doubles. To solve $4 + 4$, Bridget touched two fingers to the desk as she counted “two, four, six, eight”.

7.4.2.5 Task Groups 10, 11 and 12

When presented with the first of the ten-plus addition combinations $10 + 4$ in bare number format, Bridget responded immediately with “13”. When the researcher repeated her response back to her, Bridget immediately raised 10 fingers, and appeared to count-on-by-ones from ten and then corrected her response to “14”. For each of the remaining ten-plus tasks presented in bare number format, Bridget responded correctly and immediately.

When presented with $8 + 8$ in bare number format, Bridget raised eight fingers and counted-by-ones from one to eight, and then counted the eight fingers again by ones from nine to 16. The researcher then made $8 + 8$ on the arithmetic rack, sliding across eight beads on the upper row and eight beads on the lower row. With each row of eight beads being made up of five blue beads and three yellow beads (see Figure 5.22), Bridget stated that the total number of blue beads was 10. She then counted the yellow beads by ones from 11 to 16 to correctly determine the total.

When the researcher placed the task $10 + 10$ in bare number format in front of Bridget, she appeared to begin to count-on-by-ones. When the researcher read the expression aloud, Bridget immediately responded with “20”. To solve the remaining large doubles tasks, $6 + 6$ and $9 + 9$ presented in bare number format, Bridget made each addend on the upper row and on the lower row of the arithmetic rack. She then counted-on-by-ones
from 11, because there were 10 blue beads altogether, to correctly determine the total. To solve 7 + 7, Bridget made this large double on the arithmetic rack and counted-on-by-ones from 8 to 14.

When the final task group of one-digit addition tasks in bare number format was presented, Bridget successfully used a build-to-ten strategy and her knowledge of ten-plus combinations to help her solve each of the nine-plus tasks of 9 + 4, 9 + 5, 9 + 6, and 9 + 3. To solve 8 + 6, Bridget first raised eight fingers and then paused for a moment, before responding with “I don’t know that”. The researcher then asked if she had a way to work it out. Bridget raised eight fingers again, said “8”, then put the eight fingers down and counted-on-by-ones from 9 to 14, keeping track of the number of counts by touching a finger for each count. To solve 7 + 5, Bridget counted-on-by-ones from seven. She initially answered “11” but in explaining her strategy she self-corrected and answered “12”. Similarly, to correctly solve 8 + 3, 7 + 4, and 8 + 7 presented in bare number format, Bridget made the first addend on her fingers, then put these fingers down and counted-on-by-ones using her fingers to keep track of the required number of counts.

In summary, Bridget was facile at counting-on-by-ones to solve addition tasks presented as two screened collections or in bare number format, for which she did not yet have automatic recall. She reasoned about the partitions of 10 using her knowledge of five-plus addition combinations, and she used the grouping strategy of build-to-ten to solve nine-plus tasks. Bridget automatically recalled most of the small doubles, ten-plus addition combinations and ascribed number to patterns in the range 1 to 10 presented on five-wise ten-frame pattern cards.
7.5 The Case of Tracey

7.5.1 Pre-assessment

7.5.1.1 Task Groups 1, 2 and 3

Tracey responded correctly when asked to state the FNWS to 24. When asked to begin counting at 17, she responded “71, 72, 73”. When the researcher re-posed the task, she stated the FNWS without error to 35. Tracey correctly identified the numerals in the range 1 to 20 presented on cards. To solve $4 + 2$ presented as an addition task involving two screened collections, Tracey counted-on-by-ones from four, saying “four … five, six”, using her fingers to keep track of the two counts. Tracey was unsuccessful in using a count-on-by-ones strategy to solve $9 + 4$ presented as two screened collections. She raised nine fingers simultaneously, and then attempted to count-by-ones using fingers to keep track of the counts. She then paused for six seconds before stating “14”. She explained, “I counted from nine and then I counted four and then I figured it out and it was 14, I think”.

7.5.1.2 Task Groups 4, 5, 6 and 7

Tracey immediately and correctly ascribed number to the patterns on a flashed five-frame pattern card for all numbers in the range 1 to 5, and on a flashed ten-frame pattern card for all numbers in the range 1 to 5 in a five-wise arrangement, except for five dots, which she counted-by-ones. Tracey used the strategy of count-on-by-ones from five to determine six dots and seven dots when ascribing number to the patterns on a ten-frame pattern card in the range 6 to 10. To determine eight dots on a flashed ten-frame pattern card, Tracey said the number of dots was “10 take away two”.

In Task Group 4, with the five-frame partition card displayed, Tracey immediately arranged the numerals to make the equation for the partitions $2 + 3 = 5$ and $3 + 2 = 5$ but, for the partitions $4 + 1$ and $1 + 4$, Tracey counted the four dots by-ones from one.
Tracey moved the beads on an arithmetic rack immediately and without counting-by-ones for all numbers in the range 1 to 10, except seven, which she counted-by-ones from one.

7.5.1.3 Task Group 8

In Task Group 8, with the ten-frame partition cards displayed, Tracey used her knowledge that five dots fill the upper row of the ten-frame to then count-on-by-ones from five to determine the number of dots in the first addend for the partitions 8 + 2 and 7 + 3. Tracey then read the dot pattern for the second addend, in order to construct the appropriate equation to represent the partition of ten. Tracey used a count-by-ones from one strategy to determine the number of dots in the second addend for the partitions 6 + 4, 3 + 7, and 1 + 9. Tracey incorrectly constructed the bare number equation for the partition two and eight, as 2 + 9 = 10.

7.5.1.4 Task Groups 9 and 10

Tracey stated all of the small doubles and ten-plus addition combinations from tasks presented in bare number format immediately and correctly.

7.5.1.5 Task Groups 11 and 12

When presented with 8 + 8 in bare number format, Tracey raised eight fingers and attempted to count-on-by-ones from nine using fingers to keep track of the counts. She was incorrect in this and her responses to all large doubles tasks. As described in Section 5.1.12, Task Group 12 was not presented to any students in the pre-assessment.

In summary, Tracey demonstrated an ability to count-on-by-ones to solve addition tasks but, when the second addend was more than three or four, she was often incorrect. She had automatised some addition combinations such as small doubles and ten-plus addition combinations.
7.5.2 Post-assessment

7.5.2.1 Task Groups 1, 2 and 3
Tracey responded correctly and immediately when asked to state the FNWS to 30, beginning at various numbers. She correctly identified the numerals in the range 1 to 20 presented on cards and responded correctly to the task 4 + 2 and 9 + 4 presented as two screened collections. To solve 4 + 2, Tracey raised four fingers simultaneously on one hand, studied these for one second and stated “six”. She explained that she knew there were four fingers raised “because one was missing, and two more made six”. To solve 9 + 4, she raised nine fingers simultaneously, sub-vocally stated “nine” and then focused her attention in the distance for seven seconds and stated “13”. The researcher asked “How did you do that?” Tracey explained that she counted-on-by-ones, “9, 10, 11, 12, 13,” to solve the task.

7.5.2.2 Task Groups 4, 5, 6 and 7
Tracey immediately and correctly ascribed number to the patterns on a flashed five-frame pattern card for all numbers in the range 1 to 5, and on a flashed ten-frame pattern card for all numbers in the range 1 to 10 in a five-wise arrangement. When asked to move four beads on the arithmetic rack, Tracey began to slide the beads one at a time. When the task was re-posed with the researcher requesting her to move the beads in one slide, Tracey did so and then moved the beads immediately and without counting-by-ones for all other numbers in the range 1 to 5, and six, nine and ten beads. When asked to move seven beads, Tracey put her finger above the fifth bead on the rack and sub-vocally stated “five” and then counted-by-ones “six, seven” to determine the seventh bead. Tracey responded similarly when asked to move eight beads. In Task Group 4, with the five-frame partition cards flashed, Tracey immediately arranged the numerals to make the equation for all of the partitions.
7.5.2.3 Task Group 8

In Task Group 8, with the ten-frame partition cards flashed, Tracey immediately used the numerals to correctly make the equations for each of the partitions except three and seven. Tracey made the equation 3 + 8 to represent this partition. For seven of the nine partitions of 10, Tracey reasoned about the larger addend by considering it as “five-plus”. Protocol 12 below describes Tracey explaining her reasoning about five, as she selected the numerals to make the equation for six and four, as a partition of ten.

Protocol 12 (Task Group 8):
R: Okay are you ready, number of red dots, plus number of blue dots [flashes the ten-frame partition card representing 6 + 4]
T: [Looks up and to left, puts one finger on six numeral card and one on four numeral card and moves them into place so that the equation reads 6 + 4 = 10]
R: Can you read me that?
T: 6 + 4 = 10 [touches each symbol as she states each component of equation]
R: How did you know it was six?
T: Because 5 + 1 = 6.
R: Good. How did you know the other one was four?
T: Because five and if you take one away it makes four.

7.5.2.4 Task Groups 9, 10 and 11

Tracey stated the small doubles and ten-plus addition combinations from tasks presented in bare number format immediately and with certitude. When presented with the task 8 + 8 in bare number format, she paused for a moment before stating “17”. The researcher then made 8 + 8 on the arithmetic rack, sliding across eight beads on the upper row and eight beads on the lower row. As described in Protocol 13, the researcher mediated (Askew, 2013) the task through judicious questioning. She elicited a description from Tracey that each row of eight beads was made up of five blue beads and three yellow beads, and by grouping the 10 blue beads and the 6 yellow beads, and using the ten-plus addition combinations, Tracey correctly stated the total number of beads as 16.
Protocol 13 (Task Group 9):

R: [Places task card showing 8 + 8 as a bare number task on the table]
T: 8 + 8 = [pause, looks up and left for two seconds] 17
R: How do you know?
T: Because there’s eight and that one’s the one after, if there isn’t enough to fill up the top row, so the eight, and then plus eight, equals 17.
R: Let’s have a look [places arithmetic rack on table] Can you make eight on the top?
T: [Moves eight beads in one slide on the upper row]
R: And make eight on the bottom.
T: [Moves eight beads in one slide on the lower row]
R: Good, how many blue beads altogether?
T: [Touches each bead as she counts along upper row] five, six, seven.
R: No, no how many blue beads altogether?
T: Five.
R: And five [gestures with hand to two rows, as each five is uttered] and five, 10 blue beads.
T: 16!
R: 16? How did you see 16?
T: Because 3 + 3 = 6 [separates three beads on upper row from remaining five, and repeats on lower row]
R: Good.
T: And, five plus five [touches groups of five] equals 10, and if you put them together it equals 16.

Tracey correctly solved each of the other large doubles using the arithmetic rack to support grouping strategies, such as building-through-five and building-through-ten.

7.5.2.5 Task Group 12

When the final one-digit addition tasks were presented in bare number format, Tracey successfully used ten-plus addition combinations to solve each of the nine-plus tasks: 9 + 4, 9 + 5, 9 + 6, and 9 + 3. Tracey did not solve 8 + 6 and 7 + 5. For 8 + 7 and 8 + 3, she was incorrect initially but in explaining her strategy she self-corrected. In Protocol 14 below, it can be seen that Tracey used a grouping strategy to solve the task 8 + 7 presented in bare number format.
Protocol 14 (Task Group 12):
R: [Places task card showing 8 + 7 as a bare number task on the table]
T: 8 + 7 equals [pause for one second] 16.
R: How come?
T: Because 8 + 7 you might think it is 17 because, ‘cos there’s a seven there, but it isn’t! And I, I changed my answer now.
R: What do you think?
T: Eight [pause for four seconds] 15!
R: How come?
T: Because there’s eight and there’s seven and because there’s three taken away from there [indicating eight numeral card] It means that it would be 15.

Thus Tracey partitioned eight into three and five, added three to seven to make ten, and finally added five onto ten.

In summary, Tracey had automatic recall of small doubles, ten-plus addition combinations and could immediately ascribe number to patterns in the range 1 to 10 presented on five-wise ten-frame pattern cards. When presented with addition tasks involving two screened collections, she used a counting-on-by-ones strategy but, when addition tasks were presented in bare number format in the post-assessment, her first attempt to solve each task was to use a grouping (Band 4) strategy, as opposed to a counting-on-by-ones (Band 3) strategy, to solve these. Whilst not always correct, she consistently and correctly used a grouping strategy to solve nine-plus tasks, and attempted to use a similar strategy to solve other one-digit addition tasks by building-through-ten.

7.6 Summary

This chapter presents the combined and individual results for the three case study students. Qualitative data were reported from the pre- and post-assessment interviews, and quantitative data of frequencies of correct responses and use of grouping strategies were reported for all tasks presented in bare number format. In Chapter Eight, the results of these three case studies are discussed with reference to the five research questions of the study.
and key episodes from the teaching sequence. The aim of this discussion is to contextualise the observed changes from the pre- to the post-assessment in the teaching sequence, and to describe the emergence of PEGS as a model for the learning progression of students in their advancement towards the use of grouping strategies to solve addition tasks.
Chapter Eight: Discussion – Case Studies of Three Students

The assessment results of three case studies were described in Chapter Seven. This chapter presents an analysis of the teaching sequence in order to account for the three students’ cognitive reorganisations that were described in Chapter Seven. Jack’s case is described in Sections 8.1 and 8.2, Bridget’s in Sections 8.3 and 8.4, and Tracey’s in Sections 8.5 and 8.6. Section 8.7 reviews the five research questions posed in this study in light of the results and discussion of the three case study students. Section 8.7.6 includes a discussion of the emergence of Phases of Early Grouping Strategies (PEGS) as a progression of learning from observations of student behaviour, knowledge and skills throughout the teaching experiment.

8.1 The Case of Jack

This section presents an analysis of episodes from the teaching sequence in order to document Jack’s cognitive reorganisations as indicated by his progression from the pre-assessment to the post-assessment. Particular focus is given to the shift from using counting strategies to using grouping strategies.

As described in Section 7.1, Jack was six months younger than the two other case study students, and in the lower range of mathematical ability in the class. When recording in his workbook, Jack did not begin at the first page and sequentially work through from front to back according to normal convention. Rather, he chose a page at random and commenced recording his day’s activities there. His fine motor skills were less developed than those of most other students in the class.
8.1.1 Formal Arithmetic Notation as a Form of Written Communication

One of the research questions addressed by this study focuses on the appropriateness of introducing formal arithmetic notation to students in their first year of school. The following protocol from Lesson 3 of the teaching sequence describes Jack’s attempts to record his thinking. During Part 3 of the lesson, the students were given a tower of four unifix cubes of the same colour. Their task was to investigate the partitions of four and to record these using dots to represent the partition they had identified. Jack split his tower of four unifix cubes into two and two and Protocol 14 describes his interaction with the other students at his table.

Protocol 14 (Lesson 3):
J: 2 and 2, I’ve made 2 and 2 [picks up his pencil and goes to record his partition of 2 and 2. Thirty seconds elapses] How do you do 2 and 2? [Looks to others at table]

Figure 8.1 shows Jack’s recording of two and two as a partition of four.

In Protocol 14, Jack was asked to partition four, and record his thinking in a way that could be clearly communicated to others. Whilst Figure 8.1 does clearly show two and two as a partition of four, Jack merely copied what his peers had written. Section 8.7.3 describes the appropriateness of introducing formal arithmetic notation to students in their
first year of school; however, this episode early in the teaching sequence suggests that Jack was unable to record his thinking using formal arithmetic notation. Students’ reading and interpreting formal arithmetic notation on the one hand, and independently recording with formal notation on the other, are regarded as distinct skills.

During Part 3 of Lesson 7, Jack appeared to have difficulty connecting a written expression and a material setting. Students were asked to match five-plus configurations of beads on an arithmetic rack with the corresponding formal arithmetic notation. Jack matched the $5 + 5$ structure on the arithmetic rack (five beads on each row) with the expression $5 + 2$. Protocol 15 describes the dialogue which occurred when the researcher noticed Jack’s error.

Protocol 15 (Lesson 7):
R: *Okay, tell me Jack how many on the top row?* [Indicates upper row on picture of arithmetic rack]
J: *Um, five.*
R: *How many on the bottom row?*
J: *Um, five.*
R: *What does this sum say?* [Indicates expression that Jack has pasted next to picture of arithmetic rack]
J: *5 + 2.*
R: *Do they match?*
J: [Shakes head] *No.*

As seen in Protocol 15, even after intervention from the researcher, Jack did not solve this task independently. Either he misunderstood the task or he was not aware of the correspondence between the written expression and the configuration shown on the arithmetic rack. Thus, after seven lessons of the teaching sequence, Jack was unable to link formal arithmetic notation with the corresponding configuration on the arithmetic rack.
8.1.2 Use of Counting Strategies to Solve Addition Tasks

During Part 1 of Lesson 14, the researcher revisited ten-plus number combinations presented in bare number format. Expression cards such as 10 + 7 or 7 + 10 were shown and students were asked to state the sum. Students used the arithmetic rack to verify their answers. This knowledge of the ten-plus combinations and the commuted combinations, referred to as turnarounds, was becoming taken-as-shared (Cobb & Whitenack, 1996; McClain, 2002) by the classroom community. Part 2 of the lesson focused on building-to-ten in the setting of a twenty-frame, with problems presented in the context of passengers on a double-decker bus. This approach of embedding mathematical tasks into a realistic, or imaginable, context (Van den Heuvel-Panhuizen & Wijers, 2005) was used regularly throughout the teaching sequence. The task was posed to the class as nine passengers on the upper row of a double-decker bus, and some more passengers on the lower row. The challenge for students was to determine the total number of passengers on the bus. “Guided reinvention” (Freudenthal, 1991) was promoted through the use of careful questioning. This process was designed to encourage students to re-configure passengers on the bus to make 10 on the upper row, and then use their knowledge of ten-plus number combinations to determine the total number of passengers in preference to using a count-by-ones strategy.

During Part 3 of the lesson, students independently investigated similar tasks and had access to blank twenty-frames and counters to represent the passengers, and a template of the bus on which they could record the process they used. Jack did not use the manipulative materials provided to support him in solving the task 9 + 7. Jack coloured nine dots on the upper row of his bus and seven dots on the lower row. He determined the total number of passengers on the bus as represented by the dots by counting-on from nine. Jack then solved 9 + 5 also by counting-on-by-ones from nine. While the students worked
independently, the researcher discussed Jack’s strategy with him. Protocol 16 describes this dialogue.

Protocol 16 (Lesson 14):

R: Okay Jack, tell me how did you work out that 9 plus 5 equals 14?
J: Because, um, because 9 [indicates upper row of dots] 10, 11, 12, 13, 14 [touching each dot as he counts]

R: Good thinking. I can count by ones, can’t I? What if I push this one up here though?
[Moves one counter from lower row to upper row of twenty-frame]

R: That would make [sweeps upper row]
J: Ten … Thirteen.

R: How many on the bottom row Jack?
J: Four.

R: And 10 and 4 make?
J: 14!

R: 14.

This protocol describes the researcher providing scaffolding to enable Jack to use a build-through-ten strategy. When the researcher moved away and Jack continued to work independently, he solved 9 + 3 by drawing the dots onto the blank template and then counting-on-by-ones from ten, “10, 11, 12”. Again, Jack did not use the empty twenty-frame or the counters that were provided as a key part of the material setting. These examples suggest that, while Jack could be supported in the use of a grouping strategy to solve an addition task in the context of passengers on a bus, when working independently, he counted-on-by-ones. However, it is important to note that he did not use the strategy of count-by-ones from one, which is considered to be a lower level of mathematical sophistication, and was a possible strategy given the 12 counters were visually available to him.

8.1.3 Use of Grouping Strategies Across the Teaching Sequence

Approximately midway through the teaching experiment, it was noted that students did not appear to be routinely using five-plus number combinations as a grouping
strategy to solve addition tasks. During Lesson 12, the researcher observed that the most prevalent use of a grouping strategy was when students were asked to ascribe number to flashed ten-frame pattern cards in the range 6 to 10. For example, to solve $4 + 2$ in the material setting of a ten-frame, the student placed four blue counters on the upper row, and one red counter to fill the upper row and one red counter on the lower row. To determine the total number of counters, students either counted-by-ones from one, or counted-on-by-ones from four, to find the total of six. They did not perceive the counters as five-plus-one and ascribe the number six. A likely explanation is that many of these students only associated five-plus number combinations as a strategy to use when ascribing number to a dot pattern in the setting of a flashed ten-frame with printed dots. Even in a setting such as a ten-frame which engendered a grouping strategy, they did not use five-plus number combinations to assist them in solving addition tasks, but rather they defaulted to a strategy of a lower level of mathematical sophistication, such as count-on-by-ones.

In direct response to the observation described above, during Part 1 of Lesson 16, the researcher endeavoured to make an explicit link between addition tasks in the setting of unifix cubes and ascribing number to a five-wise configuration of dots on a ten-frame. A configuration for a number in the range 6 to 10 was displayed on a five-wise ten-frame. Also displayed were 10 unifix cubes arranged in two rows of five. In the first example, the researcher displayed a five-wise seven on the ten-frame, and seven unifix cubes were individually placed onto each of the dots on the frame. The remaining three unifix cubes were placed to one side. The class discussed the benefit of ascribing number to the seven unifix cubes if they were joined together in one row of five and one row of two, in the same configuration as the ten-frame. The seven unifix cubes were joined into a row of five and a row of two and these were displayed on the carpet next to the ten-frame. The
class agreed to use this convention for future examples. The task was repeated for five-wise configurations of eight and six.

In the fourth example, the researcher displayed a five-wise configuration of nine dots on a ten-frame, and unifix cubes configured in two rows of five. Protocol 17 describes Jack’s contribution to the whole class discussion.

Protocol 17 (Lesson 16):
R: What about that one? [Places five-wise configuration of nine dots on a ten-frame on the floor] Who could come and make my cubes look like that? Jack?
J: [Plucks up the lower row of five unifix cubes and breaks them all apart. He places one to the left and the other four cubes individually to the right, under the upper row of five unifix cubes]
R: Can you make it look the same, like we’ve been joining up the unifix cubes to make it look the same?
J: [Plucks up three unifix cubes next to one another, but not joined below the upper row of five unifix cubes]
R: Good, how many do I need here? [Indicates lower row of unifix cubes]
J: Um.
R: Look at my dots [indicates ten-frame]
J: Four.
R: I need four, how many have I got?
J: Three.
R: Three. What if we make them all face the same way, would that make it easier to see?
J: [Joins unifix cubes into a row]
R: How many unifix cubes are there, Jack?
J: Um, five and four.
R: Good, how many is that altogether?
J: Nine.
R: How do you know?
J: Because five and four makes nine, if you put one on it’s 10, if you take two away it makes seven, if you put one more it makes eight, and if you put one more onto eight it makes nine.
R: You know lots of things about numbers, thank you.
Research Question Four of this study addresses the importance of knowledge of the part-part-whole structure of numbers in supporting the use of grouping strategies. In the protocol described above, the class discussed and used a row of five and another row as part-part-whole configurations for seven, eight, six and then nine in two material settings. In Protocol 17, Jack initially had difficulty building nine as five and four with unifix cubes, yet his final statement indicated that he could quite flexibly describe numbers in terms of their parts. However, this example indicates that, in this material setting, he is not facile in his knowledge of the part-part-whole structure of the number nine, and therefore he is unlikely to use grouping strategies to solve simple addition tasks.

In Lesson 23, Jack demonstrated immediate knowledge of five plus two as seven. Protocol 18 is taken from Part 1 of Lesson 23. The students were sitting in a circle and the researcher had two sheets of black cardboard covering two cards with dots configured in canonical dice patterns. On the left hand card was the dice pattern for five, and on the other was the dice pattern for two. The researcher flashed the two cards, and students were asked to state how many dots altogether.

Protocol 18 (Lesson 23):
R: What do you think, Jack?
J: Seven.
R: How do you know?
J: Because five and two make seven.

In this material setting, Jack immediately stated “7” as the sum of five and two. In Part 1 of Lesson 24, Jack immediately described eight dots as five and three when the configuration was flashed on a five-wise ten-frame pattern card.

Similarly, in Part 3 of Lesson 24, Jack immediately stated “7” to solve the five-plus number combination 5 + 2 as an addition task. The researcher rolled two six-sided dice and, as a whole class, the students discussed strategies to combine the two addends.
Settings available for the students to use included counters, an empty ten-frame, an empty twenty-frame and two sets of five unifix cubes. As students began Part 2, the independent activity part of the lesson, Jack rolled five and two, recorded this as a written expression (5 + 2) and immediately stated the answer as seven. To solve 6 + 4 he counted-on-by-ones “6, 7, 8, 9, 10” and recorded his written response as “10”. To solve 2 + 4 he counted-on-by-ones “2, pause, 3, 4, 5, 6”. To solve 1 + 2 he immediately stated “four”, but then counted-by-ones “1, 2, 3” and recorded his written response as “3”.

During Part 1 of Lesson 22, the researcher had a twenty-frame with nine counters on the upper row, and five counters in a five-wise configuration on the lower row as shown in Figure 8.2. As described above, this material setting had emerged from previous lessons as a “model for” (Gravemeijer, 1999; Van den Heuvel-Panhuizen & Wijers, 2005) the context of passengers on a bus.

![Figure 8.2 Model for Passengers on a Bus](image)

In accordance with the pedagogical approach known as horizontal mathematisation (Van den Heuvel-Panhuizen & Wijers, 2005), formal arithmetic notation was used to denote the number of passengers on the bus. That is, for the realistic context of nine passengers on the upper level and five passengers on the lower level of a bus, students placed nine counters on the upper row and five counters on the lower row of a twenty-frame, and also wrote 9 + 5. Developed as part of the hypothetical learning trajectory (Simon, 1995), the researcher anticipated that, in the context of this material setting, students might suggest moving one counter from the lower row to fill the upper row and then ascribe number to the ten-plus pattern on a twenty-frame without counting-by-ones.
The purpose of this session was to revisit the strategy of building-through-ten to solve a nine-plus task, and to link this across the material settings of the twenty-frame and the formal arithmetic notation. The researcher facilitated discussion with the class to ensure that it was taken-as-shared (Cobb & Whitenack, 1996; McClain, 2002) that there were nine dots on the upper row and five dots on the lower row. One student collected the 9 + 5 expression card and placed it next to the twenty-frame. The researcher “distanced the setting” (Wright et al., 2012) by screening the materials and asked the students to visualise and then predict the total when nine and five were added. Once some students had stated their prediction for the total and explained their strategy, the screen was removed and another student came and moved one counter from the lower row to build-to-ten on the upper row, and collected the 10 + 4 expression card and placed it next to the twenty-frame. The researcher then asked: “So, we’ve ended up with 10 + 4. What’s 10 + 4 equal?” And another student stated correctly, “14”. Protocol 19 describes Jack’s contribution to the whole class discussion immediately after this student’s correct response of “14”.

Protocol 19 (Lesson 22):
R: So what’s 9 + 5, what must that equal, Jack?
J: Um, equals 15.
R: Not 15.
J: [Interrupts researcher] 19!

This protocol indicates that this level of reasoning was beyond Jack’s current knowledge. His response of “19” could be considered similar to his default strategy described in the pre-assessment (Section 7.3.1.1) when he stated ten more than the first addend in response to each question posed.

8.2 The Case of Jack: Summary

At the commencement of the teaching sequence, Jack was unable to solve addition tasks involving two screened collections. He was unable to solve addition tasks
presented in formal arithmetic notation, and his default strategy appeared to be to state the number 10 more than the first addend. He could ascribe number to all five-wise numbers in the range 1 to 10 presented on a ten-frame.

On occasions, Jack reasoned flexibly about the part-part-whole structure of numbers in the setting of a ten-frame. Although he could ascribe number to all five-wise patterns in the range 1 to 10 on the ten-frame in the pre-assessment, in Lesson 7 he counted-on-by-ones from five to move seven beads on the arithmetic rack. However, by the end of the teaching sequence, Jack solved 5 + 2 immediately, in two settings – flashed dot cards (Lesson 23) and two dice (Lesson 24).

In Lesson 14 and the post-assessment, Jack demonstrated that he was able to state the ten-plus number combinations immediately when they were presented to him orally and at the level of formal arithmetic. However, in Lessons 14 and 22 in the setting of a twenty-frame, Jack did not use this ten-plus knowledge as a grouping strategy to solve an addition task, but rather counted-on-by-ones.

Across the duration of the teaching sequence, Jack had a sporadic approach to the use of count-by-ones strategies to solve addition tasks. In the post-assessment, Jack did not seem to have a viable strategy for solving addition tasks with addends less than 10 and sums greater than 10 presented in bare number format. Rather, he stated the number 10 more that the first addend regardless of the second addend; that is, he answered in the same way as he did in the pre-assessment.

The teaching sequence described above involved an emphasis on the use of grouping strategies. At the completion of the sequence, when working with two addends in the range 1 to 10, Jack mostly used a count-on-by-ones strategy to solve addition tasks in a range of settings. However, when the two addends were less than 10 and the sum of the addends was greater than 10, and the addends were visually displayed in a setting such as a
twenty-frame, Jack counted-by-ones from one. When a task was presented in bare number format and there were no items available to count, Jack used his default strategy of stating the number 10 more than the first addend. By comparing the episodes from Lessons 3 and 24, it can be seen that Jack did advance in his ability to formally record addition tasks and their solutions at the level of formal arithmetic.

As the examples from the final lessons in the teaching sequence indicate, Jack was using grouping strategies to solve some simple addition tasks, using counting strategies to solve other tasks and, for tasks which involved larger addends, reverting to his default strategy of stating 10 more than the first addend.

8.3 The Case of Bridget

This section presents an analysis of episodes from the teaching sequence in order to document Bridget’s cognitive reorganisations as indicated by her progression from the pre-assessment to the post-assessment. Particular focus is given to the shift from using counting strategies to using grouping strategies. Bridget is representative of students in the middle range of ability.

8.3.1 Progression From Count-by-ones From One to Count-on-by-ones

As a result of targeted questioning, Bridget was observed adjusting the strategies she used to add two one-digit numbers during Part 3 of Lesson 6. This shift is described below. Recall that in the pre-assessment Bridget immediately and correctly ascribed number to the patterns on a ten-frame pattern card for all numbers in the range 1 to 10 in a five-wise configuration.

During Part 2 of Lesson 6, to solve simple addition tasks, students were combining two addends with a sum greater than five in the context of passengers on a double-decker bus. As described in Section 8.1.3, counters of two colours on a ten-frame were used as a “model for” (Gravemeijer, 1999; Van den Heuvel-Panhuizen & Wijers,
2005) these passengers on a bus. The first addend filled the upper row of an empty ten-frame from left to right; the second addend completed the upper row and then part filled the lower row. Targeted questioning and the use of screening were designed to elicit from students the strategy of ascribing number to the five-wise configuration on the ten-frame, rather than use of a count-by-ones strategy to find the total of the two addends. During Part 3 of the lesson, Bridget had access to the material setting of an empty ten-frame and counters, and a blank ten-frame template on which to record her working. Bridget did not use the setting to assist her in solving this task. To solve the task 4 + 3, Bridget coloured four dots on the upper row in yellow, then the last dot on the upper row and two dots on the lower row in blue. To determine the total number of dots, Bridget touched each dot in turn and counted-by-ones from one. Protocols 20 and 21 describe discussions between the researcher and Bridget as she solved the tasks 4 + 3 and 3 + 3.

Protocol 20 (Lesson 6):
R: Bridget, I just noticed you say one, two, three, four, five, six, seven [touching a dot for each count] Do you need to count that top row?

B: [Pause]

R: How many are in that top row?

B: Five.

R: Five [sweeps hand across upper row] six, seven [touching each dot on lower row as each count is made]

The next task Bridget attempted was 3 + 3. She drew three dots in one colour on the upper row, and then coloured two more dots on the upper row and one dot on the lower row. She swept her hand across the upper row and then filled in the answer to 3 + 3 as 7. Protocol 21 is a record of the conversation between the researcher and Bridget about her strategy.

Protocol 21 (Lesson 6):
R: Now let’s check, Bridget. How many in the top row?

B: Five [sweeps hand across upper row]
R: And one makes?

B: Six.

To solve $3 + 4$, Bridget coloured three yellow dots on the upper row and two blue dots in the upper row and two blue dots in the lower row. To determine the total, unaided she swept her hand across the upper row and stated “five”, then touched the two dots “six, seven” on the lower row. She spontaneously and correctly used the five-plus strategy to solve this addition task, as well as the tasks $3 + 5$ and $4 + 4$. It is likely that the questioning and scaffolding by the researcher during this session encouraged Bridget to change her strategy use from a strategy of a lower level of mathematical sophistication (i.e., count-by-ones from one), to the more mathematically sophisticated grouping approach of count-on-by-ones from five.

The sequence of interactions described here suggests that, in this case, a teaching focus on grouping strategies influenced Bridget’s method of solving simple addition tasks. Within this lesson, she moved from using the less efficient count-by-ones from one strategy, to repeatedly using a more efficient count-on-by-ones strategy.

### 8.3.2 The Role of Visualisation in Supporting Addition Strategies

The focus of Lesson 13 was providing opportunities for students to explore the use of a build-through-ten strategy in the context of passengers on a double-decker bus. This approach of “guided reinvention” (Freudenthal, 1991) was supported by a teaching focus on visualisation to assist students to solve nine-plus tasks. During Part 2 of the lesson, a twenty-frame and counters were used as a model for the bus and its passengers, as described in Section 8.1.3. One student placed nine counters on the upper row, and five counters on the lower row, as show in Figure 8.2. The material setting was screened and students were asked to visualise what the bus might look like if the top row was filled. Ascribing number to the visual image of a ten-plus pattern was discussed in Lessons 11 and
12 and reviewed during Part 1 of this lesson. The addition tasks $9 + 4$, $9 + 6$, and $9 + 3$ were completed prior to this task, $9 + 5$. Protocol 22 describes Bridget’s contribution to the whole class discussion.

Protocol 22 (Lesson 13):
R: What about this one? Nine on the top and five on the bottom? Bridget, how many?
B: 14.
R: How do you know?
B: Because if this was up there [indicates movement of one counter from lower row to upper row of twenty-frame] it would also be 14.
R: You move it and show me, so how do you know that is 14?
B: [Moves one counter from lower row to fill upper row] Because 10 and four make 14.
R: Very good thinking.

Protocol 22 indicates that, in the material setting of a twenty-frame, Bridget did visualise and then enact the use of a build-through-ten strategy, and used her knowledge of ten-plus number combinations to solve this task without the need to count-by-ones. In the post-assessment, Bridget also used build-through-ten strategies to solve nine-plus tasks presented using formal arithmetic notation. Thus, across the duration of the teaching sequence, Bridget had progressed beyond using a build-through-ten strategy in a material setting with an inherent ten structure to using the same strategy at the level of formal arithmetic.

This example indicates that visualisation can play a very important role in supporting cognitive reorganisation on the part of the student. Through setting a task in a realistic context (Van den Heuvel-Panhuizen & Wijers, 2005), creating a model for it in a material setting designed to engender the use of mathematically sophisticated strategies, encouraging students to create a visual image of the material setting and finally expecting students to check their own solutions by recreating the visual image, students can be supported in developing grouping strategies to solve simple addition tasks.
8.3.3 The Influence of Settings on Strategy Use

During Part 2 of Lesson 15, a number in the range 6 to 10 was configured as a five-plus number combination using two rows of unifix cubes. The material setting was briefly shown to students and then screened to encourage the use of visualisation strategies. Students described the configuration of the cubes and the number of cubes in total. During Part 2 of the lesson, when working independently to solve the task 5 + 3, Bridget did not make it with unifix cubes but coloured five dots on the upper row of a ten-frame template and three dots on the lower row. She raised five fingers simultaneously on her left hand, stated “five” and counted “six, seven, eight” as she sequentially raised each of three fingers on the other hand. Bridget had been an active and involved participant in the class discussion about using known number combinations to support finding the sum of unknown addition tasks in Part 1 of the lesson, but, when asked to independently solve the task 5 + 3, she counted-on by ones.

Protocol 23 is taken from Part 2 of Lesson 17, two lessons later in the teaching sequence. The researcher had two sheets of black cardboard covering two cards with dots configured in canonical dice patterns. On the left hand card was the dice pattern for five, and on the other was the dice pattern for two.

Protocol 23 (Lesson 17):
R: Under here I’ve got some dots to show you, I’m going to very quickly show you and I want you to tell me altogether, how many dots can you see. Are you ready? (Flashes dot pattern for five)
Class Group Chorus: Five.
R: Well done [flashes dot pattern for two]
Chorus: Two.
R: Two, I’ve got five dots under here [indicates first screen] and two dots under here [indicates second screen]. How many have I got altogether?
[Students discuss their strategies for determining the total number of dots. This task is repeated for 5 + 4 and then 5 + 3 as described below]
R: [Flashes dot pattern for five]
Chorus: *Five.*
R: [Flashes dot pattern for three]
Chorus: *Three.*
R: *When you know how many, put your finger on your nose. How many altogether?*
B: *Eight.*
R: *How do you know?*
B: *Because five plus um three equals eight.*
R: *You just know that, is that right?*
B: *Yep.*

In this protocol from Lesson 17, Bridget used her knowledge of five-plus number combinations in the setting of canonical dice patterns. She combined the two addends without any apparent need to count-by-ones. It is interesting to note that, as in Lesson 15, this is a five-plus task. Two lessons prior, she solved the same task presented in the setting of unifix cubes and a blank ten-frame template by counting-on-by-ones.

When presented with the addition task $4 + 2$ presented as two screened collections in the post-assessment, Bridget counted-on-by-ones, yet to solve the addition task $5 + 3$ involving two screened collections in Lesson 17, as described above, she appeared to use her knowledge of five-plus number combinations. In the post-assessment, making two counts to solve the task $4 + 2$ was very convenient for Bridget. There was no compelling reason for her to use a grouping strategy to solve this task. These examples from the post-assessment and Lesson 17 suggest that the settings in which a task is posed, as well as the materials provided and the questioning used, may keenly influence the strategies used by students to solve simple addition tasks.

Another example of Bridget using a count-on-by-ones strategy occurred during Part 1 of Lesson 24. As described in Section 8.1.3, the researcher rolled two six-sided dice and the students discussed ways to combine the two addends. Material settings available to the group included counters, an empty ten-frame, an empty twenty-frame and two sets of
five unifix cubes. Protocol 24 describes Bridget’s recording of the addition task at the level of formal arithmetic, and her description of counting-on during the whole class discussion.

Protocol 24 (Lesson 24):
R: [Rolls four and two on dice] Four and two, who can write that for me like a mathematician?
B: [Records on poster paper 4 + 2 =]
R: While Bridget is writing that, we’re going to be thinking about what is 4 + 2?
B: [Records “6”]
R: Do you think it’s six? What did you do? I think I saw you do something with your fingers, turn around and show us what you did.
B: I knew, I put up four fingers [raises four fingers simultaneously] but then I went four, five, six [touching one finger as she makes each count]

Bridget used a count-on-by-ones strategy to solve this task. As described in the previous section, in the post-assessment this was also the strategy she used to solve the same task 4 + 2 presented as two screened collections.

About ten minutes later, when working with Anabelle during Part 3 of Lesson 24, the girls were solving the task 2 + 5. Anabelle stated “seven” and Bridget counted-on-by-ones from two, using her fingers to keep track of the counts as she stated “three, four, five, six, seven”. Neither of the girls appeared to realise that 2 + 5 was the commuted fact of 5 + 2. This was a number combination which it was thought they were very familiar with, considering this was the last lesson of the teaching sequence. Given the counting-on solution strategy described, it is reasonable to assume that these two students were operating at Baroody et al.’s (2003) Level 0 and perceiving 2 + 5 as a unary operation. That is, they began with two and counted-on five, rather than thinking of two and five as the part-part-whole structure of seven.

To add the numbers two and three, Bridget recorded it as “2 + 3” and then raised two fingers simultaneously on one hand, looked at them for a moment and stated “five, 2 + 3 = 5”. In contrast to the example given above, the girls rolled three and two on the dice and
commented to each other that this was a “turnaround” of the addition task 2 + 3 which they had just completed, and stated immediately that the answer was therefore five. The next two numbers rolled were two and one. Both girls immediately recorded the total of these two numbers as three. The next two numbers rolled by the girls were two and four. Protocol 25 describes Bridget’s strategy to solve the addition task 2 + 4.

Protocol 25 (Lesson 24):
R: What is 2 + 4 girls?
B: [Reaches for empty ten-frame and places two counters on the upper row and four counters on the lower row]
R: How are you going to use that to help you?
B: [Sweeps her hand along lower row] four [moves one counter from the upper row to the lower row] six.
R: Ah, why didn’t you need to move that last one? [Indicating counter left on upper row]
B: [Nods] ’Cos five [sweeps hand along lower row], six.

Whilst the words uttered by Bridget “five, six” could be considered to be indicative of a counting-on-by-ones strategy, her actions of moving the counter to fill the lower row, and sweeping her hand along this row, indicated that she was using a build-to-five strategy. Interestingly, whenever the researcher used a build-to-five strategy with counters on a ten-frame to solve an addition task in the whole class setting, the five was always configured on the upper row. In this example, it seems that Bridget used her knowledge of the build-to-five strategy to make the five on the lower row of the ten-frame. Bridget then went on to use a similar strategy to solve the task 3 + 4. She placed three counters on the upper row of the ten-frame, and four counters on the lower row. She then moved one counter from the upper row to the lower row, swept her hand along the lower row and said “five”, and then pointed to each of the two counters remaining on the upper row “six, seven”. Bridget’s behaviour seems to indicate that she was using the grouping
approach of building-to-five, but she was still counting-on-by-ones from five to determine the total.

Bridget’s counting-on-by-ones from five in this instance is interesting because she showed in the post-assessment later this same day that she was able to ascribe number to the five-wise pattern for seven, but on this occasion she did not use this strategy to solve this task. As in Lesson 17 above, there was no compelling reason for Bridget to use a grouping strategy to solve this task, and a count-on-by-ones strategy was more convenient.

To solve the addition task 5 + 4, Bridget used yet another grouping strategy. She placed five counters on the upper row of the twenty-frame, and four counters on the lower row. She then counted the counters in the columns of the frame by twos, then added one more to determine a total of nine counters.

In considering Bridget’s strategies to solve a range of addition tasks across the duration of the teaching sequence, her strategy use was not always consistent. The strategies used appeared to be dependent upon the material setting and the way the task was posed by the researcher. This will be described in the summary below.

8.4 The Case of Bridget: Summary

As described above, Bridget used more sophisticated mathematical strategies to solve addition tasks in some settings than others. In the pre-assessment, Bridget demonstrated that she could correctly ascribe number to the five-wise configuration of dots on a flashed ten-frame pattern card without the need to count-by-ones. Yet in Lessons 6, 15 and 24 in the same setting of a displayed ten-frame, she counted-by-ones from five to determine the total number of dots. One possible explanation of this behaviour is that, without the frame being flashed, Bridget did not regard ascribing number as a viable strategy to help solve an addition task in the setting of a ten-frame. She did not appear to use the knowledge that, once the two addends were placed on the ten-frame, ascribing
number through the use of a five-plus strategy could be a quick way to find the total. In Lessons 10 and 17, Bridget appeared to have immediate recall of the five-plus number combinations. She was presented with addition tasks involving two screened collections (5 + 2 in Lesson 10, and 5 + 3 in Lesson 17) in the setting of two flashed dice patterns. Bridget immediately said “seven” in the first example and justified her answer with the statement “because five and two make seven” and immediately said “eight” in the second example and justified her answer with the statement “because five and three make eight”. A question that arises is: Why did Bridget not use this knowledge of five-plus number combinations to solve addition tasks in the setting of a displayed ten-frame, but nevertheless used it in the setting of flashed ten-frames and addition tasks involving two screened collections? It may be that Bridget only associated the act of ascribing number when dot patterns were briefly shown and she was required to visualise what she had seen. In each of the three episodes described above in Lessons 6, 15 and 24, the ten-frame was displayed and therefore count-by-ones was an available strategy, whereas when the material setting was flashed, count-by-ones was not an available strategy.

Another possible explanation is that the aural cue of stating the two addends, combined with the need for visualisation, was the catalyst that allowed Bridget to use her knowledge of five-plus number combinations to solve an addition task in this setting. In Lesson 10, the researcher stated this cue, and in Lesson 17 it was stated by another student in the class group. When working independently in the setting of the ten-frame to solve addition tasks, such as the examples in Lessons 6, 15 and 24 above, Bridget did not state aloud the number of dots in each row. It is possible that, in the absence of an aural stimulus such as the researcher stating the addends aloud, Bridget did not associate using known five-plus number combinations with the total of the two addends; whereas in Lessons 10 and 17, when Bridget did use her five-plus knowledge to solve the tasks, the two addends
of five and two and five and three were presented visually and clearly stated aloud. Presenting information in more than one way can be helpful for students as “some obstacles associated with working memory limits can be ameliorated by using dual-modality presentation techniques” (Mousavi, Low, & Sweller, 1995, p. 319). Mayer and Anderson (1992) refer to this dual presentation of information as the “contiguity principle”. This principle states that “the effectiveness of multimedia instruction increases when words and pictures are presented contiguously (rather than isolated from one another) in time or space” (p. 444). In their study, Mayer and Anderson (1992) found that “separating the animation from the narration in time (temporal noncontiguity) disrupted the building of referential connections needed to support problem-solving transfer” (p. 445). Mayer and Anderson’s research and observation of Bridget’s response and that of other students (described in Section 8.1.3) suggest that concurrently presenting the visual image of a five-plus configuration of dots on a ten-frame and the verbal statement “five and three make eight” may support students’ access to this knowledge from working memory, and facilitate its use to solve related addition tasks.

The contiguity principle may also account for one of Bridget’s responses in the post-assessment. Section 7.4.2.5 described Bridget being presented with the bare number task 10 + 10. To solve this task, she began counting-on by ones from ten. When the researcher gave the aural cue of “What is ten plus ten?” Bridget immediately stated the answer as “20”. By reading aloud the expression, the words and image were presented contiguously, and a referential connection between the formal arithmetic notation and the known number combination was re-established.

Through analysis of Bridget’s performance in the pre- and post-assessments, it seems likely that the emphasis in this teaching sequence on a grouping approach resulted in
Bridget counting-on-by-ones to solve addition tasks in the settings of screened collections, bare number tasks, ten-frame and two dice.

In the pre-assessment, Bridget counted-on-by-ones to solve ten-plus tasks presented in bare number format, and counted-by-ones to determine the larger addend on a displayed ten-frame partition card. In contrast to this Band 2 response in the pre-assessment, in the post-assessment, Bridget counted-on-by-ones (Band 3 response) to solve the addition tasks involving two screened collections (4 + 2 and 9 + 4), and to solve most tasks at the level of formal arithmetic in which the addends were less than 10 and the sum was greater than 10. Even though Bridget demonstrated the use of grouping strategies during the teaching sequence, in the post-assessment, she used count-on-by-ones. As described in Section 7.2.2, it is not unusual for students to use less sophisticated strategies than have been previously observed (Wright, 1998). Thus, evidence of cognitive reorganisation between the pre- and post-assessments is indicated by the advancement in sophistication of the default strategies that Bridget used to solve addition tasks.

In the post-assessment, further advancement in the use of strategies at a higher level of mathematical sophistication was indicated when Bridget used a grouping, build-to-ten strategy (Band 4) to solve nine-plus tasks presented in bare number format. Bridget progressed from the use of a grouping strategy to solve addition tasks in a setting with an explicit 10 structure (i.e., a ten- and twenty-frame) to the use of a grouping strategy to solve addition tasks at the level of formal arithmetic during the post-assessment. Hiebert and Carpenter (1992) suggest that as researchers we “should be concerned with how students generalize the connections they have made and extend them to unfamiliar situations” (p. 88). Thus, a reasonable claim is that Bridget’s progression from the use of grouping strategies to solve addition tasks from one setting to another is indicative of advancement in her learning.
Whilst Bridget demonstrated knowledge of a grouping approach to solve addition tasks in the way she solved nine-plus tasks presented in bare number format, she did not use a grouping strategy to solve seven-plus and eight-plus tasks presented in bare number format. Instead, she made the first addend on her fingers, and then counted-on-by-ones using her fingers to keep track. This further supports the assertion that Bridget’s default strategy had become a Band 3 response, defined by the use of a counting-on by ones strategy.

8.5 The Case of Tracey

This section presents an analysis of episodes from the teaching sequence in order to document Tracey’s cognitive reorganisations as indicated by her progression from the pre-assessment to the post-assessment. Particular focus is given to the shift from using counting strategies to using grouping strategies. Tracey is representative of students in the high range of ability.

8.5.1 Knowledge of the Part-part-whole Structure of Numbers

This section describes six episodes from the teaching sequence which highlight Tracey’s knowledge of the part-part-whole structure of numbers in the range 1 to 10. These episodes demonstrate how this knowledge supports her use of grouping strategies to solve addition tasks.

In the pre-assessment, Tracey demonstrated knowledge of the part-part-whole structure of number. As described in Section 7.5.1.2, when a five-wise eight configuration of dots on a ten-frame was shown to Tracey, she stated that she knew there were eight dots as it was “10 take away two”. This episode indicates that, even prior to the teaching sequence, Tracey was aware of the part-part-whole structure of eight and two as parts of 10.

As described in Section 8.1.2, a term that came to be taken-as-shared (Cobb & Whitenack, 1996; McClain, 2002) in this classroom community was “turnaround”. Early in
the teaching sequence, activities focused on the principle of commutativity in addition tasks. As a whole class, the students quickly became very adept at identifying formal arithmetic expressions such as $5 + 3$ and $3 + 5$ as turnarounds, but very few students moved beyond identifying turnarounds in this way to using this knowledge to assist them in solving addition tasks. Tracey was one of a small group of students who did move beyond the level of stating turnarounds to applying this knowledge of commutativity to solving addition tasks.

In an episode from Part 1 of Lesson 8, the researcher reminded the students that in recent lessons they had been investigating the partitions of five. The researcher posed a question to the whole class group: “Who knows two numbers that add together to make five?” Tracey stated, “I think I know turnarounds, there’s three and two, and two and three”.

In Part 2 of Lesson 18, the researcher made 14 as a ten-plus on the arithmetic rack by sliding across 10 beads on the upper row and four beads on the lower row. The researcher moved the beads back and then made 14 as a double seven, with seven beads on the upper row and seven beads on the lower row, and elicited a description from the students that each row of seven beads was made up of five blue beads and two yellow beads, and that in total there were ten blue beads and 4 yellow beads. Therefore, 14 beads were configured in another way, as 10 plus 4. Protocol 26 describes Tracey’s unsolicited contribution to the whole class discussion.

Protocol 26 (Lesson 18):
T: I know a different way to make 14!
R: How, what’s a different way?
T: Four at the top and 10 down the bottom.
R: I could make it like that [moves beads on arithmetic rack to show 10 beads on the upper row and four beads on the lower row]
T: And 10 up the top and four down the bottom.
R: *Well done Tracey, that is a turnaround.*

Another example of Tracey’s use of turnarounds to solve addition tasks occurred in Lesson 24. The students rolled two dice and added the two numbers together. Tracey and her partner rolled the numbers two and five, and then Tracey immediately stated, “Equal seven” and “That it is a turnaround!” Given that Tracey answered immediately after the rolling of the dice, a reasonable hypothesis is that she reasoned that $2 + 5$ was the same as $5 + 2$ and then used her knowledge of the five-plus number combination $5 + 2 = 7$.

Tracey’s spontaneous references to turnarounds indicate a strong understanding of the part-part-whole structure of numbers. She was not simply noticing that numerals are reversed when a task is presented in bare number format, she was operating at Baroody et al.’s (2003) Level 3 and perceiving addition as a binary task. Tracey was one of the few students in this class who used her part-part-whole knowledge to help solve simple addition tasks.

Further evidence of Tracey reasoning additively about numbers in the range 1 to 10 occurred during Part 2 of Lesson 8 where the focus was on the exploration of the partitions of 10 in the material setting of the upper row of the arithmetic rack. A student slid a given number of beads to the left hand end of the row, and it was anticipated that students would ascribe number to the beads remaining on the right hand end of the row, that is, the second addend of the partition of ten. This was the second example presented to the class in this part of the lesson. Protocol 27 describes Tracey’s contribution to the discussion.

Protocol 27 (Lesson 8):
R: *Billy, can you come and slide six for me please* [Student moves six beads to the left hand end of the upper row] *Billy has made six, what has he pushed? How many blue beads and how many yellow beads?*

T: *He pushed five blues and one yellow and there’s four left behind.*

In her last statement in the protocol above, Tracey referred to a five-wise pattern for six (five blue beads + one yellow bead), a partition of 10 into six and four (six beads +
four beads) and a partition of five into four and one (one yellow bead + four yellow beads left behind). This exceeded the expectations of the researcher when she posed this task. Thus, Tracey showed that she could apply her grouping knowledge in the range 1 to 10 to describe the part-part-whole structure of numbers in the quinary- and ten-based material setting of the arithmetic rack.

In Lesson 12, the students were asked to make bunny ears for seven (Appendix B). All students in the class, except Tracey, built seven by raising five fingers on one hand and two fingers on the other hand. Tracey built seven by raising four fingers on one hand and three fingers on the other.

The following excerpt from the whole class discussion during the plenary of Lesson 17 is another example of Tracey’s strong part-part-whole knowledge and her ability to flexibly partition numbers. During Part 3 of the lesson, students investigated partitioning one-digit numbers in as many ways as possible and recorded their findings in their books. The focus of the whole class discussion were the parts into which six could be partitioned. The partitions of four and two, six and zero, and two and four had been stated by other students in the class. Protocol 28 below describes Tracey’s contribution to the whole class discussion.

Protocol 28 (Lesson 17):
T: Turnaround!
R: Turnaround? What’s the turnaround?
T: It’s two and four, and four and two, and I also know another one.
R: You’ve got a different one?
T: [Nods] Um, three [touches three dots on a canonical six dot pattern] and three, and um, um six and zero.
R: [Indicates where six and zero had already been recorded]
T: Oh, I mean zero and six.
R: Ooh, you can see zero and six makes six!
In this example, Tracey identified turnarounds which had been recorded at the level of formal arithmetic, yet she also described partitions of six that had not yet been recorded. It is reasonable to assume that she was operating at Baroody et al.’s (2003) Level 3 of Commutativity Development as described above.

8.5.2 The Influence of Material Settings on Strategy Use

As described in Section 2.6.3, throughout this teaching experiment careful consideration was given to the selection of the material settings which became “models for” (Gravemeijer, 1999; Van den Heuvel-Panhuizen & Wijers, 2005) the contextualised addition task. In most cases, the materials were chosen for their quinary- and ten-based structure, which was anticipated to encourage students to use grouping strategies to solve simple addition tasks. However, in the absence of strategic questioning, scaffolding and mediating (Askew, 2013) on the part of the researcher, students often defaulted to a counting-by-ones strategy regardless of the material setting. Described above were examples of Jack and Bridget responding in this way. This episode describes Tracey responding in the same way, even though her pre-assessment and responses throughout the teaching sequence indicated that she was thinking at a higher level of mathematical sophistication than the other case study students.

In Part 2 of Lesson 14, Tracey had immediately stated “16” in response to the task 6 + 10 presented in bare number format. In Protocol 29 below, Tracey began to solve the task 9 + 7, but did not use the empty twenty-frame card and the counters available to her. The researcher refers to a twenty-frame bus with nine blue counters on the upper row and seven pink counters on the lower row configured in a similar formation to Figure 8.2. As described previously, it was anticipated that the students might use a build-through-ten strategy to solve the task. Tracey drew in nine dots on the upper row of the twenty-frame and seven dots on the lower row in the template in her book. She then counted-on-by-ones
from nine to determine there were 16 dots altogether. The researcher noticed this from across the room and came to the table.

Protocol 29 (Lesson 14):
R: Tracey, you’re right it is 16, but with 9 + 7 if I push one to the top [moves one counter from the lower row of the twenty-frame to fill the upper row] How many on top?
T: Six.
R: No, how many on top?
T: 10.
R: And?
T: Six is 16!

This protocol highlights that, even though a student may have previously used a grouping strategy to solve a task and the task is presented in a material setting designed to engender the use of a grouping strategy, the student may still use a counting-based strategy to solve the task.

However, in the absence of a setting, Tracey flexibly used a range of strategies to solve addition tasks in the range 1 to 10. In Part 3 of Lesson 24, the students were discussing strategies to solve addition tasks that were generated by the rolling of two dice. A blank ten-frame and counters were available on the floor for students to use. The task was 4 + 4 and one student described the strategy of counting-on by ones from four to determine the total of eight. Protocol 30 describes Tracey’s contribution to the whole class discussion.

Protocol 30 (Lesson 24):
R: That’s one way you could have done it. Who did it a different way for 4 + 4? What did you think Tracey?
T: I did it 5 + 3 [touches under each of the written numbers four and four as she states five and three]
R: Why did you do it 5 plus 3?
T: Because 5 + 3 = 8!
R: It does, but that says 4 + 4, how come you thought 5 + 3?
T: *Because* ...
R: *You’re not wrong!*
T: *Because I knew that there were five and then I counted on from five, and then it became eight.*

Although Tracey described using a counting-on strategy, it is likely that she actually knew five plus three is eight without having to count-on-by-ones, and was able to recompose four and four into five and three, which was a known fact for her.

Analysis of Tracey’s strategy use to solve nine-plus tasks across the duration of the teaching experiment portrays an interesting progression. In the pre-assessment, Tracey attempted to use a count-on-by-ones strategy to solve the addition task $9 + 4$ involving two screened collections, but she was incorrect. In the post-assessment, she correctly solved the addition task $9 + 4$ involving two screened collections by using a count-on-by-ones strategy. By way of contrast, however, in the post-assessment, Tracey correctly used a grouping strategy of build-through-ten to solve $9 + 4$ and other nine-plus tasks presented in bare number format.

An interesting observation is that students will use different strategies to solve similar tasks that are presented in different settings. In the previous case study, even after successfully using a grouping strategy to solve an addition task, Bridget reverted to using less sophisticated counting strategies to determine the total number of individual items in a ten-frame rather than use the structure of the setting and a grouping approach. In the nine-plus examples above, Tracey appeared to tailor her strategy use to match the setting. When presented with individual items such as counters in a screened collections task, she counted-by-ones. When presented with a formal arithmetic task, that is, a task in bare number format, she typically used a grouping strategy.

From the observations described above, a reasonable contention is that a setting involving screened collections, that is, individual items, elicits the use of counting-by-ones
strategies, whereas tasks presented in bare number format are less inclined to engender the use of counting strategies by students. Interestingly, as described in Lesson 17, an addition task in the setting of two flashed canonical dice patterns did not seem to evoke the use of a counting-by-ones strategy from Tracey. Thus, settings such as ten-frames and dice where numbers are structured also seem less likely to engender counting-by-ones strategies than screened collections where numbers are not structured. Flashing and screening of material settings also play a role in strategy use as unscreened settings allow students to use counting-based strategies whereas screened and flashed material settings encourage the student to actively reason “in the context of structured patterns” (Wright et al., 2007, p. 847).

Building-through-ten to solve nine-plus tasks is a strategy that was a focus of the teaching sequence in this study. A few students, including Tracey, were able to generate building-through-ten strategies to solve eight-plus tasks and a seven-plus task. Tracey’s ability to generate new build-through-ten strategies in this way was an important and highly desirable outcome. As Steffe and Thompson (2000) state:

In a teaching experiment, it is never the intention of the teacher–researcher that the students learn to solve a single problem, even though situations are presented to them that might be a problem for them. Rather, the interest is in understanding the students’ assimilating schemes and how these schemes might change as a result of their mathematical activity. (p. 291)

A reasonable explanation is that Tracey’s experience in using building-through-ten to solve tasks in the setting of a twenty-frame, when combined with a focus on visualisation, provided a basis for developing the strategy of building-through-ten at the level of formal arithmetic.

In the post-assessment, Tracey confidently built-through-ten to solve nine-plus tasks presented in bare number format. Her facility with partitioning numbers in the range 1 to 10 supported her use of a build-through-ten strategy to solve the addition tasks 8 + 7 and
7 + 4. Protocol 31 describes Tracey’s use of a build-through-ten strategy to solve 7 + 4 presented in bare number format.

Protocol 31 (Post-assessment):

T: 7 + 4, ’cos I knew that, ’cos there’s three taken away and there’s, four left and it, and it, and it takes a lot and, how many more to build to ten is [sub-vocally counts 8, 9, 10] three.

R: Uh huh.

T: And plus four it = [pause for two seconds] 11.

Tracey counted “8, 9, 10” when she referred to the number to build-to-ten from seven. Nevertheless, prior to this she refers to the three as “taken away”. This indicates that she knew that three is the tens complement of seven, without having to count. It can be inferred that her statement “there’s three taken away” referred to three dots missing from the ten-frame, and she realised that seven and three was the partition of ten that she could use to solve this task. This suggests that Tracey may have visualised the configuration of a five-wise seven on a ten-frame. Partitioning the second addend, four into three and one and adding the one to 10 for a result of 11 was also indicative of Tracey’s flexibility with partitioning numbers in the range 1 to 10.

In analysing responses and behaviours throughout the teaching sequence, Tracey’s willingness and ability to articulate her thinking was apparent, and could account for her strong facility with addition tasks involving two addends in the range 1 to 10. Tracey was a confident and regular contributor to whole class discussions and willingly engaged with new terminology soon after it was introduced. Described in Protocol 32 below is Tracey’s unsolicited contribution to the whole class discussion in the plenary session of Lesson 14. As described previously, the discussion focused on the use of a build-to-ten strategy to solve an addition task on a twenty-frame in the context of passengers on a bus. The task being solved was 9 + 6, and there were nine counters on the upper row and 6 counters on the lower row.
Protocol 32 (Lesson 14):
R: 9 + 6, have a look at my bus very carefully, nine on the top and six on the bottom, now I want you to close your eyes, and I’m going to move 1 up the top and I want you to try and get a picture in your mind of what that might look like [pause, moves one counter to upper row] Open your eyes and check, is that what you thought? Clare, how many on the top row?
C: 10.
R: Because I’ve built to 10, well done. How many are on the bottom?
C: Um, five.
R: Sure and what’s 10 + 5, Clare?
C: 15.
R: Great, so instead of 9 + 6, I’ve now got 10 and 5, this is called –
T: [Interrupting] Building through 10!
As described earlier, this example also highlights the importance placed on visualisation and “distancing the setting” during this teaching sequence. Students were asked to close their eyes, imagine solving the task, and then compare their answer as it was checked by manipulation of the materials. It is conceivable that visualisation of this kind may be the strategy used by Tracy to solve the bare number task 7 + 4 described in Protocol 31.

Described below is an excerpt during which Tracey demonstrated a high level of mathematical sophistication in the strategy she used to solve a large doubles addition task presented in the setting of an arithmetic rack. In Protocol 33, Tracey does not physically move beads on the arithmetic rack to solve the fourth double in the task group, 9 + 9, but uses the rack to demonstrate her strategy to the researcher.

Protocol 33 (Post-assessment):
R: [Places card, 9 + 9, on table] How about that one?
T: 9 + 9 = [pause for 1 second] 18.
R: Hmmm, how do you know?
T: Because [reaches for arithmetic rack] nine [slides nine beads on upper row] plus nine [slides nine beads on lower row] equals 18 because I knew that there was one taken away [indicates one bead left behind on lower row] and because there was five
indicates upper row of blue beads] 10 [indicates lower row of blue beads] there and four there [indicates upper and lower row together] which makes 18.

R: Good job!

Tracey demonstrated sophisticated knowledge of the part-part-whole structure of the number nine (composed of five and four) and 18 (composed of ten, four and four) in order to solve this task.

8.6 The Case of Tracey: Summary

At the beginning of the teaching sequence, Tracey could be described as emerging in her attempts to use a counting-on strategy (Band 3) to solve addition tasks in the setting of screened collections. She could ascribe number to patterns on five- and ten-frames, and could immediately state small doubles and ten-plus number combinations from tasks presented in bare number format. Tracey’s results indicate that, as a result of this teaching approach with a focus on the use of grouping strategies, she made significant progress in her ability to solve addition tasks without the need to use counting-by-ones strategies. In the post-assessment, Tracey consistently used Band 4 strategies to solve simple addition tasks presented in a range of settings.

8.7 Case Studies: Discussion

These case studies have described the progression from the pre- to the post-assessment of three students at different levels of mathematical ability. In the following sections, the results of these case study students will inform discussion around the five research questions posed as part of this teaching experiment, and the emergence of PEGs as a learning progression to describe the development of grouping strategies to solve addition tasks.
8.7.1 What Levels of Student Knowledge Are Prerequisite to the Efficient Use of Grouping Strategies to Solve Addition Tasks Involving Two Addends in the Range 1 to 10?

Throughout the teaching sequence and the assessments of the three case study students, Tracey was observed using grouping strategies to solve addition tasks with the greatest frequency. It is therefore worth considering the knowledge she demonstrated in the pre-assessment to inform a discussion of the prerequisite skills which may support advancement in the use of grouping strategies to solve addition tasks.

In the pre-assessment, both Tracey and Jack correctly identified the numerals 1 to 20 and recited the FNWS to 35. However, neither correctly solved the task 9 + 4 presented as two screened collections. Tracey did exhibit some knowledge of grouping strategies (see Section 8.5.1). Both students demonstrated Stage 1 – Perceptual Counting in SEAL (Wright, Martland, & Stafford, 2006) meaning they were able to coordinate one count for each item in a collection and know that the last count made indicated the numerosity of the collection. However, evidence of their knowledge of SEAL Stage 2 – Figurative Counting (Wright, Martland, & Stafford, 2006) or counting-by-ones from one strategies as a prerequisite for using grouping strategies to solve simple addition tasks is inconclusive. Discussion in Section 6.1.1 suggested that students may not need to progress through this counting stage in order to develop facile addition skills. However, as described in Section 6.7, Phase 3, which is distinguished by the use of a count-on strategy, is significant in both the counting and grouping strategy models. Learning to use a count-on strategy is important conceptual knowledge as described by Steffe (1992), and also supports students’ knowledge of numbers in the range 6 to 10 as five-plus number combinations. As a point in case, Tracey was not observed using a count-by-ones from one strategy to solve any task during the teaching sequence. She did articulate using a count-on-by-ones strategy.
on occasions such as in Protocol 30, but this may have been her way of describing a solution, not an account of the solution strategy she used.

Whilst Jack and Tracey were observed using the same skills in the pre-assessment, in the post-assessment Tracey’s conceptual knowledge of grouping strategies was far more advanced. Tracey could be described as cognitively ready to use operative strategies (Steffe & Cobb, 1988) and, when supported by a targeted pedagogical focus and material settings, she advanced quite quickly in her use of grouping strategies to solve addition tasks.

In contrast, although in the pre-assessment Jack and Bridget demonstrated the same basic skills of numeral identification and FNWS, neither progressed to the same level as Tracey in the use of grouping strategies to solve addition tasks. Bridget used grouping strategies to solve nine-plus tasks at the level of formal arithmetic, but counted-on-by-ones to solve other bare number tasks. Jack was unable to solve additions presented in bare number format.

8.7.2 How Does a Teaching Focus on Grouping Strategies Influence Students’ Methods of Solving Simple Addition Tasks?

The post-assessment results for Jack, Bridget and Tracey suggest that a pedagogical focus on grouping strategies does influence students’ methods of solving simple addition tasks, but this focus is not a panacea. As described in Section 8.7.1, all students, including the three case study students, advanced from the pre- to the post-assessment, some in their use of grouping strategies, and some also advanced in the sophistication of counting strategies. Tracey’s advancements were described in the previous section. Bridget advanced from the pre- to the post-assessment in the use of grouping strategies to solve three of the doubles combinations and nine-plus tasks, and in her use of counting-on to solve bare number tasks. Jack advanced from the pre- to the post-assessment
in the use of grouping strategies to solve the ten-plus number combinations and four of the
doubles combinations, yet he did not advance in his use of counting strategies to solve
screened collections tasks. Each of these students was influenced by a teaching focus on
grouping strategies, but the outcome of this influence was different for each of them.

8.7.3 To What Extent is it Appropriate and Useful to Introduce Formal Arithmetic
Notation to Students in the First Year of School Who Are Reasoning
Additively in the Settings of Quinary- and Ten-based Materials?

Jack was representative of students at the lower level of mathematical ability. Because he started from the lowest base knowledge, considering his progression through the teaching sequence serves to illuminate the minimum achieved by most students. As described in Section 8.1.1, early in the teaching sequence Jack was not able to record his thinking using formal arithmetic notation, nor in Lesson 7 was he able to match a visual configuration of the part-part-whole structure of numbers on the arithmetic rack with formal notation. However, it is suggested that repeated experiences of aligning formal arithmetic notation with a physical configuration of beads or dots in a material setting throughout the teaching sequence increased his ability to interpret the notation. Therefore, whilst the introduction of formal arithmetic notation is very appropriate for students who are reasoning additively, it can also serve to support those students who have not yet reached that stage. The following describes the way formal arithmetic notation was used to support additive reasoning.

In the whole class setting, the researcher (or students nominated by the researcher) modelled formal arithmetic notation as a way of recording strategy use in quinary- or ten-based settings. Students were also encouraged to record their strategies in this way when they completed independent work. An example of formal arithmetic notation used to record the partitions of 10 after the configurations of beads were modelled on an
arithmetic rack is shown in Figure 8.3. This example of recording occurred during Lesson 8 of the 24 lesson sequence.

Figure 8.3 Formal Arithmetic Notation of the Partitions of 10 Recorded by the Teacher and Students During the Whole Class Component of Lesson 8

In the first example shown in Figure 8.3, students reasoned that five and two could be combined to make seven, the first addend. Students then used knowledge of the partitions of five to reason that, when five blue beads were combined with two red beads to make seven, the remaining three red beads indicate the number that, when added to seven, makes 10. This can be expressed mathematically as follows: three is the tens-complement of seven, and recorded as \(7 + 3\). When considered in the setting of an arithmetic rack or a ten-frame during the teaching sequence, whole class discussion only focused on this conventional notation as describing \(7 + 3\) as a binary operation (Baroody et al., 2003). At no stage during the teaching sequence was this task interpreted as a unary operation to be solved as “start with seven and count on three”.

Introducing formal notation in this way as a means of describing and recording students’ reasoning helped to synthesise new knowledge with existing knowledge (Hiebert & Carpenter, 1992). In this way, the formal mathematical notation was used to describe children’s mathematical intuitions (Resnick, 1992) to model their own thinking processes.
To What Extent Does Knowledge of the Part-part-whole Structure of Numbers Support the Use of Grouping Strategies to Solve Addition Tasks Involving Two Addends in the Range 1 to 10?

The results of the three case study students show that knowledge of the part-part-whole structure of numbers is very important in supporting the use of grouping strategies to solve addition tasks. Tracey was the most advanced case study in terms of the level of mathematical sophistication she used to solve addition tasks presented in multifarious ways, and, as described in the discussion above, she demonstrated part-part-whole knowledge in the pre- and post-assessments and throughout the teaching sequence. Conversely, Jack was the least advanced student in terms of mathematical sophistication and he demonstrated part-part-whole knowledge only on a few occasions. Part-part-whole knowledge increases students’ flexibility in mental strategies (Bobis, 2008), and this flexibility was more frequently evident in Tracey’s articulation of strategies when compared with Jack’s.

What Role Does a Teaching Focus on Visualisation Have in Supporting the Use of Grouping Strategies to Solve Simple Addition Tasks?

In this study, students were expected to articulate their thinking and justify their reasoning in the context of visualised images. The pedagogical strategy of distancing the material setting (Wright et al., 2012) was repeatedly used to encourage re-presentation (Olive, 2001) and empirical abstraction (Steffe & Cobb, 1988) to support students’ additive reasoning. The use of aural cues described in Section 8.4, coupled with visualisation and formal arithmetic notation, were important strategies in supporting students to use efficient strategies to solve addition tasks. Whilst all students were encouraged to use visualisation strategies to mentally recreate physical settings, when compared with Jack, students, such
as Tracey, were able to do so at a higher level of sophistication and in a range of material settings, and therefore the effects were more apparent.

8.7.6 Emergence of Phases of Early Grouping Strategies (PEGS)

As described in Section 6.7, PEGS emerged from the teaching experiment as a model for student progression in learning about grouping strategies to solve addition tasks. Prior to the teaching experiment, PEGS was developed as a hypothetical learning trajectory (Simon, 1995) which was then enacted and adapted as necessary to respond to evidence of cognitive reorganisation on the part of the students. The importance of the quinary- and ten-based material settings, a strong emphasis on visualisation, and close connection between the material setting and formal arithmetic notation all emerged as part of a pedagogical strategy to support students’ use of grouping strategies. As outlined in Table 6.1, each of the PEGS as a learning progression was divided into three parts: a) acknowledging students’ use of counting-based strategies to solve the task; b) students’ use of grouping strategies in the setting of quinary- or ten-based materials; and c) students’ use of grouping strategies when presented with tasks at the level of formal arithmetic (in Phases 2, 3, 4 and 6). In order to accurately describe the developmental journey of all students, it was important to include counting strategies (even though these are not considered grouping strategies) as part of the progression towards mastery of grouping strategies to solve addition combinations with two addends in the range 1 to 10. Whilst Tracey was not observed using counting strategies to solve tasks in several of the PEGS, this was not true for Jack and Bridget. Therefore, in order to develop a progression of learning which accurately accounted for all students, the inclusion of counting strategies was necessary. Similarly, the importance of quinary- and ten-based material settings and the reflexive use of formal arithmetic notation were key elements of the learning progression.
After considering the results and ensuing discussion for all students in Chapters Five and Six, and the three case study students in Chapters Seven and Eight, Chapter Nine summarises the conclusions reached through the enactment of this teaching experiment, the limitations of the study and recommendations for future learning.
Chapter Nine: Limitations, Findings, Recommendations and Conclusion

The focus of this study was a teaching experiment enacted as a 24 lesson sequence with 20 students in their first year of school using grouping strategies to solve addition tasks involving two addends in the range 1 to 10. The purpose of this chapter is threefold. Firstly, it outlines the limitations of this study. Secondly, it describes the findings and recommendations from this study in light of the five research questions. Thirdly, it draws a conclusion. Five research questions were posed prior to the commencement of this study:

1. What levels of student knowledge are prerequisite to the efficient use of grouping strategies to solve addition tasks involving two addends in the range 1 to 10?
2. How does a teaching focus on grouping strategies influence students’ methods of solving simple addition tasks?
3. To what extent is it appropriate and useful to introduce formal arithmetic notation to students in the first year of school who are reasoning additively in the settings of quinary- and ten-based materials?
4. To what extent does knowledge of the part-part-whole structure of numbers support the use of grouping strategies to solve addition tasks involving two addends in the range 1 to 10?
5. What role does a teaching focus on visualisation have in supporting the use of grouping strategies to solve simple addition tasks?
Chapter Five documented the pre- and post-assessment results for the students as a class, and Chapter Six recounted episodes from the teaching sequence to account for these results. Chapter Seven documented the pre- and post-assessment results for three case study students, and Chapter Eight recounted episodes from the teaching sequence to account for the advancement of these three students who were representative of low, middle and high levels of mathematical ability.

In light of these results and analyses, Sections 9.2 to 9.6 address the five research questions. In responding to these research questions, it is important to acknowledge that not all students who participated in this study demonstrated efficient use of grouping strategies, either in the post-assessment or during the 24 lesson sequence. Similarly, some students used grouping strategies on tasks presented in some material settings and not in others. However, it is possible to draw some conclusions from the findings and make recommendations for future action based on the research conducted as part of this study.

**9.1 Limitations of This Study**

This exploratory study comprised a 24 lesson teaching sequence. Whilst not a limitation as such, the opportunity to extend the duration of the study would have allowed more time for the consolidation of students’ knowledge. The teaching experiment was planned as a hypothetical learning trajectory (Simon, 1995; Gravemeijer, 2004a), thus involving an ongoing cycle of teaching and reflection. However, a teaching sequence of a longer duration may have allowed greater flexibility in the number of lessons planned to support students’ progression of learning through the Phases of Early Grouping Strategies (PEGS). This extra time may have resulted in an increase in the knowledge and facility of some students.

In this study, the most advanced task group in the assessments comprised one-digit addition tasks presented in bare number format. The inclusion of some addition tasks
presented as word problems may have been useful in terms of determining the additive strategies students might have used to solve tasks set in a story context.

This teaching experiment focused on the use of grouping strategies to solve addition tasks. Research into the use of grouping strategies to solve subtraction tasks in the range 1 to 20 may serve to further advance knowledge regarding the appropriateness of the use of grouping strategies by students in the first years of school.

This study focused on 20 students from a school situated in a mid-range socioeconomic suburb in a large city. Lester, Wiliam and Lester (2002) believe that “in particular, the sensitivity of educational phenomena to small changes in detail means that it is literally impossible to put the same innovation into practice in the same way in different classrooms” (p. 491). Whilst findings from this study have resulted in recommendations regarding the pedagogical approaches used in the teaching and learning of early addition strategies in the first years of school, conducting a similar study in other classrooms is recommended. Similar studies conducted with different cohorts of students from different socioeconomic backgrounds may serve to further inform discussion regarding the appropriateness of this pedagogical approach.

**9.2 What Levels of Student Knowledge Are Prerequisite to the Efficient Use of Grouping Strategies to Solve Addition Tasks Involving Two Addends in the Range 1 to 10?**

As described previously, this study was conducted in the students’ first year of school. Discussion with the class teacher prior to the commencement of the teaching sequence indicated that she had not explicitly taught the concept of addition. Therefore, the students had not been formally taught any strategies to solve addition tasks. The findings from this study indicate that some prerequisite knowledge and skills support students in the efficient use of grouping strategies to solve simple addition tasks. It is helpful to classify

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the prerequisite skills into two categories: non-counting-based skills and counting-based skills.

9.2.1 Non-counting-based Prerequisite Knowledge

Recall from the description in Section 2.2 that a student’s ability to recite the forward number word sequence (FNWS) is considered to be distinct from their ability to coordinate the oral count with perceptual items (Steffe, 1992), and therefore knowing the FNWS is regarded as non-counting-based knowledge. This was assessed in Task Group 1. Identification of numerals in the range 1 to 20 was measured in Task Group 2 of the assessments. Sections 5.1.1 and 5.1.2 state that, collectively, the students were highly successful in their ability to recite the FNWS and identify numerals in the range 1 to 20 prior to the commencement of the teaching sequence. Rather than an indication of a ceiling effect in the assessments, these skills were considered prerequisites for solving addition tasks in the same range; however, they cannot be regarded as an indicator of readiness to do so.

While using facile knowledge of doubles is a valid non-counting strategy to solve addition tasks, this study found that students responded more positively to a focus on quinary- and ten-based strategies (Murata, 2004; Verschaffel et al., 2007; Wright, Martland, Stafford, & Stanger, 2006). Therefore, as PEGS emerged as a learning progression during the teaching experiment and as part of the ongoing “tinkering” (Gravemeijer, 2004a) with the original hypothetical learning trajectory, the time allocated to the exploration of doubles as an addition strategy was less when compared with what was originally envisioned. As a result, students had only limited knowledge of small and large doubles in the context of formal written tasks at the beginning and conclusion of the teaching cycle. Thus it was concluded that knowledge of doubles was not considered prerequisite knowledge for students’ use of quinary- and ten-based grouping strategies to solve simple addition tasks.
Knowledge of the part-part-whole structure of numbers underlies the ability to partition numbers using non-counting strategies, and the ability to use grouping strategies to solve simple addition tasks. Knowledge of the part-part-whole structure of numbers in the range 1 to 10 is considered to be prerequisite for using grouping strategies to solve simple addition tasks.

9.2.2 Counting-based Prerequisite Knowledge

The SEAL model (Wright, Martland, & Stafford, 2006), which was informed by the work of Steffe and Cobb (1988), describes the first three of five counting stages as perceptual counting, figurative counting, and initial number sequence (sees Table 2.1). The learning progression through phases of the counting stages is presented in Table 6.1. Recall that figurative counting can also be described as counting-by-ones from one, and the initial number sequence as counting-on and counting-back.

Results from the post-assessment, presented in Table 5.15, highlighted the absence of count-by-ones from one (figurative) strategies to solve 9 + 4 presented as two screened collections, small doubles, and ten-plus combinations presented as bare number tasks. The results also highlighted the very low frequency of use of this strategy to solve large doubles and other bare number tasks. This suggests that, whilst perceptual counting and counting-on (initial number sequence) are important skills for students to acquire as prerequisites to solving addition tasks using grouping skills, this may not be true for figurative counting.

As illustrated in Table 6.1, PEGS proposes an alternative model for the learning progression towards mastery of simple addition tasks which does not include all of the counting stages. Therefore, in contrast to the SEAL model, it is proposed that students may not need to master count-by-ones from one as a prerequisite skill to using efficient grouping strategies to solve simple addition tasks.
Results from this study lead to the recommendation that knowledge of the FNWS in the range 1 to 20, numeral identification to 20, part-part-whole knowledge of the structure of numbers, perceptual counting and counting-on strategies are all considered prerequisite to the efficient use of grouping strategies to solve addition tasks involving two addends in the range 1 to 10.

### 9.3 How Does a Teaching Focus on Grouping Strategies Influence Students’ Methods of Solving Simple Addition Tasks?

Grouping strategies are perceived to be at a higher level of mathematical sophistication than counting strategies, as explained in the counting-based frameworks of Carpenter and Moser (1984), Steffe and Cobb (1988) and Wright, Martland, and Stafford (2006). With structural responses describing those that use non-counting strategies, a study by Mulligan and Mitchelmore (2009) found that “… students who gave predominantly structural responses were judged by their teachers as highly competent at mathematics and … students who gave predominantly pre-structural responses were judged to be mathematically weak” (p. 45). Therefore, in the early years of schooling, the purposeful use of grouping strategies to solve simple addition tasks is considered to be a valuable skill in developing competence in mathematics.

As described in response to Research Question One, and presented in Table 5.15, the findings from this study indicate that a teaching focus on grouping strategies resulted in a low frequency of use of counting strategies in the post-assessment to solve tasks presented in bare number format. As expected, comparatively high frequencies in the use of Band 4 grouping strategies when compared to other strategies were observed. Task Groups 9, 10, 11 and 12 all consisted of addition tasks presented as bare number tasks. Addition tasks presented in bare number format have been found to be more difficult for students to
solve than addition tasks presented in other settings (Van den Heuvel-Panhuizen & Wijers, 2005).

Throughout the teaching sequence, tasks were most frequently presented—in a quinary- or ten-based material setting. Therefore, the post-assessment results suggest that students were able to apply their experience of solving addition tasks in these settings to more difficult tasks presented in bare number format. The role of visualisation in this application of knowledge to a different context will be further discussed in response to Research Question Five.

As illustrated by PEGS in Table 6.1, the development of grouping strategies to solve addition tasks was cumulative. The teaching focus was initially on grouping strategies to solve addition tasks in the range 1 to 5, then 1 to 10 and then 1 to 20. Although cumulative, this teaching focus was not invariant; students consolidated their knowledge in one phase whilst being introduced to new knowledge from another phase. Therefore, findings from this study conclude that this teaching focus did influence students to use grouping strategies to solve addition simple addition tasks.

In accordance with these findings it is recommended that the teaching of number in the first year of school should focus on grouping strategies and this should be reflected in future Australian curriculum documents.

9.4 To What Extent is it Appropriate and Useful to Introduce Formal Arithmetic Notation to Students in the First Year of School Who Are Reasoning Additively in the Settings of Quinary- and Ten-based Materials?

Research Question Three focuses on students in the first year of school who are reasoning additively. Wright (2008) states that additive reasoning “is when the child progresses from using advanced counting-by-ones (counting-on and counting-back) to reasoning that does not involve counting-by-ones” (p. 232).
Students can use additive reasoning to solve tasks in a range of settings, however, it is proposed that quinary- and ten-based settings are more likely to engender additive reasoning strategies than screened collections. An example of additive reasoning is when a student states that they know \(8 + 3\) is 11, because “\(8 + 2\) is 10 and one more makes 11”. This teaching experiment indicates that, whilst a setting such as ten-frames is more likely to engender the use of this type of grouping strategy than a setting involving two screened collections, additive reasoning can also be engendered when students are solving addition tasks at the level of formal arithmetic.

In Figure 5.26, the disaggregated data from the post-assessment showed that seven students solved all four nine-plus tasks presented in bare number format by using a grouping strategy. A hypothesis to account for this result is that the frequent use of formal arithmetic notation as a written record of additive reasoning in a quinary- or ten-based setting during the teaching sequence can engender grouping strategies.

In this study the use of formal notation as a means of recording was not a symbolic system into which students were inducted, rather it was introduced incidentally (Wright, Martland, Stafford, & Stanger, 2006) as a shared means of describing the part-part-whole structure of numbers and the conception of addition as a binary operation (Baroody et al., 2003). For instance, \(7 + 3\) was recorded at the level of formal arithmetic to illustrate a partition of ten in the setting of the arithmetic rack. The focus was always on the part-part-whole configuration of numbers recorded using formal arithmetic to highlight addition as a binary operation. Addition as a unary operation, that is, a counting-based approach to solving \(7 + 3\) by counting on three more from seven, was never a focus.

The result that a high frequency of Band 4 strategies were used to solve bare number tasks presented in Table 5.15, indicate that it is both appropriate and useful to introduce formal arithmetic notation to students in their first year of school, as a means of
describing and presenting addition tasks and it is recommended for all students regardless of whether or not they are reasoning additively in the settings of quinary- and ten-based materials.

9.5 To What Extent Does Knowledge of the Part-part-whole Structure of Numbers Support the Use of Grouping Strategies to Solve Addition Tasks Involving Two Addends in the Range 1 to 10?

As explained in Section 9.2, results from this study suggest that combining and partitioning numbers in the range 1 to 5, and then 1 to 10 constitute prerequisite skills for developing knowledge of the part-part-whole structure of numbers in the range 1 to 20. In turn, this part-part-whole knowledge of the structure of numbers underlies the use of grouping strategies to solve addition tasks.

In many lessons during this study, teaching focused on using part-part-whole knowledge of numbers to support the use of grouping strategies to solve addition tasks. However, data from the protocols describing the case study students suggest that, whilst students may mimic or use a particular strategy when working directly with the teacher, when they are left to solve tasks independently, they do not always use the more advanced strategy (Wright, 1998). As illustrated earlier, a student may use a particular strategy in one setting, but not always use the same strategy to solve all tasks in that setting. Similarly, a student who regularly uses a particular strategy in one setting will not necessarily use the same strategy in a different setting. Therefore, whilst findings from this study indicate that knowledge of the part-part-whole structure of numbers supports the use of grouping strategies to solve addition tasks, a teaching focus on the use of these strategies does not guarantee that students will use them when working independently.

The material quinary- and ten-based settings used in this study were designed to highlight the part-part-whole structure of numbers. PEGS describes a learning progression which supports students to think about the structure of numbers in increasingly more
sophisticated ways, thus encouraging students to conceive of a total as the combination of two parts, rather than as the result of adding one quantity to another. For example, in this approach to addition tasks, students conceive of 8 as the combination of 5 and 3, rather than as the result of adding 3 to 5. Therefore, a recommendation from this study, as outlined in the progression of PEGS, is that knowledge of the part-part-whole structure of numbers be a capstone in the use of grouping strategies to solve addition tasks involving two addends in the range 1 to 10, and should be a key teaching focus in the first year of school.

9.6 What Role Does a Teaching Focus on Visualisation Have in Supporting the Use of Grouping Strategies to Solve Simple Addition Tasks?

Throughout this study, students were expected to articulate their thinking and justify their reasoning in the context of visualised images. The pedagogical strategy of distancing the material setting (Wright et al., 2012) was used repeatedly to encourage representation (Olive, 2001) and empirical abstraction (Steffe & Cobb, 1988) to support students’ additive reasoning. The use of aural cues (described in Section 8.4) coupled with visualisation and formal arithmetic notation constituted important instructional strategies in supporting students to consider addition as a binary operation and to encourage the use of efficient grouping strategies to solve addition tasks.

In summary, this study found that a strong focus on visualisation strategies in the context of quinary- and ten-based material settings reduced the need for students to progress through counting levels as they acquired mastery of efficient grouping strategies to solve addition tasks. When complemented by visualisation strategies (such as distancing the setting), a focus on the quinary- and ten-base of numbers supported students in moving within and through the phases of PEGS as they advanced in their knowledge of the part-part-whole structure of numbers and their use of grouping strategies. The results presented in Tables 5.12 and 5.13 show from the pre- to the post-assessment an increase in the
frequency of both correct responses and the use of Band 4 grouping strategies to solve tasks presented in bare number format across the class group as a whole.

9.7 Conclusion

This study serves as a viable instance of design research. The teaching experiment methodology provided the opportunity to observe and learn from students in their regular learning environment whilst appraising the pedagogy of early number learning related to addition in the classroom setting. The findings from this teaching experiment indicate that foci on the part-part-whole structure of numbers and visualisation of quinary- and ten-based material settings, supported by the use of formal arithmetic notation to represent the binary relationship between the addends, can vigorously support the use of grouping strategies to solve addition tasks. Findings from this study also indicate that the figurative counting stage is not a necessary prerequisite for counting on to solve addition tasks. Recall that a review of curriculum documents indicated that neither Singapore nor Australia included any reference to the use of grouping strategies to solve one-digit addition tasks in the first year of school. In summary, if the findings from this study are replicated in like studies, it may be appropriate to recommend that in the case of the Australian curriculum, PEGS be considered as a viable learning trajectory for solving simple addition tasks, and thus PEGS could constitute an important part of the mathematics curriculum for students in their first year of school.
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Appendix A: Terms and Definitions

The following terms and definitions are described below:

- Addition Task
- Ascribing Number
- Building Perceptual Replacements
- Certitude
- Chorus
- Default
- Facile
- Flashing
- Knowledge
- Number Combinations
- Protocol
- Setting
- Task Group

**Addition Task:** This term refers to any task which requires students to determine the sum of two numbers and in which the two numbers are presented in any of the settings described in Appendix B.

**Ascribing Number:** In the context of a five- or ten-frame, this term refers to the immediate identification of the number of dots on the frame without the need for counting-by-ones (Ellemor-Collins & Wright, 2009, p. 68).
**Building Perceptual Replacements**: This is a strategy referred to in Wright, Martland, and Stafford (2006) in which a student uses items, usually, but not always, their fingers to replace other items.

**Certitude**: Wright, Martland, Stafford, and Stanger (2006) define certitude as “a child’s assuredness about the correctness of their solution to a problem” (p. 39). When a child is described as responding to a task “immediately and with certitude” in this study, they appear very confident in the correctness of their response.

**Chorus**: The whole class orally responds to a question posed by the researcher.

**Default**: In some cases, the strategy a student uses to solve an addition task is referred to as a “default” strategy. This strategy is considered to be at a lower level of mathematical sophistication than the strategy the student might have recently demonstrated. For example, a student might have previously used a build-through-ten strategy to solve an addition task, but in this instance a count-by-ones strategy was used. The count-by-ones strategy is regarded as the default strategy.

**Facile**: Wright, Martland, and Stafford (2006) define facile as “used in the sense of having good facility, that is, fluent or dexterous, for example, a facile counting-on strategy or facile with the backward number word sequence” (p. 190).

**Flashing**: According to Wright, Martland, Stafford, and Stanger (2006):

> The technique of flashing is used in the presentation of tasks which involve spatial patterns or settings for which spatial arrangement … [i]s particularly significant. The term “flash” is used in the sense of display briefly (typically for about half a second) … [T]he technique of … [f]lashing is used because we believe they are likely to support student’s imaging (used in the sense of making a picture in the head) and reflection (used in the sense of thinking about one’s thinking). (pp. 34–35)

**Knowledge**: Wright, Martland, Stafford, and Stanger (2006) use the term knowledge to encompass two meanings. The first refers to those aspects of students’ learning that are not
easily characterised as strategies; and the second refers to everything that a student knows about early number. They do not focus on a distinction between the terms concepts and procedures, and this stance can be assumed throughout this research work. The terms “understandings” and “knowledge and understandings” are used extensively in many curriculum documents and are referred to by teachers when planning their numeracy programs. Evidence of what a student understands is problematic for both teachers and researchers. An observer is more likely to be able to identify strategies and procedures used by a student than determine an accurate picture of what they understand. It is important that the reader be cognisant of this definition as the term “knowledge” is used quite specifically throughout this study.

**Number Combinations:** This refers to the addition of two one-digit numbers, for example, five-plus number combinations include the addition tasks 5 + 1, 5 + 2, 5 + 3, 5 + 4, and 5 + 5. These are also known as **basic addition facts** and **number bonds**.

**Protocol:** This term refers to a textual description of the words and actions of the teacher/researcher and a student as transcribed from a video excerpt.

**Setting:** A setting refers to devices and materials used in posing tasks to the student (Wright, Martland, & Stafford, 2006, p. 157). Detailed descriptions of the settings used in this study are found in Appendix B.

**Task Group:** In this study, each assessment consists of 12 task groups. Each task group is designed to assess the student’s knowledge on a particular topic related to addition tasks involving two addends in the range 1 to 10. Each task group consists of several closely related assessment tasks.
Appendix B: Material Settings Used in This Study

The following settings, used as part of the Task Groups and/or the instructional sequence of this study, are described below:

- Arithmetic Rack
- Bare Number Task
- Bead Strings
- Finger patterns (including bunny ears)
- Canonical Dice Pattern Cards
- Counters
- Expression Cards
- Five-frame
- Screened Collections
- Ten-frame
- Thinkboard
- Twenty-frame
- Unifix Cubes or Unifix Towers

**Arithmetic Rack**: An arithmetic rack consists of 20 beads on two rows. On the upper row there are five beads of one colour and five beads of an alternate colour. This is replicated on the lower row. Convention dictates that, in the beginning position, all beads are placed on the right hand side of the rack. Beads are then moved to the left hand side of the rack where they are considered to be active.
**Bare Number Task:** This refers to the presentation of a task using only the conventional mathematical symbols (e.g., $5 + 2$). If this bare number task includes an equals sign it is referred to as an equation. When the task appears without an equals sign, as in the example above, it can also be referred to as an expression.

**Bead Strings:** These serve the same function as the tower of ten unifix cubes with each group of five represented by a different colour. Ten beads are threaded onto a string, with two groups of five beads represented by different colours. The bead string is used by students to represent partitions of 10. One student hides some beads at one end of the string in their hand and asks the other student to state how many beads (in the range 5 to 9) they can see. Students are encouraged to use their five-plus number combinations to determine the number of visible beads. The student then determines the number of hidden beads by stating the complement to ten. As students become more familiar with the partitions of 10, the visible beads include numbers in the range 1 to 4, which means students need to state the larger complement to ten. Stating this larger complement is often a more difficult task for students.

**Finger Patterns (including bunny ears):** The purpose of this activity is for students to partition a number, generally in the range 3 to 8, by making each part with raised fingers on their hands. The researcher calls out a number and students make finger patterns in front of their bodies or “bunny ears” above their head for that number. The students are encouraged to raise their fingers simultaneously (initially some find this difficult). Making finger patterns is considered to be an easier task than making bunny ears. The researcher then invites students with various partitions to show their finger patterns or bunny ears to the rest of the class. This facilitates discussion about the commutative principle and the range of partitions that can be represented. This acknowledgement of different partitions encourages students to partition the number in a different way from that of the students around them.
Canonical Dice Pattern Cards: The numbers 1 to 6 are represented as canonical (standard) dice patterns on cards. These cards are initially used to ascribe number to the dice patterns. Later, in the instructional sequence, the cards are used to represent addends in tasks involving screened collections. Figure B1 shows five as a canonical dice pattern.

![Five as a Canonical Dice Pattern](image)

Counters: When counters are used during this study, each addend is represented by a different colour. This deliberate use of colour encourages students to see each addend as a group as well as seeing the individual counters. This is thought to support the use of at least a count-on-by-ones strategy in preference to a count-by-ones from one strategy. In the setting of two screened collections, the use of a different colour for each addend supports the students’ imaging, or is thought to support the students’ visual imagery (Wright, Martland, Stafford, & Stanger, 2006, p. 35).

Expression Cards: This term refers to addition tasks presented in bare number format, printed on individual cards.

Five-frame: There are two types of five-frames used in the assessments: five-frame pattern cards and five-frame partition cards. The five-frame is a rectangular frame, which has five boxes arranged in one row and one dot placed in each box. On a five-frame pattern card (see Figure B2), dots fill the frame from the left hand side to the right hand side. There is one five-frame pattern card for each number in the range 1 to 5.
A five-frame partition card (see Figure B3) has five dots in two colours. There is one five-frame partition card for each partition of five (i.e., one red dot and four blue dots, two red dots and three blue dots, etc.). As a convention, the red dots are used to represent the first addend and the blue dots the second addend.

**Screened Collections:** One collection of counters is placed on the table and screened by an opaque piece of card. The student is told the number of counters under the screen, and the collection is flashed in order for the student to see the counters but not have enough time to count the counters by ones. A second collection is then placed on the table under a second card and the student is told the number of counters as they are flashed. The purpose of screening the collections is to determine if the student is able to count on from the number of items in the first screened collection or whether they need to count-from-one for both collections and then count all of the counters from one again. In the Mathematics Recovery Program, colour-coding is used as a means of differentiating the two collections: “In the case of an additive task involving two collections or a missing addend task, counters of contrasting colours (for example, red and green) are used for the collections” (Wright, Martland, Stafford, & Stanger, 2006, p. 34). In the assessment task presented in Task Group 3, one collection was made up of red counters, and the other of yellow counters.

**Ten-frame:** One of the main settings used throughout the assessment tasks and teaching sessions is the ten-frame. The two types of ten-frames used are ten-frame pattern cards and
ten-frame partition cards. The ten-frame is a rectangular frame, which has ten boxes arranged in two rows of five (see Figure B4).

A ten-frame pattern card has dots placed in the boxes in a five-wise arrangement, that is, one row is filled with dots from left to right before any dots are placed in the second row. Seven, for example, is represented as five plus two (see Figure B4). Frames for numbers less than five have an incomplete row of dots on the upper row and an empty lower row. There is one ten-frame pattern card for each number in the range 1 to 10.

![Figure B4 A Five-wise Arrangement for Seven on a Ten-frame](image1)

A ten-frame partition card (see Figure B5) has ten dots in two colours. There is one ten-frame partition card for each partition of ten (i.e., one red dot and nine blue dots, two red dots and eight blue dots, etc.). As a convention, the red dots are used to represent the first addend and the blue dots the second. The dots fill the frame in a five-wise arrangement as for the ten-frame pattern cards.

![Figure B5 A Ten-frame Partition Card Configured as 1 + 9](image2)

In the teaching tasks in this study, the ten-frame with dots in a pair-wise arrangement was also used. A pair-wise arrangement of the dots means that one dot is placed in the upper row and the next dot is placed in the same position in the lower row and so on. A pair-wise seven, for example, consists of four dots on the upper row and three dots on the lower row.
Thinkboard: A thinkboard is a worksheet divided into four quadrants, each pertaining to the same addition task but presented in various forms. These forms may be: (1) at the level of formal arithmetic; (2) as a word problem; (3) in a setting; and (4) as a picture.

Twenty-frame: The setting of a twenty-frame is a rectangular frame, which has 20 boxes arranged in two rows of ten. There is a line separating the first five boxes in each row from the second five boxes. Dots are placed in the boxes in a ten-wise arrangement, that is, one row is filled with dots from left to right before any dots are placed in the second row. Twelve, for example, is represented as 10 plus 2 (see Figure B6). Frames for numbers less than 10 have an incomplete row of dots on the upper row and an empty lower row.

Unifix Cubes: These are small, square interlocking blocks which are commonly used in Australian classrooms. Throughout this study, the unifix cubes are grouped in two ways: in towers of ten with each group of five represented by a different colour; or in towers of ten with all blocks in the same colour. Unifix cubes in two groups of five are used to represent five-plus number combinations, partitions of 10 and doubles while unifix cubes in groups of ten are used to represent ten-plus number combinations.
### Appendix C: Curriculum Statements From 10 Countries

#### Table C1
Curriculum Statements for Five-Year-Olds With Regard to the Use of Counting or Grouping Strategies to Solve Addition Tasks in the Range 1 to 20

<table>
<thead>
<tr>
<th>Country</th>
<th>Curriculum Statements FIVE-YEAR-OLDS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Australia</td>
<td>• Compare, order and make correspondences between collections, initially to 20, and explain reasoning</td>
</tr>
<tr>
<td></td>
<td>• Connect number names, numerals and quantities, including zero, initially up to 10 and then beyond</td>
</tr>
<tr>
<td></td>
<td>• Subitise small collections of objects using subitising as the basis for ordering and comparing collections of numbers</td>
</tr>
<tr>
<td></td>
<td>• Represent practical situations to model addition and sharing</td>
</tr>
<tr>
<td></td>
<td>• Using a range of practical strategies for adding and subtracting small groups of numbers, such as visual displays or concrete materials</td>
</tr>
<tr>
<td>New Zealand</td>
<td>• Know groupings with five, within ten, and with ten</td>
</tr>
<tr>
<td></td>
<td>• Communicate and explain <strong>counting</strong>, grouping, and equal-sharing strategies, using words, numbers, and pictures</td>
</tr>
<tr>
<td></td>
<td>• Generalise that the next counting number gives the result of adding one object to a set and that <strong>counting</strong> the number of objects in a set tells how many</td>
</tr>
</tbody>
</table>

Source: ACARA, 2011; Ministry of Education New Zealand, 2007

#### Table C2
Curriculum Statements for Six-Year-Olds With Regard to the Use of Counting or Grouping Strategies to Solve Addition Tasks in the Range 1 to 20

<table>
<thead>
<tr>
<th>Country</th>
<th>Curriculum Statements SIX-YEAR-OLDS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Australia</td>
<td>• Carry out simple addition and subtractions using <strong>counting</strong> strategies</td>
</tr>
<tr>
<td></td>
<td>• Represent and solve simple addition and subtraction problems using a range of strategies including <strong>counting-on</strong>, partitioning and rearranging parts</td>
</tr>
<tr>
<td></td>
<td>• Developing a range of mental strategies for addition and subtraction problems</td>
</tr>
<tr>
<td>Canada</td>
<td>• Compose and decompose numbers up to 20 in a variety of ways, using concrete materials (e.g., 7 can be decomposed using connecting cubes into 6 and 1, or 5 and 2, or 4 and 3)</td>
</tr>
<tr>
<td></td>
<td>• Relate numbers to the anchors of 5 and 10 (e.g., 7 is 2 more than 5 and 3 less than 10)</td>
</tr>
</tbody>
</table>
- Solve a variety of problems involving the addition and subtraction of whole numbers to 20, using concrete materials and drawings.

- Solve problems involving the addition and subtraction of single-digit whole numbers, using a variety of mental strategies (e.g., one more than, one less than, counting-on, counting back, doubles).

- Establishing a one-to-one correspondence when counting the elements in a set; counting by 1’s, 2’s, 5’s, and 10’s; adding and subtracting numbers to 20.

- **Count** the number of objects in a set, 1-10 count objects, pushing them aside while counting; count regular arrays or rows before random groups; use number rhymes and stories.

- Explore the components of number, 1-5 identify the ways in which the numbers can be modelled using concrete objects: 4 and 1, 2 and 2, 1 and 2 and 1 identify pairs of related facts: 1 and 2 is the same as 2 and 1.

- Combine sets of objects, totals to 5 add one more to a given set combine two sets, state total, record pictorially.

- Partition sets of objects, 1-5 partition sets of objects with a pencil or straw to show component parts record pictorially.

- Subitise (tell at a glance) the number of objects in a set, 1-5 tell at a glance how many objects are in a set estimate using a known set without counting, classify the other sets as less than/about the same as/more than the given set.

**Ireland 1999**

- Use simple additive strategies with whole numbers and fractions.

- Know the basic addition and subtraction facts.

- Communicate and interpret simple additive strategies, using words, diagrams (pictures), and symbols.

- Generalise that whole numbers can be partitioned in many ways.

**New Zealand 2007**

- Building up the addition bonds up to 9 + 9 and committing to memory.

**Singapore 2007**

- Under the relationship between numbers and quantities; connect counting to cardinality.

- When counting objects, say the number names in the standard order, pairing each object with one and only one number name and each number name with one and only one object.

- Understand that the last number name said tells the number of objects counted. The number of objects is the same regardless of their arrangement or the order in which they were counted.

- Understand that each successive number name refers to a quantity that is one larger.

- Count to answer “how many?” questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1-20, count out that many objects.

**United States of America 2010**

- Understand the relationship between numbers and quantities; connect counting to cardinality.

- When counting objects, say the number names in the standard order, pairing each object with one and only one number name and each number name with one and only one object.

- Understand that the last number name said tells the number of objects counted. The number of objects is the same regardless of their arrangement or the order in which they were counted.

- Understand that each successive number name refers to a quantity that is one larger.

- Count to answer “how many?” questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1-20, count out that many objects.
Table C3
Curriculum Statements for Seven-Year-Olds With Regard to the Use of Counting or Grouping Strategies to Solve Addition Tasks in the Range 1 to 20

<table>
<thead>
<tr>
<th>Country</th>
<th>Curriculum Statements SEVEN-YEAR-OLDS</th>
</tr>
</thead>
</table>
| **Australia** Australia 2011 | • Group, partition and rearrange collections up to 1000 in hundreds, tens and ones to facilitate more efficient counting  
• Explore the connection between addition and subtraction  
• Becoming fluent with partitioning numbers to understand the connection between addition and subtraction  
• Using **counting-on** to identify the missing element in an additive problem  
• Solve simple addition and subtraction problems using a range of efficient mental and written strategies  
• Becoming fluent with a range of mental strategies for addition and subtraction problems, such as commutativity for addition, building to 10, doubles, 10 facts and adding 10  
• Modeling and representing simple additive situations using materials such as 10 frames, 20 frames and empty number lines |
| **Canada** Canada 2005 | • Solve problems involving the addition and subtraction of whole numbers to 18, using a variety of mental strategies (e.g. “To add 6 + 8, I could double 6 and get 12 and then add 2 more to get 14”) |
| **Ireland** Ireland 1999 | • **Count** the number of objects in a set, 0-20 **count** the same set several times, starting with a different object each time present different patterns and arrays of the same number  
• Explore the components of number, 1-10  
• Combine sets of objects, totals to 10 use appropriate strategies: **counting all**, **counting-on** on the number strip start at 5, **count on** 3, where am I  
• Partition sets of objects, 0-10 8 people are on my team, 6 are girls, how many are boys? record pictorially  
• Use the symbols + and = to construct word sentences involving addition  
• How many different ways can you make a pattern with 6 counters? |
| **New Zealand** New Zealand 2007 | • Use a range of additive and simple multiplicative strategies with whole numbers, fractions, decimals, and percentages  
• Generalise the properties of addition and subtraction with whole numbers |
| **Singapore** Singapore 2007 | • Addition and subtraction of numbers up to 3 digits |
| **United States of America** United States of America 2010 | • Use addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using objects, drawings,
and equations with a symbol for the unknown number to represent the problem

- Solve word problems that call for addition of three whole numbers whose sum is less than or equal to 20, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem

- Apply properties of operations as strategies to add and subtract. Examples: If $8 + 3 = 11$ is known, then $3 + 8 = 11$ is also known. (Commutative property of addition.) To add $2 + 6 + 4$, the second two numbers can be added to make a ten, so $2 + 6 + 4 = 2 + 10 = 12$ (Associative property of addition)

- Understand subtraction as an unknown-addend problem. For example, subtract $10 - 8$ by finding the number that makes 10 when added to 8

- Relate counting to addition and subtraction (e.g., by counting on 2 to add 2)

- Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., $8 + 6 = 8 + 2 + 4 = 10 + 4 = 14$); decomposing a number leading to a ten (e.g., $13 - 4 = 13 - 3 - 1 = 10 - 1 = 9$); using the relationship between addition and subtraction (e.g., knowing that $8 + 4 = 12$, one knows $12 - 8 = 4$); and creating equivalent but easier or known sums (e.g., adding $6 + 7$ by creating the known equivalent $6 + 6 + 1 = 12 + 1 = 13$)

- Understand the meaning of the equal sign, & determine if equations involving addition and subtraction are true or false. e.g., which of the following equations are true and which are false? $6 + 7 = 8 - 1$, $5 + 2 = 2 + 5$, $4 + 1 = 5 + 2$

- Determine the unknown whole number in an addition or subtraction equation relating three whole numbers. e.g., determine the unknown number that makes the equation true in each of the equations $8 + ? = 11$, $5 = _ - 3$, $6 + 6 = _$

Source: ACARA, 2011; NCCA [Ireland], 1999; NGA Center and CCSSO [USA], 2010; Ontario Ministry of Education, 2005; Ministry of Education New Zealand, 2007; Ministry of Education Singapore, 2006

Table C4
Curriculum Statements for Five to Nine+ -Year-Olds With Regard to the Use of Counting or Grouping Strategies to Solve Addition Tasks in the Range 1 to 20

<table>
<thead>
<tr>
<th>Country</th>
<th>Five years</th>
<th>Six years</th>
<th>Seven years</th>
<th>Eight years</th>
<th>Nine+ years</th>
</tr>
</thead>
<tbody>
<tr>
<td>England 2011</td>
<td>KS1 5–7-year-olds</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Understand addition and use related vocabulary; recognise that addition can be done in any order</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Develop rapid recall of number facts: know addition and subtraction facts to 10 and use these to derive facts with totals to 20</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Develop a range of mental methods for finding, from known facts, those that they cannot recall</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>KS2 7–11-year-olds</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Recall all addition and subtraction facts for each number to 20</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Hong Kong 2000

- 6–8-year-olds
- Numbers to 20
- Numbers to 100
- Basic addition and subtraction (within 18)

The Netherlands 2004

Core goals for the end of primary school (Year 6):
- Can do addition tables and multiplication tables up to ten
- Learn to understand in a general way the structure and relationships of whole numbers, decimal numbers, fractions, percentages and ratios and are able to calculate with them in practical situation
- Learn to carry out mentally and quickly the basic operations with whole numbers at least up to 100; and know the additions and subtractions up to 20 and the multiplication tables by heart
- Learn written additions, subtraction, multiplications and divisions in more or less curtailed standardized ways. Learn to add, subtract, multiply and divide in a clever way

Sweden 2011

Knowledge requirements for acceptable knowledge at the end of year 3:
- Pupils can choose and use basically functional mathematical methods with some adaptation to the context to make simple calculations with natural numbers and solve simple routine tasks with satisfactory results
- Pupils can use mental arithmetic to perform calculations using the four operations when the numbers and the answers are in the range 0–20, and also for calculations of simple numbers in higher ranges of numbers
- For addition and subtraction, pupils can choose and use written methods of calculation with satisfactory results when numbers and answers lie within an integer range of 0–20

Source: Department for Education [UK], 2011; Skolverket: The Swedish National Agency for Education, 2011; The Curriculum Development Council [Hong Kong], 2000; Van den Heuvel-Panhuizen & Wijers [the Netherlands], 2005
Appendix D: Sequence of Lessons Delivered During the Teaching Experiment

<table>
<thead>
<tr>
<th>Lesson</th>
<th>Learning Intention</th>
<th>Focus Strategy</th>
<th>Instructional Materials</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Develop students’ knowledge of partitions of 5.</td>
<td>Partitioning 5,</td>
<td>2 sets of 5 counters, 5 frame pattern cards, 5 frame partition cards, towers of 5 unifix cubes.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Commutativity.</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Reinforce partitions of 5, introduce recording building to 5 on number line.</td>
<td>Partitioning 5 Commutativity.</td>
<td>5 frame pattern cards, record on number line, towers of 5 unifix cubes.</td>
</tr>
<tr>
<td>3</td>
<td>Develop students’ knowledge of partitions of numbers 2–4.</td>
<td>Partitioning numbers 2–4.</td>
<td>towers of unifix cubes.</td>
</tr>
<tr>
<td>4</td>
<td>Students to build-through-5 to combine two numbers.</td>
<td>Addition through 5.</td>
<td>5 frames &amp; counters.</td>
</tr>
<tr>
<td>5</td>
<td>Students to record partitions of 5 and build-through-5 tasks presented at level of formal arithmetic.</td>
<td>Partitioning 5, addition through 5.</td>
<td>5 frames &amp; counters, 10 frames &amp; counters.</td>
</tr>
<tr>
<td>6</td>
<td>Students to record on a 10 frame build-through-5 for addition tasks presented at level of formal arithmetic.</td>
<td>Addition through 5.</td>
<td>5-plus 10 frame pattern cards, 10 frames &amp; counters.</td>
</tr>
<tr>
<td>7</td>
<td>Students to begin to automatically recall 5-plus combinations.</td>
<td>Numbers in the range 6 to 10 as 5-plus combinations.</td>
<td>bunny ears for numbers 6–10 as a 5-plus combination, arithmetic rack, expression cards, 5 and 5 bead string.</td>
</tr>
<tr>
<td>8</td>
<td>Explore partitions of 10–large plus small, record at level of formal arithmetic.</td>
<td>5-plus number combinations.</td>
<td>5 and 5 bead string, arithmetic rack, bare number task.</td>
</tr>
<tr>
<td>9</td>
<td>Partitions of 10–large plus small – record at level of formal arithmetic.</td>
<td>Partitioning 10,</td>
<td>arithmetic rack, expression cards, 10 frame pattern cards, 10 frame &amp; counters, bunny ears.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>commutativity.</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>Exploring all partitions of 10 – record at level of formal arithmetic.</td>
<td>Partitioning 10,</td>
<td>10 frame pattern cards, 10 frame bus &amp; counters.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>commutativity.</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>Exploring all partitions of ten</td>
<td>Partitioning 10,</td>
<td>5-plus as two screened</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5 plus as two screened</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Activity</td>
<td>Materials</td>
<td></td>
</tr>
<tr>
<td>---</td>
<td>--------------------------------------------------------------------------</td>
<td>---------------------------------------------------------------------------</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>Linking addition tasks at level of formal arithmetic with setting of 20 frame in context of passengers on a bus via a thinkboard.</td>
<td>arithmetic rack, thinkboard, 20 frame &amp; counters.</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>Linking addition tasks at level of formal arithmetic with setting of 20 frame in context of passengers on a bus via a thinkboard.</td>
<td>thinkboard, 20 frame &amp; counters, expression cards.</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>Students to use 10-plus knowledge to help solve 9-plus tasks.</td>
<td>20 frame &amp; counters, expression cards.</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>Students to use 10-plus knowledge to help solve 9-plus tasks, introduce small doubles.</td>
<td>towers of unifix, 20 frame &amp; counters, 10 frame pair-wise pattern cards, bare number task.</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>Students to represent 5-plus number combinations in 4 different ways.</td>
<td>bare number task, domino patterns, unifix blocks, 10 frame &amp; counters.</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>Students to partition – numbers 4–9</td>
<td>domino patterns, bare number task.</td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>Students to partition 6, 7, 8, 9, &amp;10, &amp; use a build-through-10 strategy</td>
<td>20 frame &amp; counters, bead string, towers of unifix, own recording.</td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>Develop students’ knowledge of and large doubles.</td>
<td>arithmetic rack, 10 frame pair-wise pattern cards, finger patterns, unifix blocks, arithmetic rack, 10 frame pair-wise pattern cards, expression cards, unifix blocks, bingo game.</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>Develop students’ knowledge of small and large doubles.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>Linking bare number doubles tasks with setting of 10 frame and word story via a thinkboard.</td>
<td>Small doubles, large doubles.</td>
<td>arithmetic rack, 10 frame &amp; counters, thinkboard.</td>
</tr>
<tr>
<td>22</td>
<td>Revision of partitions of 5 and 10.</td>
<td>Partitioning 5 and 10</td>
<td>unifix cubes, expression cards.</td>
</tr>
<tr>
<td>23</td>
<td>Develop students’ knowledge of small near doubles.</td>
<td>Small near doubles.</td>
<td>unifix cubes, expressions at level of formal arithmetic.</td>
</tr>
<tr>
<td>24</td>
<td>Students to solve one-digit addition tasks in the range 1 to 10 using strategies of their choice</td>
<td>Build-through-10, doubles or near-doubles.</td>
<td>10 frame 5-plus pattern cards, dice, expressions at level of formal arithmetic.</td>
</tr>
</tbody>
</table>

Note: The shaded area indicates that two different activities were provided during Part 3 of the lesson in order to cater for student needs.
Appendix E: Assessment Interview Script

**Prep Pre-assessment – Individual Interview**

Student Name: ___________________________  D.O.B: ___________  Age: ___________

**Task Group 1: Forward number word sequence**

*Start counting for me, and I will tell you when to stop.*

1 to 20  
13 to 24

5 to 16  
17 to 35

If student is unsuccessful with the first two items of Task 1, terminate the interview.

**Task Group 2: Numeral identification**

Place numeral cards on table.

*Read this number for me please.*

3  7  1  5  4  6  8  2  9  10

Task 2a.

If student successful, continue with

13  17  15  14  16  18  11  19  12  20

**Task Group 3: Addition tasks involving two screened collections**

Screen two collections of counters.

*There are four red counters under here, and two blue counters under here. How many counters are there all together?*

4 red counters and 2 blue counters

9 red counters and 4 blue counters
Task 3a.

If unsuccessful on Task 3, repose the first task 4 red counters and 2 blue counters, with the second collection unscreened. Do not re-pose the second task 9 red counters and 4 blue counters.

If unsuccessful on Task 3a, got to Task 3b.

Task 3b.

Place a pile of counters on the table.

*Get me fifteen counters please.*

**Task Group 4: Ascribing number to five-wise patterns on a five-frame**

Show five-frame for three to student and establish that there are three dots and two empty squares. *When all squares are filled, there are five dots, and no empty squares.*

Flash five-frame at student and ask *How many dots can you see?*

- 2 dots
- 3 dots
- 5 dots

- 4 dots
- 1 dot

**Task Group 5: Partitions of five on a five-frame**

Show one five-frame, with the red dots on the left hand side.

*Here is a five-frame that has red dots and then blue dots. Can you make a number sentence using these cards, showing the number of red dots and the number of blue dots that make five?*

- 4 + 1
- 3 + 2
- 2 + 3
- 1 + 4

**Task Group 6: Ascribing number to five-wise patterns on a ten-frame**
Familiarise student with a full ten-frame. i.e., five dots on the upper row and five dots on the lower row, ten dots in total. Flash five wise ten-frame at student and ask *How many dots can you see?*

- 3 dots
- 4 dots
- 5 dots

**Task Group 7: Moving beads in the range 1 to 10 on the arithmetic rack**

Familiarise student with the arithmetic rack i.e., five beads of each colour, ten beads on the upper and ten beads on the lower row, twenty beads in total.

*Use one slide to move the following number of beads?*

If student uses a count–by-one strategy, then question

*Do you know a quicker way to make this number?* N.B. Rack must be cleared between each task.

- 3 beads
- 1 bead
- 2 beads
- 4 beads
- 5 beads

If student is only able to use a count-by-ones strategy in Task Group 7, terminate the interview.

**Structuring Numbers in the range 1 to 10**

Task 6a – Ascribing number to a five-wise pattern on a ten-frame

If successful with Task 6, continue with

- 6 dots
- 9 dots
- 7 dots
- 10 dots
- 8 dots

Task 7a - Making five wise numbers on the arithmetic rack
If student successful with Task 7, continue with

6 beads
7 beads
8 beads
9 beads
10 beads

**Task Group 8: Partitions of ten on a ten-frame**

Show one ten-frame, with the red dots shown from the top left hand square. Here is a ten-frame that has red dots and blue dots.

*Can you make a number sentence using these cards, showing the number of red dots and the number of blue dots that make ten?*

- \[6 + 4\]  
- \[9 + 1\]  
- \[7 + 3\]  
- \[10 + 0\]  

**Task 8a.**

- \[3 + 7\]  
- \[4 + 6\]

**Task Group 9: Small doubles presented in bare number format**

Present the following as bare number tasks on cards.

*Read this to me. Tell me the answers to these please.*

- \[3 + 3\]  
- \[1 + 1\]  
- \[2 + 2\]  
- \[5 + 5\]  
- \[4 + 4\]  

If student unsuccessful on doubles as bare numbers, repeat task but with doubles represented on the arithmetic rack. Screen all of the beads. Move three beads from each row to the visible end of the rack. N.B. Rack must be cleared between each task.

*How many beads all together?*
**Task Group 10: Ten-plus number combinations presented in bare number format**

Present the following as bare number tasks on cards.

*Read this to me. Tell me the answers to these please.*

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>10 + 4</td>
<td>10 + 5</td>
</tr>
<tr>
<td>10 + 3</td>
<td>10 + 9</td>
</tr>
<tr>
<td>10 + 7</td>
<td>10 + 10</td>
</tr>
<tr>
<td>10 + 6</td>
<td>10 + 8</td>
</tr>
<tr>
<td>10 + 2</td>
<td>10 + 1</td>
</tr>
</tbody>
</table>

If student unsuccessful on ten-plus facts as bare numbers, repeat task but with ten-plus facts represented on the arithmetic rack. Slide across ten beads on the upper row and the other number on the lower row. N.B. Rack must be cleared between each task.

*How many beads all together?*

If student is only able to use a count-by-ones strategy in Task 9, terminate the interview.

**Task Group 11: Large doubles presented in bare number format**

Present the following as bare number tasks on cards.

*Read this to me. Tell me the answers to these please.*

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>8 + 8</td>
<td>10 + 10</td>
</tr>
<tr>
<td>6 + 6</td>
<td>9 + 9</td>
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<tr>
<td>7 + 7</td>
<td></td>
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**Task Group 12: One-digit additions, bridging ten, presented in bare number format**

Present the following as bare number tasks on cards.

<p>| | |</p>
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<tbody>
<tr>
<td>9 + 4</td>
<td>8 + 6</td>
</tr>
<tr>
<td>9 + 5</td>
<td>7 + 5</td>
</tr>
<tr>
<td>8 + 3</td>
<td>9 + 6</td>
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<tr>
<td>7 + 4</td>
<td>9 + 3</td>
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</table>
8 + 7
Appendix F: 24 Lesson Plans of the Instructional Sequence
<table>
<thead>
<tr>
<th>Day</th>
<th>Mathematical Objective for Lesson</th>
<th>Tool Session</th>
<th>Whole Class Focus/:</th>
<th>Activity:</th>
<th>Share Time / Summary:</th>
<th>Resources / Assessment:</th>
</tr>
</thead>
</table>
| Session 1 | Moving the children towards automaticity with partitions of 5                                     | Chn make Bunny Ears in range 1-5. Share different ways of doing it. If ok, move to range 6-10. *(didn't do)* | Today's lesson is all about the different ways we can make 5. Flash black 5 frames - chn respond with number of dots they can see. Discuss reasoning strategies i.e. I can see 4 because there is 1 missing. Change – chn now read number of spaces. Model How Many Am I Hiding game? One child breaks off some of a tower of 5 unifix and hides behind back. Show how many left, how many am I hiding? Chn play in pairs. | Chn work in pairs. Each pair has 5 red/white counters in a cup, shake and tip, organise onto 5 frame. (Discussion with chn about keeping same colours together). Chn record with coloured dots e.g. 3 and 2  
\[
\begin{array}{c}
\circ \circ \circ \circ \circ \\
3 \\
\end{array}
\]  
Share combinations found by chn. Model partitions of 5 (start with 5 red frames and record then turn one red over each time and record) using drop down recording  
\[
\begin{array}{c}
5 \\
/ \ \ / \ \\
5 \ 0 \ 4 \ 1 \\
\end{array}
\]  
in systematic order. Chn notice “turn arounds” i.e. 3 and 2 and 2 and 3 - establish that as shared language | Share time / summary: | 5 frames  
Unifix blocks  
Red/white counters  
Plastic cups  
Blank 5 frames |

Vocabulary/Taken as Shared Understandings: 5 frames, dots, spaces, How Many Am I Hiding? (how do I play this game?) turn arounds
<table>
<thead>
<tr>
<th>Day</th>
<th>Mathematical Objective for Lesson</th>
<th>Tool Session:</th>
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</tr>
</thead>
<tbody>
<tr>
<td>Session 2 27.7.10</td>
<td>Reinforce partitions of 5, introduce recording building to 5 on number line</td>
<td>How Many am I Hiding? Game in pairs on floor. Encourage chn to use language and reasoning to explain how they know how many are hiding.</td>
<td>Make to 5 Game - model with class - chn play in pairs on the floor. Provide 5 unifix chn for those who need extra support. Model use of number line for recording thinking. Start at black dot and jump to 5 - jumping by 1s is inefficient - use knowledge of partitions of 5 to know how many to jump to make 5.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Activity:</th>
<th>Share Time / Summary:</th>
<th>Resources/Assessment:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chn record jumping to 5 on worksheet. Share time - then chn try recording own jumps like mathematicians i.e. $3 + 2 = 5$</td>
<td>Writing like a mathematician Record large jump on number line as a sum i.e. $3 + 2 = 5$ Discuss inefficiency of jumping by 1 when could do bigger jumps to land on 5.</td>
<td>Make to 5 Game Card Whiteboard markers Worksheet: 1-9 numberline</td>
</tr>
</tbody>
</table>

**Vocabulary:**
<table>
<thead>
<tr>
<th>Day</th>
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</tr>
</thead>
<tbody>
<tr>
<td>Session 3 28.7.10</td>
<td>Efficient recording on numberline Partitions of numbers 1-4</td>
<td>Numerals 1 - 10 Place 5 on board and ask chn to place other numbers in appropriate positions. Encourage use of language such as before, after and in between</td>
<td>Recording of one big jump on number line as opposed to repeated jumps of 1. Model as more efficient - chn complete worksheet at seats, by jumping from black dot the number indicated in square to right, by both ones and by one big jump. When sharing include numbers on some numberlines and record as mathematicians what happened. 2+3 = 5</td>
</tr>
<tr>
<td>Preps 100 Days of School</td>
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</tr>
</tbody>
</table>

**Activity:**
Model 3 unifix – how can I break these up so that I have some in each hand? Record as dots eg. 2 1
Chn sent to seats to find how many different ways they could make 4?
Share different ways to make 4, record systematically. Discuss turnarounds. Why doesn't 2+2 have a turnaround? - this is called a double.

**Share Time / Summary:**
Share different ways to make 4, record systematically. Discuss turnarounds. Why doesn't 2+2 have a turnaround? - this is called a double.

**Resources/ Assessment:**
Numeral cards 1-10 Blank Number line worksheet Unifix

**Vocabulary:**
before, after, in between, turnarounds, double
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</thead>
</table>
| Session 4 2.8.10 | Chn to build through 5 to combine two numbers | Make to 5 - chn complete in pairs with whiteboard markers. Provide 5 cubes for chn to use if needed. | Roll dice - place this many red counters on five frame. Roll dice again - build to 5 by filling frame and then count on from 5 to determine total. | Chn complete 5 frame activity at tables. Model recording. If start with 3 and roll 4, +2 +2 3 → 5 → 7  
Match it game - chn to match written expression with 5 frame - first addend being number of dots, second addend number of spaces. | Make link between knowing partitions of 5 and being able to use this to support addition in the range 1 to 10 by building through 5. | Make 5 Whiteboard markers  
Blank 5 frames  
Red/white counters  
Dice numbered 2 - 4  
Template for recording build through 5  
Match It Game cards |

**Vocabulary:** build to 5, build through 5
<table>
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<tbody>
<tr>
<td>Session 5 3.8.10</td>
<td>Chn to record partitions of 5 as expressions Chn to build through 5 to solve bare addition tasks</td>
<td>Flash five wise ten frames, chn bark at number of dots they can see</td>
<td>Model a 3 five frame. Ask child to write equation for number of dots and number of spaces. i.e. 3 + 2 = 5 Repeat with a 1 five frame. 1 + 4 = 5 Chn complete worksheet in their books to record equation to match frame. Can you find any turnarounds?</td>
<td>Regroup with whole class. Roll dice, place dots on top row of ten frame. Roll again, how many to build to five - how many will there be all together? Model filling on ten frame. Try this session with no recording.</td>
<td>How could we know the answers to the following? 4 + 3 3 + 3 2 + 4 without having to count by ones?</td>
<td>Five wise ten frames Worksheet 1 Blank ten frames</td>
</tr>
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</table>

**Vocabulary:** build to five, build through five
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<tbody>
<tr>
<td>Session 6 3.8.10</td>
<td>Chn to record on a ten frame building through 5 for bare number addition tasks</td>
<td>Flash five plus ten frames - chn to state bare number fact to match i.e. 5+2 =7</td>
<td>Solving bare number tasks by building through 5 on blank ten frame. Pose question 4 + 3 - fill 4 in five wise arrangement on ten frame in red - then 3 in yellow - discuss build through 5.</td>
<td>Chn complete Worksheet Lesson 6 to practise building through 5. Game of Hide the5 Cubes</td>
<td>How could we know the answers to the following? 4 + 3 3 + 3 2 + 4 without having to count by ones? Model recording on Number line by building through 5.</td>
<td>Red/white counters Worksheet Five wise ten frames</td>
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**Vocabulary:** build to five, build through five
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<tbody>
<tr>
<td>Session 7 6.8.10</td>
<td>Chn to begin to automatically recall 5+ combinations</td>
<td>Bunny ears but must be 5 plus</td>
<td>Revisit idea of five plus from previous lesson Model five plus on top and bottom row of arithmetic rack. Hold up five plus card eg. 5+2 - ask child to come and slide 5 beads on top and 2 beads on bottom - how many beads all together? Chn complete cut and paste activity matching five plus expression with image of arithmetic rack. Record total number of beads for themselves</td>
<td>Look at 5 and 5 bead string. Discuss how many beads of each colour, how many beads all together. Discuss non-count-by-ones strategy. One child rolls ten sided dice and the other must show only this many beads on the strung. Repeat. Look for non-count-by-ones.</td>
<td>Ask child to move 7 beads on arithmetic rack - who can record this as a 5 plus? Repeat this with other numbers in the range 6 - 10. Does it matter whether we slide 5 plus' on both rows or all in one row? Why/Why not?</td>
<td>Arithmetic rack Worksheet bead strings 10 sided dice</td>
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<tbody>
<tr>
<td>Session 8 10.8.10</td>
<td>Explore partitions of ten - large + small with chn</td>
<td>Show 5+ written expression card - how many does this make? Check on arithmetic rack.</td>
<td>Revisit idea of partitions of 5. Record systematically to have on display in room ie. 5 + 0, 4 + 1, 3 + 2 etc. Today we are going to use these 5+ to help us work up to 10. Make a 5+ on bead string w/o counting-by-ones. How many are left hiding in your hand? How do you know? Practise with bead string with partner.</td>
<td>Chn complete worksheet where they need to draw in the beads on the arithmetic rack to build to ten, and finish this as a bare number sentence. Ext: Ask chn to write the turnaround of each bare number sentence</td>
<td>Review partitions of ten on arithmetic rack (possibly screening one end) and make link with using partitions of five to help.</td>
<td>Arithmetic rack Worksheet Bead strings</td>
</tr>
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Vocabulary:
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<tbody>
<tr>
<td>Session 9 12.8.10</td>
<td>Partitions of ten - large and small - recording</td>
<td>Make numbers on rack - how many left behind? Record down LHS of paper. If had 7 and 3 what would be prediction for 3 beads on LHS frame - HDYK?</td>
<td>Play match up ten frames with number of dots Video Sam and Joshua Ask chn to put into some sort of order - how do they sort and why?</td>
<td>Gp 1 - how many different ways can you make 10? Explore different partitions of 10 - have you found them all HDYK? Gp 2 - Colour and record numbers in range 5 - 10 as 5+ and record expression to match</td>
<td>Bunny ears seen - put up 7 fingers - how many are still down - link with partitions of 10.</td>
<td>Arithmetic rack Ten frame matching cards</td>
</tr>
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Lisa Marie - colour dots to match numeral - make towers to match numeral

Gp 1 - Partitions of ten Gp 2 - 5 plus with expression -
<table>
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<tr>
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<tbody>
<tr>
<td>Session 10</td>
<td>Partitions of ten - large and small</td>
<td>Make numbers on rack - how many left behind? Record down LHS of paper as</td>
<td>Model 10 people on</td>
<td>Gp 1 - there are 10 people on the double decker bus - how many different ways can 10 people be arranged on</td>
<td>Covered collections 5 plus</td>
<td>Arithmetic rack</td>
</tr>
<tr>
<td>13.8.10</td>
<td>- recording</td>
<td>equations. If had 7 and 3 what would be prediction for 3 beads on LHS frame</td>
<td>the bus - need to</td>
<td>the bus? Gp 2 - Ten Frame bus - how many people can you have on the bus if the top row has to be full?</td>
<td></td>
<td>Bus templates</td>
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<td></td>
<td></td>
<td>- HDYK?</td>
<td>have people all sitting together, no empty seats in between people.</td>
<td>Make 5 game board with white board markers</td>
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<td>Worksheets</td>
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<td></td>
<td>How can I organise 10 people to be on the bus?</td>
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<td>Day</td>
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<tr>
<td>Session 11 13.8.10</td>
<td>Partitions of ten - large and small - recording</td>
<td>Make numbers on rack - how many left behind? Record down LHS of paper as equations. If had 7 and 3 what would be prediction for 3 beads on LHS frame - HDYK?</td>
<td>Model 10 people on the bus - need to have people all sitting together, no empty seats in between people. How can I organise 10 people to be on the bus?</td>
<td>Gp 1 - there are 10 people on the double decker bus - how many different ways can 10 people be arranged on the bus? Gp 2 - Ten Frame bus - how many people can you have on the bus if the top row has to be full? Make 5 game board with white board markers</td>
<td>Covered collections 5 plus</td>
<td>Arithmetic rack Bus templates Worksheets</td>
</tr>
</tbody>
</table>

Lisa Marie - colour dots to match numeral - make towers to match numeral
Gp 1 - Partitions of ten
Gp 2 - 5 plus with expression -
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<tr>
<th>Day</th>
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</thead>
</table>
| Session 12 17.8.10 | Linking addition statements from setting of ten frame with concept of joining together via a thinkboard | 10 plus - make on arithmetic rack and record as 10+                          | Model a thinkboard using Uncle Henry's bus with 4 people on the top row and "some" on the bottom row. How many people all together? | Gp 1 - 9+ on Uncle Henry's (twenty frame) bus  
Gp 2 - 4+ on Uncle Henry's (ten frame) bus | Review 10 + and link with 9+ on bus                                                               | counters ten frame and twenty frame bus thinkboards |

Lisa Marie - circle dots to match numeral and trace digit- make towers to match numeral  
Gp 1 - Twenty frame thinkboard,  
Gp 2 - Ten frame thinkboard -
<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>Session 13</td>
<td>Linking addition statements from setting of ten frame with concept of joining together via a thinkboard</td>
<td>10 plus - show expression, chan predict answer - check on arithmetic rack</td>
<td>Model a thinkboard using Uncle Henry’s bus with 9 people on the top row and “some” on the bottom row. How many people all together? Model build through 10 strategy</td>
<td>-9+ on Uncle Henry’s (twenty frame) Bus</td>
<td>Review 10 + and link with 9+ on bus Discuss build through 10</td>
<td>counters ten frame and twenty frame bus thinkboards</td>
</tr>
</tbody>
</table>

2 sts - colour dots to match story
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</thead>
<tbody>
<tr>
<td>Session 14 19.8.10</td>
<td>Making link between 9+ and 10+</td>
<td>10 plus and plus 10 mixed up - show expression, chn predict answer - check on arithmetic rack if necessary, discuss turn arounds</td>
<td>Model 9 + 4 on Uncle Henrys Bus - move one from bottom row to top to build to ten - then use 10 plus to work out how many on bus all together</td>
<td>Worksheet to transform 9+ to 10+</td>
<td></td>
<td>counters twenty frame bus worksheet</td>
</tr>
</tbody>
</table>

2 stts - 4+ on ten frame bus
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<tr>
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</thead>
<tbody>
<tr>
<td>Session 15 23.8.10</td>
<td>Making link between 10+ and 9+</td>
<td>Small doubles on arithmetic rack - chn make on fingers with a partner</td>
<td>Place 6 teddies under a cover - remove one - how many left? Repeat with different numbers covered. What is happening each time? Show card - 10 + 4 Who can come and make that for me with unifix - what is the easiest way? i.e. Tower of 10 plus four more How many unifix all together? HDYK? Take one unifix off tower - now how many all together? (Make link with teddies task if necessary)</td>
<td>Chn complete worksheet in pairs, making 10+ with worksheet and then removing one to make 9+</td>
<td>Revisit 9+ and make link with 10+. Show chn expression card with 9+ and see if they can use 10+ remove 1 to help them</td>
<td>Unifix in tens with 5 shown in two colours</td>
</tr>
</tbody>
</table>

2 sts - 4+ on ten frame bus
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<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>Session 16 24.8.10</td>
<td>Try to support chn in making links between settings</td>
<td>Small doubles on rack and chn make finger patterns and join fingers with partner to make small doubles</td>
<td>Flash domino patterns - How many dots do you see? How do you know? Use domino cards - how many dots all together? How do you know?</td>
<td>Chn to be given a sheet with the number 6, 7, 8, or 9 on it. They need to represent this as a 5+ in 4 different ways - eg. One as a bare number sum, ten frame, domino cards, make with unifix blocks</td>
<td>How can we know that 5 + 2 is 7? We can 'prove' it lots of ways - counters, ten frames, arithmetic rack, unifix, domino cards. Repeat</td>
<td>Worksheet Unifix Domino cards Blank ten frames</td>
</tr>
</tbody>
</table>

- complete same task but just for the number 6 - no expectation of it as a +
<table>
<thead>
<tr>
<th>Day</th>
<th>Mathematical Objective for Lesson</th>
<th>Tool Session: Use domino patterns (5+) screened to work out how many dots all together, chn model with their own unifix</th>
<th>Whole Class Focus:/ Look at domino pattern for 6. How many dots can you see? How do you know it is 6? Cut up pattern and stick on sheet to model &quot;splits&quot; of 6.</th>
<th>Activity: Chn to be given domino patterns of a number in range 4 - 10. Cut this up into splits and record these as a bare number sentence. Eg. 3 + 4 = 7</th>
<th>Share Time / Summary: Review small doubles - look at 10 in five wise on rack</th>
<th>Resources/ Assessment: Worksheet Unifix Domino cards Blank ten frames</th>
</tr>
</thead>
<tbody>
<tr>
<td>Session 17 25.8.10</td>
<td>Partitions (splits) of numbers 4 - 9</td>
<td></td>
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</table>

- complete same task but just for the number 4
- complete same task but just for the number 5
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<thead>
<tr>
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<tbody>
<tr>
<td>Session 18 30.8.10</td>
<td>Splits of 6, 7, 8, 9, and 10 Building though 10</td>
<td>Model 10+ on AR in a doubles pattern - then use this to introduce large doubles</td>
<td>Group 1 - Explore splits of 6, 7, 8, 9, or 10 using coloured pegs on plastic plates Group 2 - Splits of 10 on bead string and then building though 10 on 9+ ten frame with second empty ten frame below</td>
<td>Chn to independently work on given task - able to determine own method of recording</td>
<td>Model 9+ as build through 10 with whole class, record on ENL</td>
<td>Plastic plates Pegs AR Double ten frames templates and counters</td>
</tr>
</tbody>
</table>

Group 1 -  
Group 2 -
<table>
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<tbody>
<tr>
<td>Session 19 1.9.10</td>
<td>Chn to practise their small and large doubles</td>
<td>Splits of 5 - Introduce with paper plate and 5 pegs activity then play hide cubes game</td>
<td>Review large doubles on arithmetic rack - chn make it with fingers with a partner Match with expression card</td>
<td>Chn to match written expression with total and with representation of AR. Encourage use of blocks to represent and use of non c1s strategies Group 1 - small doubles Group 2 - large doubles</td>
<td>Review small and large doubles</td>
<td>Plastic plates Pegs AR Worksheet 2 sets of unifix</td>
</tr>
</tbody>
</table>

Group 1 -  
Group 2 -
<table>
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<tbody>
<tr>
<td>Session 20 3.9.10</td>
<td>Chn to practise their small and large doubles</td>
<td>Make small double on AR - child to select matching expression card</td>
<td>Review using small doubles to help us work out our large doubles - match fingers with a partner to show small and then large doubles</td>
<td>Groups 1, 2, 3 - play bingo with small doubles - one child (or teacher) makes with unifix other chn cover one of the counters on the game board Group 4 - play in pairs, match up large doubles cards as quickly as possible, can then play snap</td>
<td>Review small doubles on AR and then introduce idea of small near doubles</td>
<td>AR Unifix Bingo boards Match cards</td>
</tr>
</tbody>
</table>

Group 1 -  
Group 2 -  
Group 3 -  
Group 4 -  

YEAR: 2010  
TERM: 3  
WEEK: 8  
CLASS: Prep F
| Day  | Mathematical Objective for Lesson | Tool Session: | Whole Class Focus:/: | Activity: | Share Time / Summary: | Resources/Assessment:
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<tbody>
<tr>
<td>Session 21 6.9.10</td>
<td>Chn to practise their small and large doubles</td>
<td>Make a 9+ on board chn find correct expression - then change to a 10+ and find correct expression</td>
<td>Chn play match up activity for either small or large doubles</td>
<td>Chn complete a thinkboard for either a small or large double</td>
<td>Review small/large doubles on AR and then chn use idea of double fingers with a partner</td>
<td>AR Match cards Thinkboard template</td>
</tr>
</tbody>
</table>

Small doubles: Large doubles:
<table>
<thead>
<tr>
<th>Day</th>
<th>Mathematical Objective for Lesson</th>
<th>Tool Session:</th>
<th>Whole Class Focus/:</th>
<th>Activity:</th>
<th>Share Time / Summary:</th>
<th>Resources/Assessment:</th>
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</table>
| Session 22  
7.9.10 | Chn to revise splits of 5 and splits of 10 | Make a 9+ on board chn find correct expression - then change to a 10+ and find correct expression | Review splits of 5 or 10 using unifix cubes in 5s - chn play make 5 or make 10 with gameboard and whiteboard markers | Chn cut up numbers and make mathematical statements about the splits of 5 or 10. | Review splits of 10 AR Worksheet unifix |

Splits of 5:  
Large doubles:
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<tr>
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<tbody>
<tr>
<td>Session 23 8.9.10</td>
<td>Chn to work with small near doubles, and record build through 10 on ENL</td>
<td>Flash dot patterns - how many dots all together - start with 5+ but do other combinations as well</td>
<td>Build through 9 using two ten frames - model thinking on ENL</td>
<td>Group 1 - practise building through 10 and recording on ENL Group 2 - small near doubles Group 3 - Draw small doubles on AR template</td>
<td>Chn make small double with unifix and then add another cube to top row to make a near double</td>
<td>Dot patterns Worksheet</td>
</tr>
</tbody>
</table>

Group 2:  
Group 1:  
Group 3:
<table>
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<tr>
<td>Session 24 10.9.10</td>
<td>Chn to solve single digit addition tasks in the range 1 to 10 using strategies of their choice</td>
<td>Flash five wise black ten frames and ask 4 qns - How many on top row? How many on bottom row? How many all together? How many more to make ten?</td>
<td>Roll two dice - ask chn to add two numbers together - what equipment could we use to help us check our answers?</td>
<td>Chn complete activity at tables. Roll 2 dice and add 2 numbers together - observe strategies</td>
<td>Ask chn following qns - what is answer HDYK? 4 + 2 = 5 + 3 = 6 + 2 = 3 + 5 =</td>
<td>6 sided dice 2,3,4 dice Counters Blank ten frames Blank Unifix in groups of 5</td>
</tr>
</tbody>
</table>

Lisamarie - complete sheet tracing numbers and drawing correct number of items

Note: Students’ names have been removed to preserve anonymity.